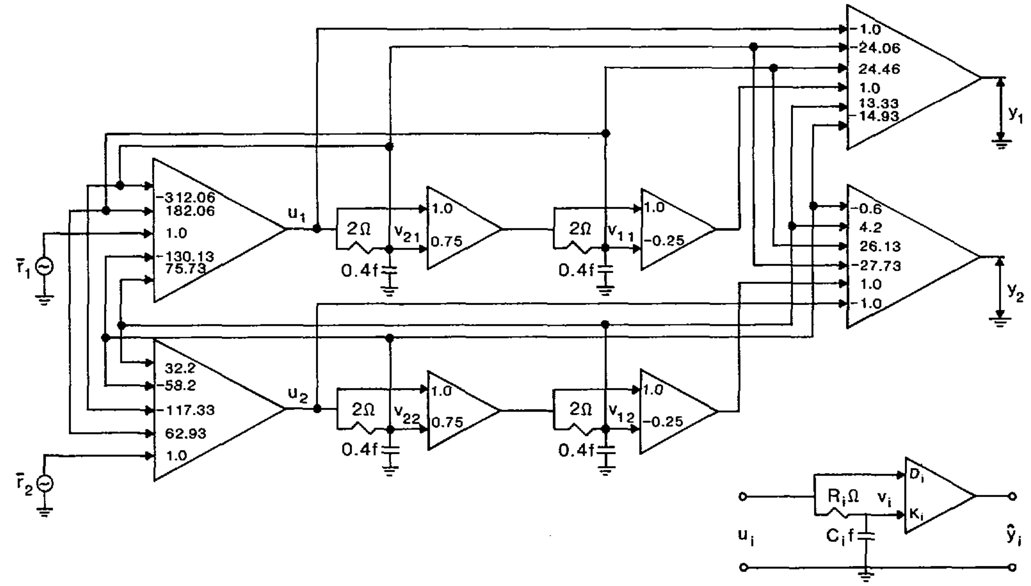

Fig. 1. Cascade realization of $T(s)$.

km where k is an integer, a set of nondominant eigenvalues can be placed at the diagonal entries of the system matrix A in (1) to enlarge the dimension of A from n to $\hat{n}$ such that $\hat{n} = km$. As a result, the proposed techniques can still be applied to carry out the block-decomposition and realization of the modified systems.

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Matrix Fraction Construction of Linear Compensators

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Abstract—**A construction is given using polynomial matrix fractions of the compensator resulting from a combination of a linear feedback control law and estimator design.**

Introduction

The aim of this note is to exhibit a matrix fraction description for the development of a compensator for a linear system using a state feedback law and estimator design for that system.

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To fix ideas, suppose that we appeal to linear-quadratic design methods. In state-variable terms, the plant is

$$\dot{x} = Fx + Gu + w \qquad z = Hx + v \tag{1.1}$$

where $[F, G, H]$ is minimal, $E[ww'] = \bar{Q}\delta(t)$, $G[uu'] = \bar{R}\delta(t)$, $E[wv'] = 0$, w and v being zero mean, Gaussian, white noise processes, and the performance index is

$$J = \lim_{t \to \infty} E[u'Ru + x'Qx]. \tag{1.2}$$

If the feedback gain is K and the estimator gain is L, then the compensator transfer function matrix is

$$W_c(s) = K'(sI - F + LH' + GK')^{-1}L. \tag{1.3}$$

Sometimes, the compensator is defined using the two transfer function matrices

$$W_{c1}(s) = K'(sI - F + LH')^{-1}G \qquad W_{c2}(s) = K'(sI - F + LH')^{-1}L \tag{1.4}$$

(see Fig. 1). Actually, the transfer function matrix from the reference input to the output is different for the two arrangements; the equivalence is limited to closed-loop models.

II. Polynomial Matrix Solution of the Feedback Law and Estimator Design Problems

Following, for example, [1] let us define the following polynomial matrix fraction descriptions, all coprime:

$$(sI - F)^{-1}G = B_1A^{-1} \qquad H'(sI - F)^{-1}G = BA^{-1} \tag{2.1}$$

$$H'(sI - F)^{-1} = \bar{A}^{-1}\bar{C}_1 \qquad H'(sI - F)^{-1}G = \bar{A}^{-1}\bar{B}. \tag{2.2}$$

Define also square polynomial matrices M, $\bar{N}$ with Hurwitz determinants by

$$A_*RA + B_{1*}QB_1 = M_*RM \quad \lim_{s \to \infty} MA^{-1} = I \tag{2.3}$$

$$\overline{ARA}_* + \bar{C}_1\bar{Q}\bar{C}_{1*} = \overline{NRN}_* \quad \lim_{s \to \infty} \bar{A}^{-1}\bar{N} = I. \tag{2.4}$$

Then the connection between M, $\bar{N}$, and the state variable quantities is provided by

$$MA^{-1}=I+K'(sI-F)^{-1}G \tag{2.5}$$

$$\bar{A}^{-1}\bar{N}+I+H'(sI-F)^{-1}L. \tag{2.6}$$

Our aim now is to define the compensator $W_c(s)$ using M, $\bar{N}$, A, B, etc.

III. A Key Lemma

By way of motivation of the lemma below, let us observe that in the 2×2 block matrix

$$W_e(s) \triangleq \begin{bmatrix} H' \\ K' \end{bmatrix} (sI-F)^{-1}[G \quad L] \tag{3.1}$$

we have MFD's for the $1-1$, $1-2$, and $2-1$ entries [see (2.1), (2.2), (2.5), and (2.6)]. We also know the McMillan degree of the whole matrix is identical to that for the $1-1$ entry. The lemma below explains how we can construct a minimal MFD for the whole matrix, and is presumably of independent interest.

Lemma 3.1: Consider the 2×2 block matrix (3.1), with MFD's for the individual blocks defined by

$$H'(sI-F)^{-1}G=BA^{-1}=\bar{A}^{-1}\bar{B} \qquad K'(sI-F)^{-1}G=(M-A)A^{-1}$$

$$H'(sI-F)^{-1}L=\bar{A}^{-1}(\bar{N}-\bar{A}). \tag{3.2}$$

Let X, Y define any polynomial solutions of

$$\bar{A}X+\bar{B}Y=I \tag{3.3}$$

(which exist because $\bar{A}$, $\bar{B}$ are left coprime) and let

$$Z=\Pi(MA^{-1}Y\bar{N}) \tag{3.4}$$

where $\Pi(\cdot)$ denotes taking the polynomial part. Then a coprime MFD of $W_e(s)$ is provided by

$$W_e(s)=\begin{bmatrix} B & X\bar{N}-I \\ M-A & Y\bar{N}-Z \end{bmatrix}\begin{bmatrix} A & -Y\bar{N} \\ 0 & I \end{bmatrix}^{-1}. \tag{3.5}$$

Proof: Observe that

$$\begin{bmatrix} B & X\bar{N}-I \\ M-A & Y\bar{N}-Z \end{bmatrix}\begin{bmatrix} A & -Y\bar{N} \\ 0 & I \end{bmatrix}^{-1}$$

$$=\begin{bmatrix} B & X\bar{N}-I \\ M-A & Y\bar{N}-Z \end{bmatrix}\begin{bmatrix} A^{-1} & A^{-1}Y\bar{N} \\ 0 & I \end{bmatrix}$$

$$=\begin{bmatrix} BA^{-1} & BA^{-1}Y\bar{N}+X\bar{N}-I \\ (M-A)A^{-1} & MA^{-1}Y\bar{N}-Y\bar{N}+Y\bar{N}-Z \end{bmatrix}$$

$$=\begin{bmatrix} H'(sI-F)^{-1}G & \bar{A}^{-1}(\bar{B}Y+\bar{A}X)\bar{N}-I \\ K'(sI-F)^{-1}G & MA^{-1}Y\bar{N}-Z \end{bmatrix}$$

$$=\begin{bmatrix} H'(sI-F)^{-1}G & H'(sI-F)^{-1}L \\ K'(sI-F)^{-1}G & \text{strictly proper} \end{bmatrix}. \tag{3.6}$$

Because the denominator matrix of $W_e(s)$ has determinantal degree equal to dim F, and $H'(sI-F)^{-1}G$ is a submatrix of $W_e(s)$ with $[F, G, H]$ minimal, (3.5) must be coprime. It remains to be shown that the $2-2$ block entry in (3.6) is necessarily $K'(sI-F)^{-1}L$. Now consider the following problem: given rational transfer function matrices $X_1(s)$, $X_2(s)$, $X_3(s)$ of dimensions such that the following statement can be made concerning McMillan degrees:

$$\delta[X_1]=\delta[X_1 : X_2]=\delta\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \tag{3.7}$$

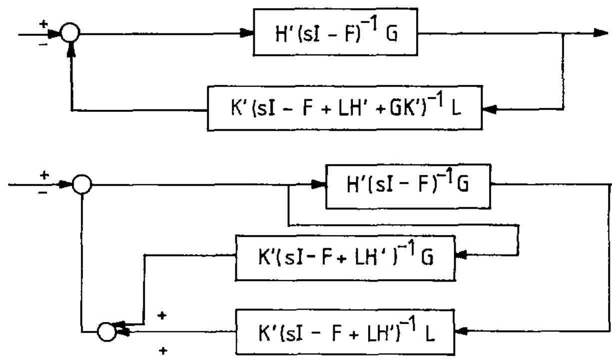

Fig. 1. Plant with two different compensator arrangements.

find X_4 such that

$$\delta\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}=\delta[X_1] \tag{3.8}$$

and X_4 is strictly proper. This problem has a unique solution, as can be seen by working with the relevant Hankel matrices formed from the Markov coefficients of X_1, X_2, X_3. Identifying X_1 with $H'(sI-F)^{-1}G$, etc., above, we see that the bottom right block in (3.6) can only be $K'(sI-F)^{-1}L$.

IV. Construction of the Compensator

From the MFD defined via (3.1) and (3.5), we construct an MFD of a further transfer function matrix which includes the compensator transfer function matrix.

Lemma 4.1: With $W_e(s)$ as in (3.1) and possessing the MFD (3.5), there holds

$$\begin{bmatrix} H' \\ K' \end{bmatrix}(sI-F+LH'+GK')^{-1}[G \quad L]$$

$$=\begin{bmatrix} B & X\bar{N}-I \\ M-A & Y\bar{N}-Z \end{bmatrix}\begin{bmatrix} M & -Z \\ B & X\bar{N} \end{bmatrix}^{-1}. \tag{4.1}$$

Proof: If, in Laplace transform notation, $y_e(s) = W_e(s)u_e(s)$, then both sides of (4.1) result from using output feedback

$$u_e=v_e-\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}y. \tag{4.2}$$

Finally, we can state the following.

Lemma 4.2: With quantities as defined in Section II and the statement of Lemma 3.1, there holds

$$K'(sI-F+LH'+GK')^{-1}L=\tilde{Y}\tilde{X}^{-1} \tag{4.3}$$

where

$$\tilde{Y}=Y\bar{N}-AM^{-1}Z \qquad \tilde{X}=X\bar{N}+BM^{-1}Z. \tag{4.4}$$

Proof: By the matrix inversion lemma,

$$\begin{bmatrix} M & -Z \\ B & X\bar{N} \end{bmatrix}^{-1}=\begin{bmatrix} M^{-1}-M^{-1}Z\tilde{X}^{-1}BM^{-1} & M^{-1}Z\tilde{X}^{-1} \\ -\tilde{X}^{-1}BM^{-1} & \tilde{X}^{-1} \end{bmatrix}. \tag{4.5}$$

Then (4.1) yields

$$K'(sI-F+LH'+GK')^{-1}L=(M-A)M^{-1}Z\tilde{X}^{-1}+(Y\bar{N}-Z)\tilde{X}^{-1}$$
$$=(Y\bar{N}-AM^{-1}Z)\tilde{X}^{-1}$$

and the lemma is proved.

Remark 4.1: We can form a polynomial MFD for the compensator transfer function by introducing a coprime pair M_1, Z_1 for which

$$M^{-1}Z=Z_1M_1^{-1}. \tag{4.6}$$

Then the MFD is $\hat{Y}\hat{X}^{-1}$ where

$$\hat{Y}=\tilde{Y}M_1=Y\tilde{N}M_1-AZ_1 \qquad \hat{X}=\tilde{X}M_1=X\tilde{N}M_1+BZ_1. \tag{4.7}$$

Observe then the "characteristic equation"

$$\bar{A}\hat{X}+\bar{B}\hat{Y}=(\bar{A}X+\bar{B}Y)\tilde{N}M_1+(\bar{A}B-\bar{B}A)Z_1=\tilde{N}M_1. \tag{4.8}$$

This exhibits the closed-loop modes in terms of the estimator modes (zeros of det $\tilde{N}$) and "control" modes (zeros of det M_1 = zeros of det M in general).

Remark 4.2: In the scalar plant case, A, B, etc., are all scalar polynomials, $\bar{A}=A$, $\bar{B}=B$, $M_1=M$, and (4.8) modified can be used to define the controller $\hat{Y}\hat{X}^{-1}$ by

$$A\hat{X}+B\hat{Y}=\tilde{N}M, \quad \deg \hat{Y}<\deg A. \tag{4.9}$$

As might be expected, the scalar plant results can be derived also *ab initio* by a simpler procedure, see, e.g., [2].

Remark 4.3: A dual version of Lemma 3.1 and the later ideas is possible. Let $\bar{X}$, $\bar{Y}$ be any solution of

$$\bar{X}A+\bar{Y}B=I \tag{4.10}$$

and let

$$\bar{Z}=\Pi(M\bar{Y}\bar{A}^{-1}\bar{N}). \tag{4.11}$$

Then

$$W_e(s)=\begin{bmatrix}\bar{A} & 0\\ -M\bar{Y} & I\end{bmatrix}^{-1}\begin{bmatrix}\bar{B} & \bar{N}-\bar{A}\\ M\bar{X}-I & M\bar{Y}-\bar{Z}\end{bmatrix} \tag{4.12}$$

$$\begin{bmatrix}H'\\K'\end{bmatrix}(sI-F+LH'+GK')^{-1}[G\quad L]$$

$$=\begin{bmatrix}\bar{N} & \bar{B}\\ -\bar{Z} & M\bar{X}\end{bmatrix}^{-1}\begin{bmatrix}\bar{B} & \bar{N}-\bar{A}\\ M\bar{X}-I & M\bar{Y}-\bar{Z}\end{bmatrix} \tag{4.13}$$

and

$$K'(sI-F+LH'+GK')^{-1}L=\bar{\bar{X}}^{-1}\bar{\bar{Y}} \tag{4.14}$$

where

$$\bar{\bar{X}}=M\bar{X}+\bar{Z}\bar{N}^{-1}\bar{B} \qquad \bar{\bar{Y}}=M\bar{Y}-\bar{Z}\bar{N}^{-1}\bar{A}. \tag{4.15}$$

With $\overline{ZN}^{-1}=\bar{N}_1^{-1}\bar{Z}_1$, we have

$$K'(sI-F+LH'+GK')^{-1}L=\hat{\bar{X}}^{-1}\hat{\bar{Y}} \tag{4.16}$$

where

$$\hat{\bar{X}}=\bar{N}_1M\bar{X}+\bar{Z}_1\bar{B} \qquad \hat{\bar{Y}}=\bar{N}_1M\bar{Y}-\bar{Z}_1\bar{A} \tag{4.17}$$

and

$$\hat{\bar{X}}A+\hat{\bar{Y}}B=\bar{N}_1M. \tag{4.18}$$

Remark 4.4: Further insight into the use of the dual follows by combining $\bar{B}A=\bar{A}B$, (3.3) and (4.10), to yield

$$\begin{bmatrix}\bar{A} & \bar{B}\\ -\bar{Y} & \bar{X}\end{bmatrix}\begin{bmatrix}X & -B\\ Y & A\end{bmatrix}=\begin{bmatrix}I & 0\\ J & I\end{bmatrix}$$

for some J. Now notice that without loss of generality we can replace $\bar{X}$, $\bar{Y}$ by $\bar{X}-J\bar{B}$, $\bar{Y}+J\bar{A}$; then (4.10) still holds while we secure

$$\begin{bmatrix}\bar{A} & \bar{B}\\ -\bar{Y} & \bar{X}\end{bmatrix}\begin{bmatrix}X & -B\\ Y & A\end{bmatrix}=\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}. \tag{4.19}$$

This embodies $\bar{B}A=\bar{A}B$, (3.3), (4.10) and

$$\bar{Y}X=\bar{X}Y. \tag{4.20}$$

By interchanging the order of multiplication of the two matrices on the left side of (4.19), we can derive also $A\bar{Y}=Y\bar{A}$ or

$$A^{-1}Y=\overline{YA}^{-1} \tag{4.21}$$

and then (3.4) and (4.11) show that

$$Z=\bar{Z}. \tag{4.22}$$

Remark 4.5: When the controller is defined by the two transfer function matrices of (1.4), we can get MFD descriptions. Starting with the MFD of Lemma 3.1, arguments like those yielding the result of Lemma 4.1 establish that

$$\begin{bmatrix}H'\\K'\end{bmatrix}(sI-F+LH')^{-1}[G\quad L]$$

$$=\begin{bmatrix}B & X\bar{N}-I\\ M-A & Y\bar{N}-Z\end{bmatrix}\begin{bmatrix}A & -Y\bar{N}\\ B & X\bar{N}\end{bmatrix}^{-1}.$$

Assume that the special $\bar{X}$, $\bar{Y}$ pair is used which satisfies (4.19). Then

$$\begin{bmatrix}A & -Y\bar{N}\\ B & X\bar{N}\end{bmatrix}^{-1}=\begin{bmatrix}\bar{X} & \bar{Y}\\ -\bar{N}^{-1}\bar{B} & \bar{N}^{-1}\bar{A}\end{bmatrix} \tag{4.23}$$

so that

$$K'(sI-F+LH')^{-1}G$$

$$=(M-A)\bar{X}-(Y\bar{N}-Z)\bar{N}^{-1}\bar{B}$$

$$=M\bar{X}-I+Z\bar{N}^{-1}\bar{B} \tag{4.24a}$$

$$=\text{strictly proper part of } Z\bar{N}^{-1}\bar{B} \tag{4.24b}$$

$$=\bar{N}_1^{-1}\hat{\bar{X}}-I. \tag{4.24c}$$

(Equation (4.24c) follows from (4.17) using minor manipulation.)

Similary to (4.24), we have

$$K'(sI-F+LH')^{-1}L$$

$$=(M-A)\bar{Y}+(Y\bar{N}-Z)\bar{N}^{-1}\bar{A}$$

$$=M\bar{Y}-A\bar{Y}+Y\bar{A}-\bar{Z}\bar{N}^{-1}\bar{A}$$

$$=M\bar{Y}-Z\bar{N}^{-1}A \tag{4.25a}$$

$$=\text{strictly proper part of } Z\bar{N}^{-1}\bar{A} \tag{4.25b}$$

$$=\bar{N}_1^{-1}\hat{\bar{Y}}. \tag{4.25c}$$

V. Conclusions

We have presented a comparatively simple procedure for constructing, using polynomial MFD ideas, a compensator which is a combination of a state feedback law and a state estimator. If the calculation can be carried through also for MFD's constructed using the Euclidean domain of proper stable transfer functions, it may be possible to establish some robustness results for closed-loop systems designed this way. In connection with this latter type of representation, the ideas of [3] may be relevant.

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