CIRCUITS SYSTEMS SIGNAL PROCESS
VOL. 4, NO. 4, 1985

WAVENUMBER SUPPORT REGIONS FOR ARBITRARY ARRAYS*

Brian D.O. Anderson[1]

Abstract. We set out a procedure for defining the region of wavenumber support relevant for maximum entropy spectral estimation of a noise field incident on a planar array. We consider first linear, nonequispaced arrays, and argue that the problem of defining a region of wavenumber support is only well-posed when either position uncertainty for the array elements is postulated, or phase uncertainty in the measurement of the signals at the array elements is postulated, or one acknowledges that a polar diagram that is almost periodic defines an aliasing frequency in virtually the same way as a polar diagram which is periodic. For planar arrays, the region of wavenumber support can be approximately circular but in general has to be defined with reference to the particular array.

1. Introduction

The purpose of this work is to consider a question which arises in multidimensional spectral estimation, particularly maximum entropy spectral estimation see e.g. [1]. It is common to define spectra using wavenumber as an independent variable. Then quantities such as correlations or entropy can be defined using the spectrum and an integration operation. The integration is over "the regions of wavevector space in which power is assumed to be present," and this is a "compact subset" of some Euclidean space [1]. Again [2], the integration is over "the known region spaces in which the power is confined." In [3] a rectangular region (actually cubic) is assumed for an arbitrary three dimensional array; since such a region is not rotationally invariant, and since there seems no a priori reason for selecting any particular three mutually orthogonal directions as coordinate axes (relative to which the region is defined), there seems to be an element of arbitrariness in the definition.

The question thus arises: how should the region of wavenumber support

* Received June 1, 1984; revised June 1, 1985. Work supported by Australian Department of Defence.

[1] Department of Systems Engineering, Research School of Physical Sciences, Australian National University, Canberra, Australia.

be defined for an arbitrary array? Here we attempt to present a defining procedure for an arbitrary planar array (which is readily extendable to an arbitrary three dimensional array), since we have been unable to find a convincing description of such a procedure in the literature.

To begin, we recall what is known for a linear equispaced array, and then we consider linear nonequispaced arrays, with interelement spacings both commensurate and incommensurate, i.e. expressible as a ratio of two integers or not so expressible. In this last case, formal analysis suggests there is no spatial aliasing so that it would be appropriate to regard the region of wave number space in which power was confined as the whole space – a plainly unattractive situation. One way round this difficulty is to postulate uncertainty in our knowledge of element positions; one can then associate a spatial aliasing frequency with an arbitrary linear array (and then, as seems standard, the region of wavenumber space referred to earlier is limited by the aliasing frequency.)

As it turns out, alternative and related approaches to defining an aliasing frequency can be obtained by postulating phase uncertainty in the measurements at the different elements, and by studying the array polar diagram, which is always "almost periodic." The existence of an "almost period" then implies existence of (for practical purposes) an aliasing frequency.

We considered planar as opposed to linear arrays by examining all possible directions of wave front arrival using the linear array idea to define a spatial aliasing frequency for each arrival direction. We find that in general, this frequency may not be the same for all directions, and that there are directions in which it takes an easily computable maximum value. For certain arrays, this maximum value is approximately achieved in all directions; we give a simple condition on a circularly symmetric array for this to be so. In this situation of course, the region of wavenumber space required for maximum entropy theory is approximately circular.

Let us indicate finally what is not involved here. There is a substantial body of literature, see e.g. [4–6] on nonuniform sampling of time waveforms. One can, for example, find results that suggest that the aliasing frequency is identical with that which would be obtained with uniform sampling at the same rate as the rate of the nonuniform sampling. These ideas are not applicable to the problems of this paper. The reason is that the problem of interest is really a different one. In [4–6], the least frequency f_A is sought such that any sinusoid of frequency greater than f_A is always indistinguishable from a *linear combination* of sinusoids of frequency less than f_A, rather than a single sinusoid. For a *finite* array, any positive frequency has the property that a single sinusoid of greater frequency is a linear combination of sinusoids of lower frequency (generically, the number of lower frequency sinusoids is the number of array elements).

2. Linear arrays with exact knowledge of array element position

We shall consider in this section array elements located along the x-axis at $0, x_1, \ldots, x_n$, and we shall study the problem of determining the spatial frequency $\omega_x x$ of a wave $\cos(\omega_x x - \omega_t t + \theta_0)$ travelling in the positive or negative x direction. We assume $\omega_t > 0$; motion in the positive x direction corresponds to $\omega_x > 0$, in the negative x direction to $\omega_x < 0$.

Case 1: $x_i = iL$ for integer i, L being the interelement spacing. What happens with a uniform array is of course well known, but we review the situation briefly in preparation for considering other situations. Notice first that for fixed x_i, $\cos(\omega_x x_i - \omega_t t + \theta_0)$ is simply a function of time, and from looking at this function of time, one immediately knows $(\omega_x x_i + \theta_0) \bmod 2\pi$. [If we are given the value of $\cos(\omega_x x_i - \omega_t t + \theta_0)$ at one particular instant of time, with also the value of ω_t, we do *not* immediately know $(\omega_x x_i + \theta_0) \bmod 2\pi$. We can only find a $\phi_i \in [0, \pi]$ such that either $\omega_x x_i + \theta_0 - \omega_t t = \phi_i \bmod 2\pi$ or $\omega_x x_i + \theta_0 - \omega_t t = -\phi_i \bmod 2\pi$. The resolution of which sign applies is achieved if we know the function for all t or, equivalently, we know $\sin(\omega_x x_i - \omega_t t + \theta_0)$ at the same particular instant of time].

Now let $\omega_x x_i + \theta_0 = \phi_i \bmod 2\pi$ and $\theta_0 = \phi_0 \bmod 2\pi$ where ϕ_0 and ϕ_i lie in $(0, 2\pi)$. Then

$$\omega_x x_i = (\phi_i - \phi_0) \bmod 2\pi \tag{2.1}$$

In the ideal situation where there is no noise or other perturbation in the assumptions, this leads to

$$\omega_x = \frac{\phi_i - \phi_0}{L} - n\frac{2\pi}{L} \tag{2.2}$$

for some integer n. Accordingly, if it is known that $\omega_x \in [-\pi/L, \pi/L]$, then ω_x is uniquely defined. The frequency $\Omega = \pi/L$ is the *aliasing* frequency* Signals with spatial frequency outside $(-\pi/L, \pi/L)$ are indistinguishable from signals within this band.

Notice that if there is a priori information about the direction of arrival of a wave, i.e. a priori information about the sign of ω_x, the aliasing frequency becomes $2\pi/L$ or $-2\pi/L$ and any ω_x known to lie in $(0, 2\pi/L)$ or $(0, -2\pi/L)$ can be uniquely identified.

* We regard an aliasing frequency as an end point of an interval of frequencies within which measurements allow unambiguous identification of a frequency.

Case 2: $x_i = m_i L$ for integer m_i, L being a type of unit distance relative to which interlement spacings are defined. Then we assert

Lemma 2.1. With $x_i = m_i L$, for m_i integer, $i = 1,2, \ldots, n$ the aliasing frequency Ω involved in determining ω_x from measurements of $\cos(\omega_x x_i - \omega_t t + \theta_0)$ is

$$\Omega = \frac{\pi}{\bar{m}L} \tag{2.3}$$

$$\bar{m} = \text{greatest common divisor of } m_i, \ldots, m_n. \tag{2.4}$$

Proof. Measurement of $\cos(\omega_x x_i - \omega_t t + \theta_0)$ pins down the phase to mod 2π, i.e. we learn values $\phi_i \in [0,2\pi)$ such that

$$\omega_x m_i L + \theta_0 = \phi_i \bmod 2\pi$$

or

$$\omega_x m_i L = (\phi_i - \phi_0) \bmod 2\pi \qquad i = 1, \ldots, n. \tag{2.5}$$

With $\bar{m}$ the greatest common divisor of $m_1, \ldots, m_n$, there exist integers α_i such that $\bar{m} = \Sigma_{i=1}^{n} \alpha_i m_i$. Then

$$\omega_x \bar{m} L = \Sigma \alpha_i (\phi_i - \phi_0) \bmod 2\pi \tag{2.6}$$

If Ω is such that in addition to (2.5), there holds

$$(\omega_x + 2\Omega) m_i L = (\phi_i - \phi_0) \bmod 2\pi \tag{2.7}$$

we see immediately that

$$(\omega_x + 2\Omega) \bar{m} L = \Sigma \alpha_i (\phi_i - \phi_0) \bmod 2\pi. \tag{2.8}$$

Conversely, if (2.8) holds for some Ω and if $m_i = k_i \bar{m}$ for some integer k_i, (2.6) and (2.8) yield

$$\omega_x m_i L = k_i \Sigma \alpha_i (\phi_i - \phi_0) \bmod 2\pi \tag{2.9}$$

$$(\omega_x + 2\Omega) m_i L = k_i \Sigma \alpha_i (\phi_i - \phi_0) \bmod 2\pi \tag{2.10}$$

This means that if Ω is an aliasing frequency for (2.5), it must be an aliasing frequency for (2.6) and conversely. The aliasing frequency for (2.6) is clearly given by (2.3). Hence Ω in (2.3) is the aliasing frequency for (2.5), i.e. that associated with the original problem.

As an illustration of this idea, suppose that $x_1 = 19L$, $x_2 = 39L$. Then the greatest common divisor of 19 and 39 is 1, so the aliasing frequency is π/L. If we had instead $x_1 = 19L$, $x_2 = 38L$, then the aliasing frequency would be $\pi/19L$.

Case 3: $x_1 = L_1$, $x_2 = L_2$ with L_1, L_2 incommensurate (i.e. there do not exist integers m_1, m_2 such that $L_1 = m_1 L$ and $L_2 = m_2 L$ for some L). This is of course an idealized situation. We assert:

Lemma 2.2. *With $x_1 = L_1$, $x_2 = L_2$ and L_1, L_2 incommensurate, there is no finite aliasing frequency.*

Proof. If there were a finite aliasing frequency Ω, we should have

$$(\omega_x + 2\Omega)L_1 = (\phi_1 - \phi_0) \bmod 2\pi$$

and

$$\omega_x L_1 = (\phi_1 - \phi_0) \bmod 2\pi;$$

hence

$$2\Omega L_1 = 0 \bmod 2\pi$$

or

$$2\Omega_1 L_1 = 2\pi N_1 \quad \text{for some integer } N_1$$

Similarly,

$$2\Omega L_2 = 2\pi N_2 \quad \text{for some integer } N_2.$$

It follows that $L_1/L_2 = N_1/N_2$; a contradiction of the incommensurate assumption.

The apparent implication of this result is that if three (or more) array elements are on a straight line, but otherwise randomly located, no aliasing will occur (with probability 1). In practical terms, this is nonsense. The reason why such a result has been derived is that our assumptions have oversimplified the problem – for example we have assumed that the element positions are precisely known when in practice they can never really be so.

When we allow for uncertainty, an aliasing frequency does occur, as we now examine.

3. Linear arrays with inaccurate knowledge of array element positions

Let us suppose that no interelement distance is precisely known, but rather, there is an error of up to $\pm\delta/2$ in defining any distance from one element of the array to any other element. We assume that $\delta/2$ is significantly smaller than the interelement distance, and that these distances are $L_1, L_2, \ldots, L_n$, (measured from the zero-th element) with the elements forming a straight line.

It follows that ω_x must satisfy

$$\omega_x(L_i + \delta_i) = (\phi_i - \phi_0) \bmod 2\pi \tag{3.1}$$

for all i, and with $|\delta_i| < \delta/2$.

Before considering the question of aliasing frequencies, let us consider the structure of solutions to (3.1). For the moment, suppose there is only one equation in the set (3.1). Part of the solution set is depicted in Fig. 1 for positive ω_x, the solution set comprises the intervals $I_1, I_2, I_3, \ldots$ There are a finite number of finite length intervals, together with one infinite interval, in view of the following Lemma.

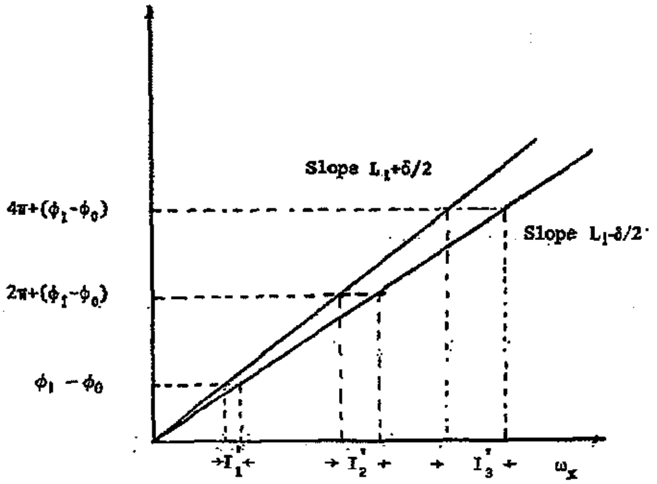

Figure 1. Solutions of a modular inequality determining spatial frequency.

Lemma 3.1 For arbitrary $\omega_x > 2\pi/\delta$, there exists δ_1 with $|\delta_1| < \delta/2$ such that the first equation of (3.1) holds.

Proof. Define $\delta_1 = [(\phi_1 - \phi_0/\omega_x) - L_1] \bmod (2\pi / \omega_x)$. Since $\omega_x > 2\pi/\delta$, $2\pi/\omega_x < \delta$, δ_i can be chosen to lie within $(-\delta/2, \delta/2)$. Then multiplication of the defining equation for δ_1 by ω_x yields the first equation of (3.1).

Now to solve (3.1) for all i we construct the intervals I_k^i associated with $i=1$, $i=2$, ..., for both negative and positive ω_x. Any point in the set

$$S(\delta) = \cap_i[\cup_k I_k^i] \tag{3.2}$$

is a solution of every equation of (3.1), i.e., is a legitimate ω_x. Lemma 3.1 guarantees that the intersection above is nonempty, and that the least $|\omega_x|$ in the intersection set $S(\delta)$ certainly satisfies $|\omega_x| \leq 2\pi/\delta$.

Notice that if the above procedure is carried out for two values δ, δ' with $\delta < \delta'$, one has $I_k^i(\delta) \subset I_k^i(\delta')$ (Fig. 1) for all i and k and so

$$S(\delta) \subset S(\delta') \tag{3.3}$$

It is logical to regard ω_x in $S(\delta)$ with the smallest absolute value as the notionally correct spatial frequency. To the extent that $S(\delta)$ itself is a union of intervals, it would in fact not be unreasonable to label *any* ω_x in the interval nearest the origin as the spatial frequency. In this sense, there is a basic uncertainty in ω_x which depends on the size of the interval: it can be thought of as due to resolution rather than aliasing problems.

Now consider the determination of an aliasing frequency Ω. Observe first that a positive frequency Ω will serve as an aliasing frequency for *one* set of measurements, position errors and spatial frequency ω_x if there holds.

$$\omega_x(L_i + \delta_i) = \phi_i \bmod 2\pi \qquad \text{for all } i \tag{3.4}$$

$$(\omega_x + 2\overline{\Omega})(L_i + \delta_i'') = \phi_i \bmod 2\pi \qquad \text{for all } i \text{ if } \omega_x \leq 0 \tag{3.5a}$$

$$(\omega_x - 2\overline{\Omega})(L_i + \delta_i'') = \phi_i \bmod 2\pi \qquad \text{for all } i \text{ if } \omega_x \geq 0 \tag{3.5b}$$

$$|\delta_i| \leq \delta/2, \quad |\delta_i''| \leq \delta/2, \quad |\omega_x| \leq \overline{\Omega}. \tag{3.6}$$

Also, in defining an aliasing frequency Ω applicable to one situation, we must exclude taking Ω to be zero, or to be a very small frequency, so that $\omega_x \pm 2\Omega$ differs from ω_x by a trivial amount since we agreed above there was a basic resolution uncertainty in ω_x.

Equivalently, we need to insist that there cannot hold $(\omega_x + 2\overline{\Omega})(L_i + \delta_i'') = \omega_x(L_i + \delta_i')$ where there is no mod 2π. This will be so provided that

$$\Omega = 0\left[\frac{\omega_x \delta}{L_i + \delta}\right] \tag{3.7}$$

fails.

Now to define an aliasing frequency applicable to *all* measurement possibilities, we shall take the minimum Ω above, where the minimum is over all possible ω_x, δ_i, ϕ_i. We shall prove

Theorem 3.1 The aliasing frequency Ω is defined as the smallest positive Ω for which

$$2\Omega\,(L_i + \delta_i'') = \bmod 2\pi \qquad \text{for all } i \tag{3.8}$$

$$|\delta_i''| < \delta/2\,. \tag{3.9}$$

Before proving this result, we offer the following intuitive interpretation of (3.8) and (3.9).

Recall all array element positions have an error of $\pm\delta/2$. Suppose we find a length L and integers m_i such that

$$|L_i - m_i L| < \delta/2 \qquad \text{for all } i \tag{3.10}$$

Then our length measurements *could* have arisen from spacings $m_i L$. For such spacings, the aliasing frequency would be $\Omega = \pi/(mL)$, with m the greatest common divisor of the m_i.

Select the largest such L, (thus finding the "worst case" or lowest possible aliasing frequency). Then the greatest common divisor of the m_i must be unity, and $\Omega = \pi/L$. Thus we are finding the smallest positive Ω such that for nonzero integer m_i,

$$|L_i - m_i \frac{\pi}{\Omega}| < \delta/2 \qquad \text{for all } i. \tag{3.11}$$

Equivalently, we are finding the smallest positive Ω for which

$$\Omega(L_i + \delta_i) = 0 \bmod \pi \qquad \text{with } \Omega(L_i + \delta_i) \neq 0$$

or

$$2\Omega(L_i + \delta_i) = 0 \bmod 2\pi \qquad \text{with } \Omega \neq 0$$

for some δ_i *with* $|\delta_i| < \delta/2$ *and all* i.

Evidently the theorem identifies the aliasing frequency defined in a first principles way with an aliasing frequency defined by approximating in a worst case way the element spacings with multiples of some common base length. This observation allows us to see quickly that the results are consistent with the equal-spacing analysis of the previous section. For suppose that in (3.10), we can have $m_i = i$ (implying that the measurements of positions are consistent with equal spacing). Then obviously $m = 1$, and $\Omega = \pi/L$.

Fig. 2 illustrates how Ω can be determined grahically, depicting the intervals $I_1, I_2, \ldots$ in which a solution of the first equation of (3.8) must lie. It is not hard to check that there are a *finite number of finite width intervals*, and that the half line $\Omega > (\pi/\delta)[(L-\delta)/(L+\delta/2)]$ also defines a solution of the first equation (3.8. Thus $\pi/2$ always satisfies the first equation of (3.8). This reflects the fact that with $L = \delta$, we can always find integer m_i to satisfy (3.10). Since $\overline{m} \geq 1$, it follows that *the aliasing frequency can be at most* π/δ.

The proof of theorem 3.1 depends on the following lemma.

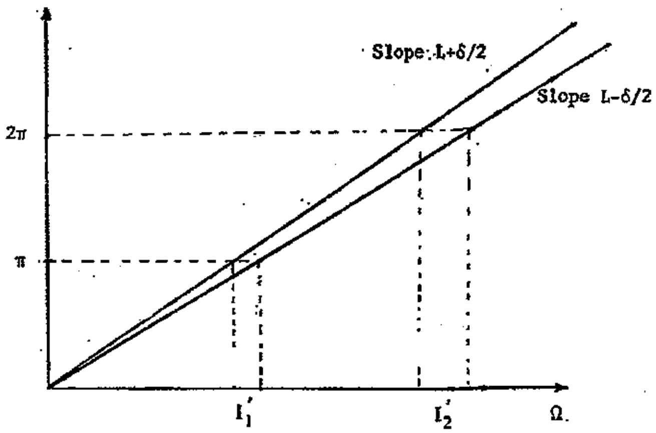

Figure 2. Solutions of a modular inequality determining the aliasing frequency.

Lemma 3.2. Suppose Ω is an aliasing frequency for one set of data ω_x, ϕ_i, δ_i, δ_i', δ_i'' i.e. (3.4) through (3.6) hold. Then it is also an aliasing frequency for data defined by

$$\tilde{\omega}_x = (\operatorname{sgn}\omega_x)\bar{\Omega} \tag{3.12}$$

$$\tilde{\delta}_i = \frac{\omega_x}{\tilde{\omega}_x}(\delta_i - \delta_i'') + \delta_i'' \qquad \tilde{\delta}_i'' = \delta_i'' \tag{3.13}$$

$$\tilde{\phi}_i = \tilde{\omega}_x(L_i + \tilde{\delta}_i) \bmod 2\pi. \tag{3.14}$$

Proof. From (3.13), we have $\tilde{\delta}_i = \alpha\delta_i + (1-\alpha)\delta_i''$ for $\alpha = \omega_x / \tilde{\omega}_x$ and so $\alpha \in [0,1]$. Hence $|\tilde{\delta}_i| < \delta/2$. This with (3.12) and (3.13) implies that (3.6) with δ_i replaced by $\tilde{\delta}_i$ etc. all hold. Equation (3.14) is the replacement of (3.4). Finally

$$\begin{aligned}
(\tilde{\omega}_x \pm 2\bar{\Omega})(L_i + \tilde{\delta}_i'') &= \tilde{\omega}_x(L_i + \delta_i'') \pm 2\bar{\Omega}(L_i + \delta_i'') \\
&= \tilde{\omega}_x(L_i + \tilde{\delta}_i) - \tilde{\omega}_x(\tilde{\delta}_i - \delta_i'') \pm 2\bar{\Omega}(L_i + \delta_i'') \\
&= \tilde{\phi}_i + (\omega_x \pm 2\bar{\Omega})(L_i + \delta_i'') - \omega_x(L_i + \delta_i) \bmod 2\pi \\
&= \tilde{\phi}_i + \phi_i - \phi_i \bmod 2\pi \\
&= \tilde{\phi}_i \bmod 2\pi
\end{aligned}$$

as required.

Now let us prove Theorem 3.1. The lemma shows that in finding the smallest positive $\bar{\Omega}$, we can restrict attention to situations where $\omega_x = \pm \bar{\Omega}$. Thus we want the smallest Ω such that

$$\pm \bar{\Omega}(L_i + \delta_i') = \phi_i \bmod 2\pi \tag{3.15}$$

$$\mp \bar{\Omega}(L_i + \delta_i'') = \phi_i \bmod 2\pi \tag{3.16}$$

$$|\delta_i'| \leq \delta/2, \quad |\delta_i''| \leq \delta/2. \tag{3.17}$$

Now set $\tilde{\delta}_i = 1/2(\delta_i' - \delta_i'')$. Then (3.15) and (3.16) imply

$$2\bar{\Omega}(L_i + \tilde{\delta}_i) = 0 \bmod 2\pi \tag{3.18}$$

while (3.17) implies

$$|\tilde{\delta}_i| \leq \delta/2. \tag{3.19}$$

Conversely, if (3.8) and (3.9) hold, one can easily find $\tilde{\delta}_i$, $\tilde{\delta}_i''$ with $\delta_i'' = 1/2(\tilde{\delta}_i - \tilde{\delta}_i'')$, $|\delta_i| < \delta/2$; $|\tilde{\delta}_i''| < \delta/2$. Define ϕ_i by

$$\bar{\Omega}(L_i + \tilde{\delta}_i) = \phi_i \bmod 2\pi.$$

Then

$$-\bar{\Omega}(L_i + \tilde{\delta}_i'') = \phi_i \bmod 2\pi.$$

Hence the smallest positive Ω satisfying (3.15) through (3.17) must be the smallest positive Ω satisfying (3.8) and (3.9), and Theorem 3.1 is proved.

Example. Consider an array with spacing at 0,1 and $\pi = 3.1415\ldots$ Suppose that distances are known to within $\pm$ 0.0015. Note that if $L = 1/7$, we have

$$\left|1 - 7.\frac{1}{7}\right| < 0.0015$$

$$\left|\pi - 22.\frac{1}{7}\right| < 0.0015.$$

So the aliasing frequencies will certainly be no greater than $\pi/(1/7) = 21.99$ rad/sec.

4. Linear arrays with noisy knowledge of phase

In this section we shall see that if there is uncertainty in the measurement of phase differences, it becomes possible to identify an aliasing frequency.

Suppose that there are $(n + 1)$ elements of a linear array located at $0, x_1, x_2, \ldots, x_n$, and that, with the true phase differences between the signals at the i-th and zero-th element being $\phi_i - \phi_0$, we have available a noisy measurement

$$\psi_i = \phi_i - \phi_0 + \eta_i \tag{4.1}$$

where

$$|\eta_i| < \eta \quad \text{for all } i. \tag{4.2}$$

A candidate for the spatial frequency ω_x of the incoming signal is provided by any ω_x for which

$$\omega_x L_i = (\psi_i + \eta_i) \bmod 2\pi \tag{4.3}$$

where $|\eta_i'| < \eta$, since the measurement ψ_i tells us that the true phase difference $\phi_i - \phi_0$ has to be of the form $\psi_i + \eta_i$ for some $|\eta_i| < \eta$. In general, the solutions ω_x of these inequalities will be in a set of nonoverlapping intervals, rather than just comprising a set of discrete points. Our primary interest would be in the interval closest to the origin. The width of the interval depends on η, but also the quantities L_i, ψ_i. The maximum width is min (η/L_i).

The gap between the interval closest to the origin and next closest to the origin is a measure of twice the aliasing frequency.

A candidate for the aliasing frequency Ω is one which satisfies

$$(\omega_x \pm 2\Omega)L_i = (\psi_i + \eta_i'') \bmod 2\pi \quad \text{for all } i \tag{4.4}$$

with $|\eta_i''| < \eta$. Evidently, such candidates Ω are equivalently characterized by

$$\Omega L_i = \eta_i \bmod \pi \quad \text{for all } i \tag{4.5}$$

where $|\eta_i| < \eta$. The solutions to any one of these question will again (in general) comprise a number of intervals, with the origin being a member of one interval. Our interest is in those Ω which are in the nearest interval to the origin, excluding that interval which contains the origin. Hence we seek the smallest solution to (4.5), subject to the constraint that

$$\Omega > \min_i \left[\frac{\pi - \eta}{L_i}\right], \tag{4.6}$$

this being the condition that Ω cannot belong to the interval containing the origin.

The characterization by an explicit analytic formula is not possible, except when all the L_i are integer multiples of some common L, i.e., $L_i = m_i L$. Then with $\bar{m}$ as the greatest common divisor of $m_i, \ldots, m_n$, we have

$$\left[\frac{\pi}{\bar{m}L}\right]L_i = 0 \bmod \pi$$

and for small enough η_i the aliasing frequency will be

$$\frac{\pi}{mL} - \min\left[\frac{\eta}{L_i}\right] .$$

In many cases this will naturally be approximately $\pi/(\bar{m}L)$. In the case of an equispaced array, $m = 1$, and one recovers the standard result, modulo the "resolution error" $\min(\eta/L_i)$.

5. Polar diagram behavior for the linear array

In this section we shall describe the occurrence of periodicities, and near periodicities in the polar diagram of a linear array. Periodicities and near periodicities can be used to define aliasing frequencies, as will be indicated.

Consider an $(n + 1)$ element array with element locations at $0, x_1, x_2, \ldots, x_n$. Suppose the weights for the various elements are chosen so that the array is pointed in the broadside direction (all shading coefficients are unity). As it turns out, this assumption contributes no loss of generality. Now if an incoming wave has spatial frequency ω_x then the response (array factor) will be

$$a(\psi_x) = 1 + \sum_{L=1}^{n} \exp\,(j\omega_x L_i) \tag{5.1}$$

and the polar diagram will be obtained by plotting against ω_x the function

$$S(\omega_x) = |a(\omega_x)|^2. \tag{5.2}$$

In case the L_i are all commensurate, i.e., $L_i = m_i L$ for integer m_i, then calculations of the type of Section 2 show that

$$a(\omega_x) = a(\omega_x + 2\Omega) \tag{5.3}$$

where $\Omega = \frac{\pi}{\bar{m}L}$, $\bar{m}$ = greatest common divisor of the m_i, and for equal spacings, $\bar{m} = 1$. It is logical to term Ω an aliasing frequency.

Now suppose the L_i are incommensurate. Then $a(\omega_x)$ is no longer periodic, but it is *almost periodic* [7–9] as is $S(\omega_x)$. (If $S(\omega_x)$ is never zero, $\ln S(\omega_x)$, an often plotted quantity is also almost periodic.) This means in particular that given arbitrary $\epsilon > 0$, we can find a positive Ω, and thus a least positive Ω, such that

$$|a(\omega_x) - a(\omega_x + 2\Omega)| < \epsilon \quad \text{for all } \omega_x. \tag{5.4}$$

Thus 2Ω is an almost-period of $a(\omega_x)$. Obviously the least Ω depends on ϵ, and for some value of ϵ, it will be practical to regard Ω as an aliasing frequency. Obviously also, 2Ω is also an almost period for $S(\omega_x)$, of course with a different ϵ.

Fig. 3 shows the polar diagram for the case $x_1 = 1, x_2 = \pi$. The existence of an effective aliasing frequency of $1/2 \times (6.95 \times 2\pi) = 21.8$ rad/sec is evident. (If the spacing had been at $x_1 = 1, x_2 = 22/7$, then we would have had $L = 1/7, x_1 = 7, x_2 = 22, m = 1, \Omega = 7\pi = 21.99$, which is obviously close.)

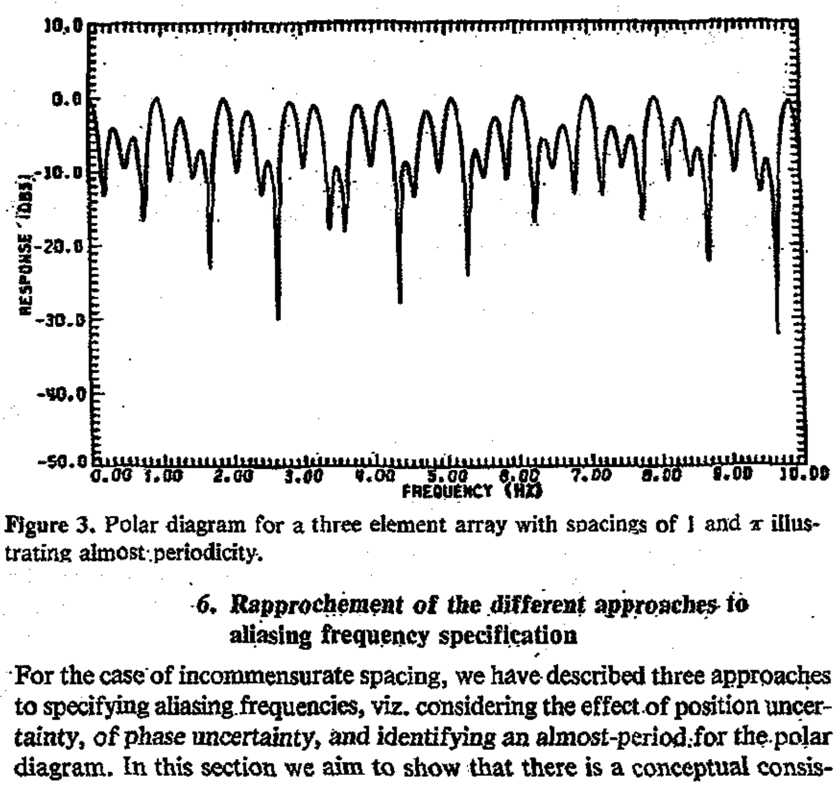

Figure 3. Polar diagram for a three element array with spacings of 1 and π illustrating almost periodicity.

6. *Rapprochement of the different approaches to aliasing frequency specification*

For the case of incommensurate spacing, we have described three approaches to specifying aliasing frequencies, viz. considering the effect of position uncertainty, of phase uncertainty, and identifying an almost-period for the polar diagram. In this section we aim to show that there is a conceptual consistency between these approaches.

The connection between position and phase uncertainty. With position uncertainties of up to $\pm\delta/2$, we found in Section 3 that the aliasing frequency $\Omega > 0$ is the least positive solution of

$$2\Omega(L_i + \delta_i) = 0 \bmod 2\pi \tag{6.1}$$

$$|\delta_i| < \delta/2. \tag{6.2}$$

In considering phase uncertainties, we found that $\Omega \leq 0$ is the least solution of

$$\Omega L_i = \eta_i \bmod \pi \tag{6.3}$$

$$|\eta_i| < \eta \tag{6.4}$$

$$\Omega \geq \max_i \left[\frac{\pi - \eta}{L_i}\right] \tag{6.5}$$

It is easy to check the following: if Ω is the least positive solution to (6.3) through (6.5) and δ is defined as $2\eta/\Omega$, then Ω will also be the least positive solution of (6.1) and (6.2). On the other hand, if Ω is the least positive solution to (6.1) and (6.2) and one defines $\eta = \Omega\delta/2$, then there may be a least positive solution Ω_1 to (6.3) through (6.5) with $\Omega_1 < \Omega$. In a sense then, the definition of aliasing frequency using phase uncertainty could be regarded as more fundamental.

The connection between phase uncertainty and almost periodicity of the polar diagrams. Suppose that Ω satisfies the phase uncertainty relations (6.3) through (6.5). Then observe that

$$\begin{aligned} a(\omega_x + 2\Omega) - a(\omega_x) &= \sum_{i=1}^{n} \exp j(\omega_x + 2\Omega) L_i - \sum_{i=1}^{n} \exp j\omega_x L_i \\ &= \sum_{i=1}^{n} \exp (j\omega_x L_i)[\exp j(2\Omega L_i) - 1] \\ |a(\omega_x + 2\Omega) - a(\omega_x)| &< n[(2\Omega L_i) \bmod 2\pi] \\ &< 2n\eta \end{aligned} \tag{6.6}$$

This calculation shows that the aliasing frequency defined using phase errors is also an aliasing frequency in the polar diagram sense, with the maximum phase error η defining the error level ϵ tolerable in the definition of almost periodicity through $\epsilon = 2n\eta$. The theory of almost periodic functions establishes that for mutually incommensurate L_i, $\sup |a(\omega_x)| = 1 + n$. So 2η is a measure of percentage departure from true periodicity of $a(\omega_x)$. Conversely, suppose that for all ω_x, some Ω and some ϵ, there holds

$$|a(\omega_x + 2\Omega) - a(\omega_x)| < \epsilon.$$

Suppose that the L_i are incommensurate (to avoid a more complicated argurment). Now

$$a(\omega_x + 2\Omega) - a(\omega_x) = \sum_{i=1}^{n} \exp (j\omega_x L_i)[\exp j(2\Omega L_i) - 1]$$

and, as proved in [7, p. 21],

$$\sup_{\omega_x} | a(\omega_x + 2\Omega) - a(\omega_x) | = \sum_l | \exp j(2\Omega L_l) - 1 |$$

Some manipulations using the facts $| \exp j\theta - 1 | = 2 | \sin(\theta/2) | \geq 2\theta/\pi$, for $0 \leq \theta \leq \pi$ and $| \exp j\theta - 1 | > 2(2\pi - \theta)/\pi$ for $\pi \leq \theta \leq 2\pi$, leads to

$$\sum [|2\Omega L_l \bmod 2\pi|] < \epsilon\pi/2.$$

This implies the following variant on (6.3) and (6.4):

$$\Omega L_l = \eta_l \bmod \pi \qquad (6.3)$$

$$\Sigma|\eta_l| < \epsilon\pi/4. \qquad (6.4')$$

Thus the sum of the magnitudes of the phase errors is bounded, rather than each one being separately bounded. Hence the aliasing frequency concepts for phase errors and polar diagrams are closely linked, but not identical. An alternative approach which might be contemplated to defining an almost period Ω for the polar diagram would be to require that

$$I(\Omega) = \lim_{W \to \infty} \frac{1}{W} \int_0^W |a(\omega_x) - a(\omega_x + 2\pi)|^2 d\omega_x \qquad (6.7)$$

was suitably small. It is not hard to verify that

$$I(\Omega) = \sum_{i=1}^{n} |1 - \exp j(2\Omega L_i)|^2 \qquad (6.8)$$

and it is clear that very similar considerations will hold.

Because of the connections between the three types of aliasing, it may be most convenient from the practical point of view to regard the aliasing frequency (or aliasing frequency candidate) to first be defined using a polar diagram (from which good candidates are easy to discern). One can then make at least limited connections to putative causes, viz. phase or position uncertainty.

7. Planar arrays with noisy knowledge of array element position

Let us now suppose that the nominal positions of the array elements are defined by $(0,0), (x_1, y_1), \ldots, (x_n, y_n)$ in a two-dimensional plane.

We shall now associate with each direction an aliasing frequency obtained in the following way. With reference to Fig. 4, suppose that we are seeking the aliasing frequency Ω_ϕ associated with the direction defined by a line making an angle ϕ with the positive x-axis. (Thus we imagine plane waves with wavefronts perpendicular to the line OQ). In effect, we are seeking an aliasing frequency associated with the lengths $x_1\cos\phi + y_1\sin\phi$, $x_2\cos\phi + y_2\sin\phi, \ldots$. Let us suppose that the error in the position of (x_i, y_i) is such

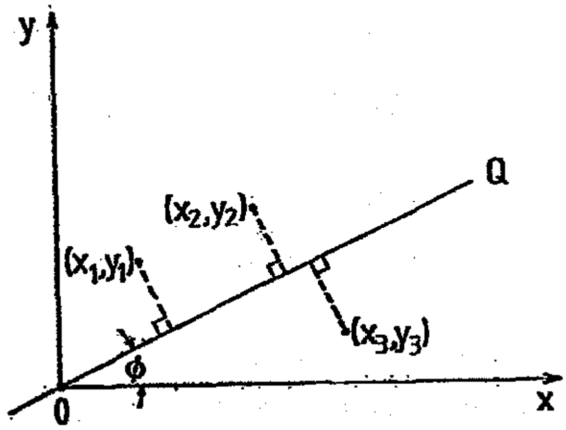

Figure 4. Determining the aliasing frequency in a given direction by using projected distances.

that the real point, of which (x_i,y_i) are the nominal coordinates, lies in a circle with the center (x_i,y_i) and radius $\delta/2$. Then the lengths $x_i\cos\phi + y_i\sin\phi$ again differ from the true lengths by up to $\delta/2$, whatever ϕ is. We can then define Ω_ϕ using the ideas of Section 3.

In Section 3, we argued that the maximum value which Ω_ϕ could have is π/δ. We now assert that there exists a value of ϕ for which π/δ must be the aliasing frequency. Choose ϕ so that the projection along the ϕ direction of the distance of one element from the notional zero element is $L_i = \delta/2 + \epsilon$ where ϵ will be specified shortly. Recall that a necessary condition on Ω_ϕ is

$$2\Omega_\phi(L_i + \delta_i) = 0 \bmod 2\pi$$

for $|\delta_i| < \delta/2$. Observe that the smallest Ω satisfying this equation is

$$\Omega_\phi = \frac{\pi}{\delta + \epsilon}.$$

By taking ϵ arbitrarily small but positive, we see that Ω_ϕ is arbitrarily close to π/δ. On the other hand, as we argued in Section 3, the aliasing frequency never exceeds π/δ. So in one direction it becomes equal to π/δ.

Needless to say, one can also define a direction-dependent aliasing frequency Ω_ϕ by postulating phase uncertainty, or by working with the polar diagram response.

8. Circularly symmetric arrays with direction-independent aliasing frequencies

It is interesting to ask whether it is possible for a planar array to have a direction-independent aliasing frequency. If the aliasing frequency is defined on the supposition of imprecisely known element positions, it would have to be π/δ. Whether or not this is possible is unknown. However, we can assume that such a situation is, at least roughly, approximated, by considering a circularly symmetric array.

Consider an array of elements equally spaced around the circumference of a circle. Let R be the radius of the circle, and ϕ the angle subtended at the center of the circle by the line joining two neighboring array elements. Define a direction α by a line through the circle center, and one element. The projected interelement distances include $L_{12} = R(1 - \cos\alpha)$, $L_{23} = R(\cos\alpha - \cos 2\alpha)$, $L_{13} = R(1 - \cos 2\alpha)$ (Fig. 5).

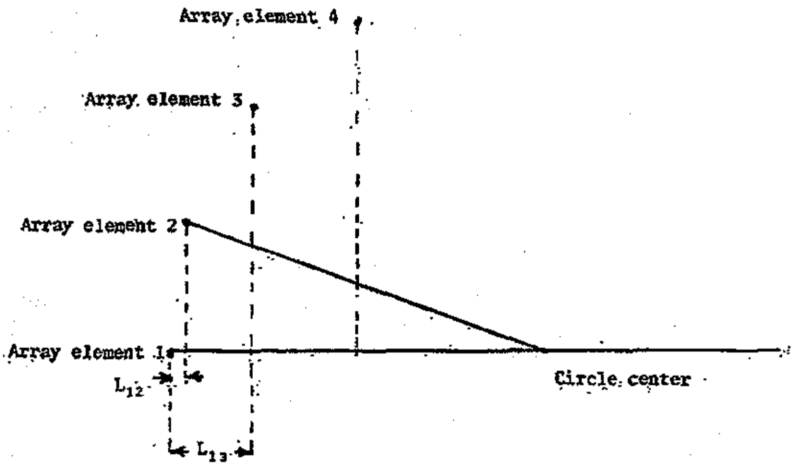

Figure 5. Part of a circularly symmetric array.

If ϕ now is defined by a line through the circle center and the midpont of the line joining two neighboring elements, we set $L_{12} = R(\cos\alpha/2 - \cos 3\alpha/2)$, $L_{23} = R(\cos 3\alpha/2 - \cos 5\alpha/2)$, $L_{13} = R(\cos\alpha/2 - \cos 5\alpha/2)$ etc.

Now suppose small δ, ϵ are prescribed. Select α and an integer k so that

$$\frac{\delta}{2R} < \sin\frac{k\alpha}{2} < \left[\frac{\delta+\epsilon}{2R}\right]\frac{k}{k+1}$$

This ensures that

$$\frac{\delta}{2R} < \sin\frac{(k+1)\alpha}{2} < \left[\frac{\delta+\epsilon}{2R}\right]$$

and therefore that

$$L_{1,k+1} = R(1 - \cos k\alpha) = 2R\sin^2\frac{k\alpha}{2} \in [\delta,\delta + \epsilon]$$

and

$$\bar{L}_{1,k+1} = R\left[\cos\frac{\alpha}{2} - \cos\frac{2k+1}{2}\alpha\right] = 2R\sin\frac{k\alpha}{2}\sin\frac{\overline{k+1}\alpha}{2} \in [\delta,\delta + \epsilon].$$

It follows that the aliasing frequency in the two directions so far nominated lies in $\left[\frac{\pi}{\delta + \epsilon},\frac{\pi}{\delta}\right]$. It is easy to see that for all directions between these two directions, the projection of the distance between elements 1 and $k + 1$ is still in $[\delta,\delta + \epsilon]$, and so again, the aliasing frequency is in $\left[\frac{\pi}{\delta + \epsilon},\frac{\pi}{\delta}\right]$. By symmetry, this will then be true for all directions.

It may well of course be the case that the implied density and/or number of array elements to secure an approximately constant aliasing frequency is impractically large.

9. Concluding remarks

There are several summarizing remarks to make. First, a methodology for determining regions of wave vector support which relies on perfect knowledge of positions seems flawed: our solution here has been to assume uncertain knowledge of positions. We could also have proceeded by allowing for uncertain knowledge of phase. Second, in dealing with planar arrays, it is in some ways more natural to be looking for a circular region of wavevector support, of the form $\omega_x^2 + \omega_y^2 \leq \Omega^2$ rather than a rectangular region of the form $|\omega_x| < \Omega_x, |\omega_y| < \Omega_y$. However, it may be that a highly noncircular region is required, since the aliasing frequency shows directional dependence; in this instance there is no guarantee that the appropriate region is rectangular, even if the array elements might be located on a rectangular grid.

As a further extension of the ideas, one could consider the definition of aliasing frequency given simultaneous phase and position errors. It would not appear difficult to combine the mechanisms given in the paper in the separate discussions of the two types of error.

10. Acknowledgements

Helpful discussions with R. Bitmead, L. Godara, D. Gray and D. Sweet are gratefully acknowledged.

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