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CONTINUITY OF THE SPECTRAL FACTORIZATION OPERATION

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ABSTRACT: *The problem is considered of passing from a power spectral density matrix (positive definite on the unit circle) to a canonical (stable, minimum phase, normalized) spectral factor. It is shown that small L^∞ perturbations in the power spectrum will lead to small L^2 perturbations in the spectral factor, but can lead to large L^∞ perturbations, unless the derivative of the power spectrum is controlled during the perturbation process.*

KEY WORDS: Spectral factorization • canonical factorization • power spectrum factorization

RESUMO: *Considera-se o mapeamento de uma matriz densidade espectral de potência (definida positiva no círculo unitário) em seu fator espectral canônico (estável, de fase mínima e normalizado). Demonstra-se que a pequenas perturbações, no sentido L^∞, do espectro de potência correspondem perturbações pequenas, no sentido L^2, do fator espectral, mas podem conduzir a perturbações relativamente grandes no sentido L^∞, a menos que o valor da derivada do espectro de potência se mantenha limitado no processo de perturbação.*

PALAVRAS-CHAVE: Fatorização espectral • fatorização canônica • fatorização do espectro de potência

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1. INTRODUCTION

Consider a linear, time-invariant system excited by white noise from the infinitely remote past. With an appropriate stability assumption on the system, the output process is stationary, and possesses a power spectrum matrix. The task of finding a system which gives rise to a prescribed power spectrum matrix is known as spectral factorization. In mathematical terms, there is prescribed a matrix $\Phi(e^{j\theta})$, $-\pi\leq\theta\leq\pi$ which is nonnegative hermitian, and one seeks matrices $W(e^{j\theta})$ for which

$$\Phi(e^{j\theta}) = W(e^{j\theta})QW'(e^{-j\theta}) \ . \tag{1}$$

Typically, $W(e^{j\theta})$ is the evaluation on the unit circle of a matrix function $W(\lambda)$ analytic in $|\lambda|<1$; sometimes, $W(\lambda)$ is square and nonsingular and $W^{-1}(\lambda)$ is required to be analytic in $|\lambda|<1$ as well, and the value of $W(0)$ is constrained in some way; normally, Q is nonnegative hermitian. With sufficient conditions on $W(\cdot)$, one can obtain a uniqueness theorem, with the associated $W(\cdot)$ being termed a canonical spectral factor.

There are many references dealing with the problem. Let us note in particular [11,12,9], the first two because of their historical significance, and the third because it provides a concise statement of the method of spectral factorization used in this paper. The task of spectral factorization as it turns out arises in contexts other than stochastic processes, for example, in linear-quadratic control design cf. [1] and network synthesis cf. [2].

A question which naturally arises is the extent to which the map from $\Phi(\cdot)$ to a canonical $W(\cdot)$ is continuous. Let us note that in [3] a continuity result is established to the effect that if two spectra are close in the L^∞ norm, then the associated canonical spectral factors are close in an L^2 norm. We shall also note subsequently that if two spectra are close in the L^∞ norm, with one continuous and the other discontinuous, then the associated spectral factors are not close in the L^∞ norm. This fact has important consequences in control theory. Linear quadratic control problems can be solved by spectral factorization cf. [1], with the to-be-controlled plant defining the spectrum and the spectral factor defining the controller. In control applications, it is typically the L^∞ norm which matters most cf. [7], and so the question naturally arises as to what norm should be used to measure closeness of the spectra in order to get L^∞ closeness of the spectral factors.

The main conclusion of the paper is that if two spectra are close in the L^∞

sense, and have bounded derivatives, then the associated canonical spectral factors are also close.

The paper is structured as follows. After notational preliminaries in Section 2, the spectral factorization scheme of [9] is reviewed in Section 3, an example of nonclose canonical factors of close spectra is considered in Section 4. Continuity results are derived in Sections 5 and 6, and concluding remarks are given in Section 7.

2. NOTATIONAL AND RELATED PRELIMINARIES

We denote by ℓ^2 the set of all complex sequences $\{f_k: -\infty<k<\infty\}$ for which $\Sigma|f_k|^2<\infty$, and equip ℓ^2 with the inner product $<\{f_k\},\{g_k\}>=\Sigma\bar{f}_k g_k$ making it a Hilbert space with norm $\|\{f_k\}\|_2=(\Sigma|f_k|^2)^{1/2}$. The subset of ℓ^2 comprising sequences zero for $k<0$ is denoted by ℓ^2_+. It is a closed subspace and thus a Hilbert space.

From $\{f_k\} \in \ell^2_+$ we construct $f(\lambda)=\Sigma f_k\lambda^k$ analytic in $\{\lambda \in \mathbb{C}: |\lambda|<1\}$. The set of all such $f(\cdot)$ is the Hardy space H^2, with inner product equal to the inner product of the corresponding sequences. The mapping $\{f_k\}\to f: \ell^2_+ \to H^2$ is a Hilbert space isomorphism.

We denote by L^2 the class of all complex-valued functions defined on the unit circle $\{e^{j\theta}: -\pi\leq\theta<\pi\}$ which are square integrable with respect to θ. The space L^2 becomes a Hilbert space with inner product

$$<f,g>: = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(e^{j\theta})}g(e^{j\theta})d\theta \; . \tag{2}$$

We get another Hilbert space isomorphism $\{f_k\}\to f(e^{j\theta})=\Sigma f_k e^{jk\theta}: \ell^2\to L^2$, with inverse mapping $f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{j\theta})e^{-jk\theta}d\theta$.

For each $f \in H^2$, the limit

$$\lim_{\rho\uparrow 1} f(\rho e^{j\theta})$$

exists for almost all θ; the limit is a unique L^2 function called the boundary function of f, and this embeds H^2 in L^2, allowing identification of an H^2 function with its boundary function (in the subspace of L^2-functions whose Fourier coefficients are zero for negative indices). The inner product of the two H^2 functions becomes the L^2-inner product of the two boundary functions.

The space of all complex-valued functions on the unit circle bounded almost everywhere is denoted by L^∞, with norm

$$\| f\|_\infty = \text{ess}_\theta \sup|f(e^{j\theta})| \ . \tag{3}$$

Note that L^∞ is a subspace (not closed) of L^2. The Hardy space H^∞ consists of all complex-valued functions which are analytic and of bounded modulus on $|\lambda|<1$, with norm

$$\| f\|_\infty = \sup\{f(\lambda): |\lambda|<1\} \ . \tag{4}$$

Again, each f in H^∞ yields a unique L^∞ boundary function with the two norms equal. The set of such boundary functions is the subspace of L^∞-functions with Fourier coefficients zero for negative indices, and we can regard H^∞ as a closed-subspace of L^∞. We note that any $f \in H^\infty$ yields an operator on H^2, *viz.* $g \to fg$, with operator norm equal to the H^∞-norm of f.

There are natural matrix generalizations of ℓ^2, ℓ^2_+, L^2, H^2, L^∞ and H^∞. If X denotes any one space, and $X^{m\times n}$ the space of all m×n matrices with elements in X, then on $X^{m\times n}$,

$$\langle F,G\rangle = \sum_{i,j} \langle f_{ij}, g_{ij}\rangle_X \ .$$

A superscript transpose will denote transposition, overbar complex conjugation and superscript asterisk complex conjugate transpose. Then, for example, the norm of F in $(H^2)^{n\times n}$ with

$$F(\lambda) = \sum_{k=0}^{\infty} F_k \lambda^k$$

is

$$\| F\|_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{trace}[F^*(e^{j\theta})F(e^{j\theta})]d\theta = \sum_k \text{trace } (F_k^* F_k) \tag{5}$$

and the norm of F in L^∞ is

$$\| F\|_\infty = \sup_{\theta\in[-\pi,\pi]} \{[\text{max eigenvalue of } F^*(e^{j\theta})F(e^{j\theta})]^{1/2}\} \ . \tag{6}$$

For a complex matrix G, we shall write

$$\|G\|_M = [\text{max eigenvalue of } G^*G]^{1/2} \ . \tag{7}$$

Notice that if G depends on θ, then

$$\|G\|_\infty = \sup_\theta \|G(e^{j\theta})\|_M \ . \tag{8}$$

An n×m matrix function of $W(\lambda)$ will be termed causal, strictly causal, stable, strictly stable according as $W(0)$ is finite, $W(0)=0$, $W(\cdot) \in (H^2)^{n\times m}$ and $W(\lambda)$ has analytic entries in $|\lambda|\leq 1$. In case $W(\lambda)$ is square and invertible, it will be termed minimum phase, strictly minimum phase, according as $W^{-1}(\lambda)$ is stable, strictly stable.

3. REVIEW OF MATRIX SPECTRAL FACTORIZATION

We shall first of all review the construction of the minimum phase spectral factor, following the approach of Rozanov in [9].

Assume there is given a nonsingular n×n power spectrum matrix $\Phi(e^{j\theta})$, *i.e.* a positive definite Hermitian matrix for $-\pi\leq\theta\leq\pi$. Suppose further that, through scaling if necessary, we have

$$\Phi(e^{j\theta}) = I + M(e^{j\theta}) \quad , \tag{9}$$

$$\|M(e^{j\theta})\|_\infty \leq q < 1 \ . \tag{10}$$

The task of spectral factorization is to find a $W(\lambda)$, stable and minimum phase (and sometimes strictly stable and strictly minimum phase) such that

$$\Phi(e^{j\theta}) = W(e^{j\theta})QW^*(e^{j\theta}) \tag{11}$$

with Q a positive definite Hermitian matrix. By requiring that $W(0)=I$, a normalization condition, it turns out that there is a unique solution to this problem, obtained in the following way.

Step 1 (Definition of the operators B_-^M, B_+^M on $(L^2)^{n\times n}$). For any function $x(e^{j\theta}) \in (L^2)^{n\times n}$, define

$$P_+[x(e^{j\theta})] = \sum_1^\infty x(n)e^{jn\theta} \text{ where } x(e^{j\theta}) = \sum_{-\infty}^{+\infty} x(n)e^{jn\theta} \ ,$$

$$P_-[x(e^{j\theta})] = \sum_{-\infty}^{-1} x(n)e^{jn\theta} .$$

Noting that $M(e^{j\theta}) \in (L^\infty)^{n\times n}$, it follows that for any $x(e^{j\theta}) \in (L^2)^{n\times n}$ there is defined

$$B_+^M[x(e^{j\theta})] = P_+[x(e^{j\theta})M(e^{j\theta})] ,$$

$$B_-^M[x(e^{j\theta})] = P_-[M(e^{j\theta})x(e^{j\theta})] .$$

Step 2 (*Properties of the Operators* B_-^M, B_+^M). It is straightforward to check that the operator *norms are bounded* as

$$\| B_\pm^M \| \leq q \tag{12}$$

and accordingly, $(I+B_-^M)^{-1}$ and $(I+B_+^M)^{-1}$ exist as operators on $(L^2)^{n\times n}$ with overbounds on their norm of $(1-q)^{-1}$. Notice that B_+^M maps $(L^2)^{n\times n}$ into that subspace of $(H^2)^{n\times n}$ of functions with Fourier coefficient for the index zero equal to zero. A related statement holds for B_-^M. *Regarding* $(I+B_-^M)^{-1}$ *as* $I-B_+^M+(B_+^M)^2 \ldots$, it follows that $(I+B_+^M)^{-1}$ is an operator on $(H^2)^{n\times n}$, with an equivalent statement for $(I+B_-^M)^{-1}$. As operators on these *subspaces*, *the norm* is still overbounded by $(1-q)^{-1}$.

Step 3 (Construction of quantities appearing in the spectral factorization). *Make the* following definitions:

$$V_+(e^{j\theta}) = (I+B_+^M)^{-1}I , \qquad V_-(e^{j\theta}) = (I+B_-^M)^{-1}I \ ; \tag{13}$$

$$Q = V_+(I+M)V_- \ . \tag{14}$$

It follows that $V_+ \in (H^2)^{n\times n}$, and $V_-(e^{j\theta})=\sum_{-\infty}^{0} v_-(k)e^{jk\theta}$ with $\{v_-(k)\} \in \ell_2$.

Step 4 (Spectral Factorization Theorem, cf. [9]). With the above constructions,

(i) $V_+, V_+^{-1} \in (H^\infty)^{n\times n}$

(ii) $V_+(e^{j\theta}) = V_-^*(e^{j\theta})$

(iii) Q is constant

(iv) $I+M = V_+^{-1}Q(V_+^{-1})^*$

(v) $V_+^{-1}(0) = I$.

Evidently, $W(\lambda)=V_+^{-1}(\lambda)$ is the normalized minimum phase stable spectral factor.

It turns out that for scalar spectra, a formula is available for a scaled version of the canonical factor. Using lower case letters to emphasize the scalar nature of the quantities involved, there holds, see [9], Theorem II.5.2:

$$w(z)q^{1/2} = \sqrt{2\pi}\,\exp\{\frac{1}{4\pi}\int_{-\pi}^{+\pi}[\log\Phi(e^{j\omega})]\,\frac{e^{j\omega}+z}{e^{j\omega}-z}\,d\omega\} \quad , \tag{15a}$$

$$q^{1/2} = \sqrt{2\pi}\,\exp\{\frac{1}{4\pi}\int_{-\pi}^{+\pi}[\log\Phi(e^{j\omega})]d\omega\} \; . \tag{15b}$$

4. A DISCONTINUITY PROPERTY OF THE SPECTRAL FACTORIZATION OPERATION

We shall show how an arbitrary small L^∞ perturbation in a spectrum can produce a substantial perturbation in the canonical spectral factor, by way of a specific example. Our tool is the following lemma:

Lemma 1. Suppose $T(z)$ is analytic in $|z|<1$, $T(z^{-1})=T^*(z)$ for $|z|=1$, $T(z)$ possesses a finite number of singularities on $|z|=1$ with, at each singularity $e^{j\omega_0}$, $(e^{j\omega}-e^{j\omega_0})T(e^{j\omega})\to 0$, as $\omega-\omega_0\to 0$, and $T(e^{j\omega})$ is bounded for $-\pi\leq\theta\leq\pi$. Let

$$T(e^{j\theta}) = R(e^{j\theta})+jI(e^{j\theta}) \tag{16}$$

where $R(\cdot)$, $I(\cdot)$ are real for $-\pi\leq\theta\leq\pi$. Then

$$I(e^{j\theta}) = \frac{\sin\theta}{2\pi}\int_{-\pi}^{\pi}\frac{R(e^{j\omega})-R(e^{j\theta})}{\cos\omega-\cos\theta}\,d\omega \tag{17}$$

and if $R(\cdot)$ is differentiable,

$$I(e^{j\theta}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{dR(e^{j\omega})}{d\omega}\,\ln\left|\frac{\sin\frac{\theta+\omega}{2}}{\sin\frac{\theta-\omega}{2}}\right|dw \; . \tag{18}$$

A result similar to this lemma, with the unit circle replaced by the imaginary axis, can be found in [13], see Section 9.6, and with interchange of the roles of R(·) and I(·), a similar result (involving the unit circle) can be found in [4]. The proof is therefore omitted.

We remark that the quantity

$$g(\theta,\omega) = \ell n\left|\frac{\sin\frac{\theta+\omega}{2}}{\sin\frac{\theta-\omega}{2}}\right| \tag{19}$$

for fixed θ is L^1 over $-\pi\leq\omega\leq\pi$ even though it is not bounded, so that with $dR/d\omega$ bounded the integral in (18) is well behaved.

To apply this theorem to the scalar spectral factorization problem, let us identify $T(z)$ with $\ell n\, w(z)q^{1/2}$. Then

$$R(e^{j\theta}) = \ell n|w(e^{j\theta})q^{1/2}| = \frac{1}{2}\,\ell n\,\Phi(e^{j\theta}) \tag{20}$$

and is known. The Lemma tells us how to construct the imaginary part of $T(e^{j\theta})$, which is the argument of $w(e^{j\theta})$, given $\Phi(e^{j\theta})$.

Consider two spectra, $\Phi_1(e^{j\theta})\equiv 1$ [for which $w(e^{j\theta})\equiv 1$] and $\Phi_2(e^{j\theta})$ differing from $\Phi_1(e^{j\theta})$ by the addition of a trapezoidal *bump*, say,

$$\left.\begin{aligned} \Phi_2(e^{j\theta}) &= \begin{cases} 1+\varepsilon & 0\leq\theta\leq\eta_1 \\ 1+\varepsilon-\dfrac{(\theta-\eta_1)\varepsilon}{\eta_2-\eta_1} & \eta_1\leq\theta\leq\eta_2 \\ 1 & \eta_2\leq\theta\leq\pi \end{cases} \\ &\text{and} \\ \Phi_2(e^{-j\theta}) &= \Phi_2(e^{j\theta})\,. \end{aligned}\right] \tag{21}$$

Notice that

$$\|\Phi_2(e^{j\theta})-\Phi_1(e^{j\theta})\|_\infty = \varepsilon \tag{22a}$$

and

$$\left\|\frac{d\Phi_2(e^{j\theta})}{d\theta}\right\|_\infty = \frac{\varepsilon}{\eta_2-\eta_1}\,. \tag{22b}$$

By taking $\eta_2-\eta_1$ *sufficiently small, we can make* $\left\|\frac{d\Phi_2(e^{j\theta})}{d\theta}\right\|_\infty$ as large as we like. With $\eta=(\eta_1+\eta_2)/2$ it is then verified in the Appendix using the formula (18) that $I_2(e^{j\theta})$ can become arbitrarily large, while $R_2(e^{j\theta})$ varies between $\frac{1}{2}\ \ell n(1+\varepsilon)^{1/2}$ and 0. This indicates that $T_1(e^{j\theta})$ and $T_2(e^{j\theta})$ will differ but little in real part but can differ by an arbitrarily large amount in imaginary part, and $w_1(e^{j\theta})$, $w_2(e^{j\theta})$ will differ but little in magnitude but can differ by an arbitrarily large amount (mod 2π) in phase.

There is another view of the same phenomenon. The argument of Rozanov guarantees that if we let $\eta_2=\eta_1$, so that $\Phi_2(e^{j\theta})$ is discontinuous at η, there nevertheless exists a canonical spectral factor $w_2(e^{j\theta})$ with $w_2(z)$ in H^∞. By Lemma 17.5.5 in [8], $w_2(e^{j\theta})$ cannot have a simple discontinuity anywhere on $-\pi\leq\theta\leq\pi$, while the requirement that $\Phi_2(e^{j\theta})=|w_2(e^{j\theta})|^2$ certainly forces w_2 to have some kind of discontinuity at $\theta=\eta_1$. Hence the phase of w_2 cannot be smoothly defined as $\theta\to\eta_1$ from above or below.

Let us also note that the formulas (15a) and (17) also suggest the discontinuity result noted above. Let $f_1(\theta)$ be a scalar complex (real) valued function of θ for $0\leq\theta\leq2\pi$, and consider a linear mapping $f_1\to f_2$ defined by

$$f_2(\theta) = \int_0^{2\pi} g(\theta,\omega)f_1(\omega)d\omega \ . \tag{23}$$

where $g(\cdot,\cdot)$ is some complex (real) valued integral kernel. Then (23) defines a bounded mapping of complex (real) L^∞ functions into complex (real) L^∞ functions if and only if (cf. [6])

$$\sup_\theta \int_0^{2\pi} |g(\theta,\omega)|d\omega < \infty \ . \tag{24}$$

We can identify $g(\theta,\omega)$ with $(e^{j\omega}+e^{j\theta})\ (e^{j\omega}-e^{j\theta})^{-1}$ in (15a) and with $\sin\theta(\cos\omega-\cos\theta)^{-1}$ in (17), provided that we agree to replace $f_1(\omega)$ in (23) by $f_1(\omega)-f_1(\theta)$. In neither case does (24) hold. But one cannot immediately draw the conclusion that a small variation in $\ell n\phi(e^{j\omega})$ in (15a) or $R(e^{j\omega})$ in (17) will lead to a large variation in $w(e^{j\theta})$ or $I(e^{j\theta})$, because in the first case, the variation is restricted to being real while $g(\cdot,\cdot)$ is complex, and in the second case, we are not actually working with (23) since $f_1(\omega)$ must be replaced by $f_1(\omega)-f_1(\theta)$. Nevertheless, the divergence of (24) undoubtedly raises the possibility of difficulties with (15a) or (17).

We note also that if we consider (18) and identify $g(\theta,\omega)$ with

$\ell n|[\sin\frac{1}{2}\ (\theta+\omega)][\sin\frac{1}{2}\ (\theta-\omega)]^{-1}|$, the condition (24) can be verified. So small variations L^∞ in the derivative of $R(e^{j\omega})$ lead to small L^∞ variations in $I(e^{j\omega})$.

The formula (15a) does not have a simple matrix generalization. Similarly, the formula (18) with the interpretation of (20) cannot be given a matrix extension which captures the continuity problem for matrix spectral factors. Accordingly, a different method is required. The scalar results however suggest the style of matrix result we should expect.

5. CONTINUITY PROPERTIES BASED ON AN L^∞ NORM FOR THE SPECTRUM MATRIX

The mapping from $\Phi(e^{j\theta})$ to $W(z)$ is not a linear mapping. Accordingly, we shall establish a continuity result of the form "small change in Φ produces small change in W". The first norm used for W is the $(L^2)^{n\times n}$ norm, and the result we prove is a refinement of one in [3].

Theorem 1. Let $\Phi_i(e^{j\theta})=I+M_i(e^{j\theta})$, i=1,2 be two positive definite Hermitian matrices defined on $-\pi\leq\theta\leq\pi$ with

$$\| M_i(e^{j\theta})\|_\infty \leq q < 1 \; . \tag{25}$$

Let W_1, W_2 be the associated normalized, minimum phase, stable spectral factors. Then there exists some K depending only on q such that

$$\| W_1-W_2 \|_2 \leq K\| M_1-M_2 \|_\infty \; . \tag{26}$$

Proof. We use the notation of the previous section. Observe first that because $\| M_i\| \leq q$ for i=1,2 it is enough to prove the theorem for $\| M_1-M_2 \|$ suitably small. Now if X_1, X_2 are two operators on $(L^2)^{n\times n}$ for which X_1^{-1}, X_2^{-1} exist and for which (in operator norms)

$$\| X_1-X_2 \| \leq \frac{1}{2\| X_1^{-1}\|}$$

then one can show, see e.g. [10], p. 306 that

$$\| X_1^{-1}-X_2^{-1} \| \leq 2\| X_1^{-1}\|^2 \| X_1-X_2 \| \; .$$

Identifying X_i with $I+B_-^{M_i}$, it follows that

$$\| (I+B_-^{M_1})^{-1}-(I+B_-^{M_2})^{-1}\| \leq 2(1-q)^{-2} \| M_1-M_2 \|$$

tinuity result on the spectral factor can involve the L_∞ norm.

The main result to be proven is as follows:

Theorem 2. Adopt the same hypothesis as in Theorem 1, and assume in addition that $dM_i(e^{j\theta})/de^{j\theta} \in (L^2)^{n\times n}$ for i=1,2. Then there exists some $\overline{K}$ dependent only on q and $\| dM_i(e^{j\theta})/d\theta \|_2$ such that

$$\| W_1 - W_2 \|_\infty \leq \overline{K}[\| M_1 - M_2 \|_\infty]^{1/2} \; . \tag{29}$$

To prove this result, we shall first relate the differentiability property of the M_i to a differentiability property of the associated spectral factor.

Lemma 2. Suppose that $\Phi(e^{j\theta}) = I + M(e^{j\theta})$ is a positive definite Hermitian matrix defined on $-\pi \leq \theta \leq \pi$ with $\| M(e^{j\theta}) \|_\infty \leq q \leq 1$. Suppose further that $\| dM(e^{j\theta})/de^{j\theta} \|_2 \leq q'$ and let $V_+(e^{j\theta})$ be as defined in Section 3. Then $dV_+(\lambda)/d\lambda$ is an $(H^2)^{n\times n}$ *function for which*

$$\frac{dV_+(e^{j\theta})}{d\theta} = -(I+B_+^M)^{-1} P_+ \left\{ V_+(e^{j\theta}) \frac{dM(e^{j\theta})}{d\theta} \right\} . \tag{30}$$

Proof. From the definition of $V_+(\cdot)$, we have

$$V_+(e^{j\theta}) + P_+[V_+(e^{j\theta})M(e^{j\theta})] = I$$

$$V_+(e^{j(\theta+\Delta\theta)}) + P_+[V_+(e^{j(\theta+\Delta\theta)})M(e^{j(\theta+\Delta\theta)})] = I$$

and so

$$\Delta V(e^{j\theta}) + P_+[\Delta V(e^{j\theta})M(e^{j(\theta+\Delta\theta)})] + P_+[V(e^{j\theta})\Delta M(e^{j\theta})] = 0$$

with the notation ΔV, M self explanatory. The equation is

$$(I+B^{M+\Delta M}) \frac{\Delta V(e^{j\theta})}{\Delta\theta} = -P_+[V_+(e^{j\theta}) \frac{\Delta M(e^{j\theta})}{\Delta\theta}]$$

or

$$\frac{\Delta V(e^{j\theta})}{\Delta\theta} = -(I+B^{M+\Delta M})^{-} P_+[V_+(e^{j\theta}) \frac{\Delta M(e^{j\theta})}{\Delta\theta}] \; .$$

Let us note at this point that $\Delta M(e^{j\theta})/\Delta\theta \in (L^2)^{n\times n}$, that because provided $\| M_1-M_2 \|_\infty \leq \frac{1}{2}(1-q)$, and then

$$\| V_{-1}-V_{-2} \|_2 \leq \frac{2}{(1-q)^2} \| M_1-M_2 \|_\infty \tag{27}$$

with an identical bound for $\| V_{+1}-V_{+2} \|$. Now (14) implies

$$V_{+i}^{-1}Q_i = (I+M_i)V_{-i} \ .$$

The left side is in $(H^2)^{n\times n}$, and the constant term in the Fourier expansion of the left side is Q_i. So

$$Q_i = (I-P_+)(I+M_i)V_{-i} \ .$$

Now as an operator on $(H^2)^{n\times n}$, $I-P_+$ has norm 1. Hence

$$\| Q_1-Q_2 \|_M \leq \| (I+M_1)V_{-1}-(I+M_2)V_{-2} \| = \| (I+M_2)(V_{-1}-V_{-2})+(M_1-M_2)V_{-1} \|_2 \leq$$

$$\leq (1+q)\| V_{-1}-V_{-2} \|_2 + \frac{1}{1-q} \| M_1-M_2 \|_\infty \leq$$

$$\leq \left(\frac{2(1+q)}{(1-q)^2} + \frac{1}{1-q}\right) \| M_1-M_2 \|_\infty \ . \tag{28}$$

[The matrix norm of Q_1-Q_2 is overbounded by the L_2-norm of a sequence with one nonzero entry equal to Q_1-Q_2]. Now with

$$W_1 - W_2 = V_{+1}^{-} - V_{+2}^{-} = (I+M_1)V_{-1}Q_1^{-1} - (I+M_2)V_{-2}Q_2^{-1}$$

use of (27) and (28) yields

$$\| W_1-W_2 \|_2 \leq K\| M_1-M_2 \|_\infty$$

for some $K(q)$, and for $\| M_1-M_2 \| \leq \frac{1}{2}(1-q)$. A bound of this type, possibly with different K, remains valid for all M_1, M_2 with $\| M_i \| \leq q$.

6. CONTINUITY PROPERTY BASED ON EXISTENCE OF DERIVATIVES OF THE SPECTRUM MATRIX

In this section, our aim is to show that if, in addition to the assumptions of the previous section, the spectrum matrix has a derivative, then the con-

$V_+(e^{j\theta}) \in (H^\infty)^{n\times n}$, the product $V_+(e^{j\theta}) \frac{\Delta M(e^{j\theta})}{\Delta\theta} \in (L^2)^{n\times n}$, that $P_+[V_+(e^{j\theta}) \frac{\Delta M(e^{j\theta})}{\Delta\theta}] \in (L^2)^{n\times n}$, and that $(I+B^{M+\Delta M})^{-1}$ maps $(L^2)^{n\times n}$ into $(L^2)^{n\times n}$, with a norm no greater than $(1-q)^{-1}$. Now let $\Delta\theta\to 0$. Because the derivative of M is $(L^2)^{n\times n}$, it is also $(L^1)^{n\times n}$, and so M is absolutely continuous. Hence

$$\lim_{\Delta\theta\to 0} \| M(e^{j(\theta+\Delta\theta)})-M(e^{j\theta})\|_{L_\infty} = 0$$

and so the operator $B^{M+\Delta M}$ approaches B^M in the norm topology, and it is trivial to conclude then that $(I+B^{M+\Delta M})^{-1}\to(I+B^M)^{-1}$ in the topology. By hypothesis, $\frac{\Delta M(e^{j\theta})}{\Delta\theta} \to \frac{dM(e^{j\theta})}{d\theta}$, which is in $(L^2)^{n\times n}$. Then

$$-(I+B^{M+\Delta M})^{-1}P_+\left[V_+(e^{j\theta}) \frac{\Delta M(e^{j\theta})}{\Delta\theta}\right] \to$$

$$-(I+B^M)^{-1}P_+\left[V_+(e^{j\theta}) \frac{dM(e^{j\theta})}{d\theta}\right]$$

as $\Delta\theta\to 0$, proving the lemma.

Notice that the equation

$$(I+B_+^M) \frac{dV_+(e^{j\theta})}{d\theta} = -P_+\left[V_+(e^{j\theta}) \frac{dM(e^{j\theta})}{d\theta}\right]$$

[which is a rewrite of (30)] is just a differentiated version of

$$(I+B_+)V_+(e^{j\theta}) = I \ . \tag{31}$$

The main work in the Lemma is to establish the existence of the derivative of $V_+(\cdot)$. By way of digression, we note that similar arguments can be used in case second, third, fourth,... derivatives of M exist, to allow the following conclusion:

<u>Lemma 3</u>. Assume the same hypothesis as Lemma 2, save that derivatives of $M(e^{j\theta})$ up to order r-1 exist as $(L^\infty)^{n\times n}$ functions, with the r^{th} derivative of $M(e^{j\theta})$ square integrable. Then derivatives of $V_+(\lambda)$ with order (r-1) exist as $(H^\infty)^{n\times n}$ functions and the r^{th} derivative exists as an $(H^2)^{n\times n}$ function.

Since $W(\lambda)=V_+^{-1}(\lambda)$, and $W_+(\lambda)$ is an $(H^\infty)^{n\times n}$ function, it follows that derivatives of W inherit the same properties as the derivatives of V_+.

Also, as might be expected, if $M(e^{j\theta})$ is the evaluation on the unit circle of a function which is analytic in an annulus $\rho<|z|<\rho^{-1}$ for some $\rho<1$, then $V_+(z)$ and $W(z)$ are analytic in $|z|\leq 1$, (see [5], p. 61, Cor. 6.1).

We now return to the case when $M(e^{j\theta})$ possesses an $(L^2)^{n\times n}$ derivative.

Proof of Theorem 2. As in the previous section, it is enough to prove the Theorem for $\| M_1-M_2\|$ *suitably small. Suppose that* $\| dM_i(e^{j\theta})/d\theta\|_2\leq q'$ for $i=1,2$. The result of Lemma 2 ensures that

$$\left\|\frac{dV_{i+}(e^{j\theta})}{d\theta}\right\|_2 \leq \frac{q'}{1-q}\, \| V_{i+}(e^{j\theta})\|_\infty .$$

Now V_{i+} and V_{i+}^- are both $(H^\infty)^{n\times n}$ functions, see Section 3, and since

$$\frac{dW_i}{d\theta} = \frac{dV_{i+}^-}{d\theta} = -W_i\, \frac{dV_{i+}}{d\theta}\, W_i$$

it is evident that for some bound K' depending on q' and q,

$$\left\|\frac{dW_i}{d\theta}\right\|_2 \leq K' .$$

Now observe that with $W:=W_1-W_2$,

$$\frac{1}{2\pi}\, \| W(e^{j\omega})W^*(e^{j\omega})-W(e^{-j\pi})W^*(e^{-j\pi})\|_M = \left\| \frac{1}{2\pi}\int_{-\pi}^{\infty} \frac{dW(e^{j\theta})}{d\theta}\, W^*(e^{j\theta})d\theta + \right.$$

$$\left. + \frac{1}{2\pi}\int_{-\pi}^{\omega} W(e^{j\theta})\, \frac{dW^*(e^{j\theta})}{d\theta}\, d\theta\right\|_M \leq$$

$$\leq 2\int_{-\pi}^{\omega} \frac{1}{2\pi} \left\|\frac{dW(e^{j\theta})}{d\theta}\right\|_M \| W(e^{j\theta})\|_M d\theta \leq$$

$$\leq 2\left[\int_{-\pi}^{\omega} \frac{1}{2\pi} \left\|\frac{dW(e^{j\theta})}{d\theta}\right\|_M^2 d\theta\right]^{1/2} \left[\int_{-\pi}^{\omega} \frac{1}{2\pi}\, \| W(e^{j\theta})\|_M^2 d\theta\right]^{1/2} \leq$$

$$\leq 2[\| W_1\|_2+\| W_2\|_2][\| W_1-W_2\|_2] \leq 4K'K\| M_1-M_2\|_\infty ,$$

using (26). Integrating the inequality over $(-\pi,\pi)$ and using (26) again we show that

$$W(e^{-j\pi})W^*(e^{-j\pi}) \le L_1[\| M_1-M_2 \|_\infty] + L_2[\| M_1-M_2 \|_\infty]^2$$

for some L_1, L_2, and then finally,

$$W(e^{j\omega})W^*(e^{j\omega}) \le L_3 \| M_1-M_2 \|_\infty + L_3[\| M_1-M_2 \|_\infty]^2 .$$

The desired bound is immediate.

We comment that the bound can be made tighter if we impose a smallness assumption on $dM_1(e^{j\theta})/d\theta - dM_2(e^{j\theta})/d\theta$.

In Section 4, in our analysis of scalar spectral factorization, we demonstrated what was in effect a scalar version of Theorem 2, with the requirement that the spectrum have an L^∞ derivative. The result of Theorem 2 suggests that the calculations of Section 4 may be simply variable to achieve a result for the case of a spectrum with L^2 derivative. Indeed this is so, and we outline the changes now. Using the formula (18), repeated for convenience as

$$I(e^{j\theta}) = -\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{dR(e^{j\omega})}{d\omega}\, \ell n \left|\frac{\sin\frac{1}{2}(\theta+\omega)}{\sin\frac{1}{2}(\theta-\omega)}\right| d\omega \tag{32}$$

a simple argument based on the Schwarz inequality shows that if the derivative of $R(\cdot)$ is square integrable, $I(e^{j\theta})$ will be in L^∞, with a bound determined by the L^2 bound of the derivative, provided that

$$\int_{-\pi}^{\pi} \left[\ell n \left|\frac{\sin\frac{1}{2}(\theta+\omega)}{\sin\frac{1}{2}(\theta-\omega)}\right|\right]^2 d\omega$$

is bounded as a function of θ. It is not hard to check that this will follow provided that

$$\int_0^\pi [\ell n|\sin\omega|]^2 d\omega$$

is bounded, and then provided that

$$\int_0^\pi [\ell n\,\omega]^2 d\omega$$

is bounded, a fact which can be verified through direct integration. This brings the results of Section 4 and this section into harmony.

7. CONCLUSIONS

The key conclusion of this paper is that small L_∞ perturbations in a spectrum will produce small L_∞ perturbations in the associated spectral factor if, and in effect only if, the perturbed and unperturbed spectra have bounded derivatives.

It is possible to conjecture a refinement of this result. With the aid of the formula (18), one can argue in the scalar case that perturbation of a spectrum in a limited frequency range affects the spectral factor primarily in that same frequency range. One can conjecture the validity of this result also in the matrix case.

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9. APPENDIX - DETAIL OF THE CALCULATION IN SECTION 4

We wish to show that

$$I(e^{j\theta}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{dR(e^{j\omega})}{d\omega}\,\ell n\left|\frac{\sin\frac{\theta+\omega}{2}}{\sin\frac{\theta-\omega}{2}}\right|d\omega \tag{33}$$

for θ set equal to $(\eta_1+\eta_2)/2$ becomes unbounded when

$$R(e^{j\omega}) = \begin{cases}\frac{1}{2}\,\ell n(1+\varepsilon) & 0\leq\theta\leq\eta_1\\ \frac{1}{2}\,\ell n(1+\varepsilon-\frac{(\omega-\eta_1)\varepsilon}{\eta_2-\eta_1}) & \eta_1\leq\theta\leq\eta_2\\ 0 & \eta_2\leq\omega\leq\pi\end{cases}$$

$$R(e^{-j\omega}) = R(e^{j\omega})$$

and we let $\eta_2-\eta_1\to 0$, keeping $(\eta_1+\eta_2)/2$ fixed. Using the evenness of the integral in (33), it is quickly checked that with $\eta := \frac{\eta_1+\eta_2}{2}$,

$$I(e^{j\eta}) = -\frac{1}{2\pi}\int_{\eta_1}^{\eta_2}\frac{-\frac{\varepsilon}{\eta_2-\eta_1}}{1+\varepsilon-\frac{(\omega-\eta_1)\varepsilon}{\eta_2-\eta_1}}\,\ell n\left|\frac{\sin\frac{\eta+\omega}{2}}{\sin\frac{\eta-\omega}{2}}\right|d\omega \; .$$

It is clear that $I(e^{j\eta})$ becomes unbounded if and only if

$$J = \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\ell n\left|\sin\frac{\omega-\eta}{2}\right|d\omega$$

becomes unbounded as $\eta_2-\eta_1\to 0$. Set $\omega' = (\omega-\eta)/2$, $\gamma = \eta_2-\eta_1$. Then

$$J = \frac{2}{\gamma}\int_{-\gamma/2}^{+\gamma/2} \ln|\sin\omega'|d\omega' = \frac{4}{\gamma}\int_{0}^{\gamma/2} \ln|\sin\omega'|d\omega' .$$

Now

$$\sin\omega' > \frac{2\omega'}{\pi}$$

for $\omega' \in (0,\gamma/2)$, and $\ln x$ is monotonic in x, so J will be unbounded (in the negative direction) as $\gamma \to 0$ if

$$K = \frac{1}{\gamma}\int_{0}^{\gamma/2} \ln\omega' d\omega'$$

is unbounded as $\gamma \to 0$. The integral can be explicitly computed, using the fact that $\lim_{x\to 0} x\ln x = 0$, to yield

$$K = \frac{1}{\gamma}\left[\frac{\gamma}{2}\left(\ln\frac{\gamma}{2}\right) - 1\right]$$

and it is clear then that J is unbounded as $\gamma \to 0$.