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PHASE LAG AND LEAD WEIGHTING IN LINEAR-QUADRATIC CONTROL

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SUMMARY

The effect of introducing a first order lag or lead weighting function into an otherwise conventional linear-quadratic optimization problem is examined. Properties of the optimal compensator are derived, and the effect on robustness of the optimal system in different frequency ranges is examined.

KEY WORDS Optimal control

1. INTRODUCTION

The use of frequency-weighted performance indices in linear quadratic (LQ) optimal control has been the subject of recent reseach[1–5]. The primary motivation for frequency-weighting is the possibility of enhancing the robustness of LQ designs in critical frequency bands. A preliminary study[3] of the case where high frequencies in the control are weighted more heavily has shown that the optimal controller in this case comprises a stable compensator, nominally introducing phase lag, in tandem with the state feedback term. In this paper, further properties of the optimal dynamic compensator are derived with a view to developing design guidelines for such compensators. A specific question addressed is the determination of conditions under which the optimal compensator is indeed a lag, and not a lead.

Attention is also focused on the optimal (nominally) phase lead compensator case which arises when low frequencies are weighted more heavily in the performance index. A detailed analysis of a first-order single-input plant with both lag and lead compensation is undertaken in order to gain insight and to help in the formulation of design rules.

An alternative to frequency-weighting the control term in the qaudratic performance index is to apply an (inverse) frequency-weighting to the state term[5]. Further relations between these two types of weighting are explored in this paper, especially as regards to their robustness properties.

The generalization of the result to the multi-input case is investigated in the final section of the paper.

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2. FREQUENCY-SHAPED LQ PROBLEM

The general formulation of the problem is to minimize the performance index

$$V = \int_0^\infty [u' \ R(j\omega)u + x' \ Q(j\omega)x] \ \mathrm{d}t \tag{2.1}$$

for the linear time-invariant system

$$\dot{x} = Ax + Bu \tag{2.2}$$

The notation in (2.1) which mixes frequency-domain and time-domain quantities is now almost standard; its precise meaning is given in, e.g. Reference 4 and can be summarized as follows. Let $U(j\omega), X(j\omega)$ be the Fourier transforms of time functions zero for $t < 0$ and equal to $u(t)$, $x(t)$ for $t > 0$. Then (2.1) can be regarded as

$$V = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [U'(-j\omega)R(j\omega)U(j\omega) + X'(-j\omega)Q(j\omega)X(j\omega)] \ \mathrm{d}\omega \tag{2.1}$$

This presupposes that the transforms exist, a condition which can be ensured in practice.

The restrictions imposed on R and Q are: R must be rational, and uniformly positive definite Hermitian for all real ω and Q must be rational, bounded and nonnegative definite Hermitian for all real ω.

Initially attention will be confined to the single-input plant

$$\dot{x} = Ax + bu \tag{2.3}$$

and the performance indices:

Control weighting

$$V_c = \int_0^\infty \left[\left| \frac{1+\beta j\omega}{1+\alpha j\omega} \right|^2 u^2 + \varrho^2 (c'x)^2 \right] \mathrm{d}t \tag{2.4}$$

State weighting

$$V_s = \int_0^\infty \left[u^2 + \varrho^2 \left| \frac{1+\alpha j\omega}{1+\beta j\omega} \right|^2 (c'x)^2 \right] \mathrm{d}t \tag{2.5}$$

where c is a vector, usually chosen such that $c'(sI - A)^{-1}b$ is minimum-phase without loss of generality[6], and ϱ is a scalar.

Two cases can be distinguished:

$\beta > \alpha$: the 'lag' compensation case

In this case there is extra high-frequency weighting on the control in the index (2.4). It is intuitive that this will lead to a reduction in high-frequency control activity compared to the nominal (unweighted) case, i.e. that the effect will be equivalent to the introduction of lag compensation in the optimal feedback loop. The validity of the intuitive deduction is one of the questions that will be addressed.

In the alternative state-weighted index (2.5), $\beta > \alpha$ corresponds to decreased high-frequency weighting on the state. This will mean that high-frequency components of the state will be tolerated more, and so will be less subject to corrective control action. Again, a reduction in loop gain at high frequencies would be expected. However, despite the superficial similarity between anticipated effects of control weighting and state weighting, it is by no means obvious that the robustness properties of the corresponding optimal systems will also be similar. This question is examined in Section 6.

3. THE (NOMINALLY) LAG COMPENSATOR CASE

The optimal compensator

In this section, the results of reference 3 for the case $\beta > \alpha$ of the control-weighted problem (2.3), (2.4) are reviewed and extended. As pointed out[3], this problem is equivalent to the conventional problem of minimizing

$$V_c = \int_0^\infty [v^2 + \varrho^2 y^2]\ \mathrm{d}t \tag{3.1}$$

with the augumented state-variable equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} -\beta^{-1} & 0 \\ b(\beta^{-1} - \alpha\beta^{-2}) & A \end{bmatrix}\begin{bmatrix} z \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ b\alpha\beta^{-1} \end{bmatrix} v \tag{3.2}$$

where $y = c'x$, z is the state of a first-order compensator, and v is related to the plant input u by

$$v = (\beta^{-1} - \alpha^{-1})z + \beta\alpha^{-1}u \tag{3.3}$$

(see also Figure 1). Suppose (A, b) and (A, c) are respectively stabilizable and detectable; then stabilizability and detectability carry over to the conventional problem with the augmented state vector, and it has a solution[6]. The optimal control law is stabilizing, and is given by

$$v = k_z z + k_x' x \tag{3.4}$$

where

$$\begin{bmatrix} k_z \\ k_x \end{bmatrix} = -P\begin{bmatrix} 1 \\ \alpha\beta^{-1} b \end{bmatrix} \tag{3.5}$$

and P is the nonnegative definite solution of

$$P\begin{bmatrix} -\beta^{-1} & 0 \\ b(\beta^{-1} - \alpha\beta^{-2}) & A \end{bmatrix} + \begin{bmatrix} -\beta^{-1} & b'(\beta^{-1} - \alpha\beta^{-2}) \\ 0 & A' \end{bmatrix} P - P\begin{bmatrix} 1 \\ \alpha\beta^{-1} b \end{bmatrix}[1\ \alpha\beta^{-1} b']P + \begin{bmatrix} 0 & 0 \\ 0 & \varrho^2 cc' \end{bmatrix} = 0 \tag{3.6}$$

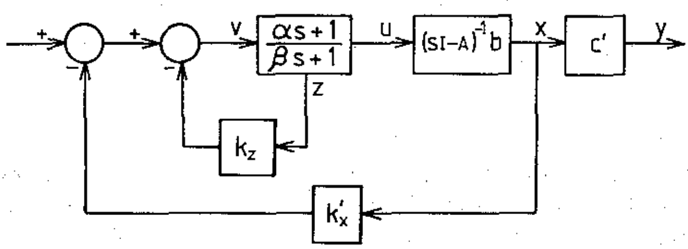

Figure 1. Structure of control weighted problem

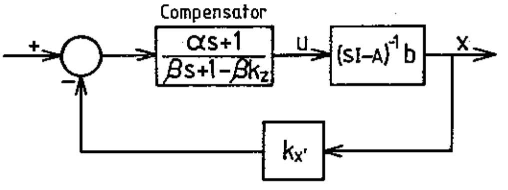

Figure 2. Optimal feedback system: control weighted problem

Assuming $z(0) = 0$, (3.2) – (3.4) yield

$$u = \frac{j\omega\alpha + 1}{j\omega\beta + (1 - \beta k_z)} k_x' x \tag{3.7}$$

That is, the optimal controller a first-order comprises a first-order compensator in tandem with the state-feedback term (Figure 2). Note that the state feedback gain k_x is different from that which would be obtained in the no frequency weighting case.

Bounds on k_z and k_x

The 1–1 entry of the algebraic Riccati equation (3.6) yields

$$-2P_{11}\beta^{-1} + 2P_{12}b(\beta^{-1} - \alpha\beta^{-2}) - (P_{11} + \alpha\beta^{-1}P_{12}b)^2 = 0 \tag{3.8}$$

Also, from (3.5)

$$k_z = -(P_{11} + \alpha\beta^{-1}P_{12}b) \tag{3.9}$$

Eliminating $P_{12}b$ between (3.8) and (3.9) yields

$$k_z^2 - 2(\beta^{-1} - \alpha^{-1})k_z = -2\alpha^{-1}P_{11} \tag{3.10}$$

Now $P_{11} \geqslant 0$ since P is nonnegative definite. Also, $\beta^{-1} - \alpha^{-1} < 0$ since $\beta > \alpha$. Hence (3.10) implies that

$$2(\beta^{-1} - \alpha^{-1}) \leqslant k_z \leqslant 0 \tag{3.11}$$

The upper bound in (3.11) was established[3] but the lower bound is new. A partial bound on $k_x = -(P_{12}' + \alpha\beta^{-1}P_{22}b)$ can also be obtained. Inspection of (3.8) shows that in view of the facts that $P_{11} \geqslant 0$ and $\beta > \alpha$,

$$P_{12}b \geqslant 0$$

Hence

$$k_x'b = -(P_{12}b + \alpha\beta^{-1}b'P_{22}b) \leqslant 0 \tag{3.12}$$

since $b'P_{22}b \geqslant 0$ as P is nonnegative definite

Condition for the optimal compensator to be a lag

Inspection of the compensator transfer function in (3.7) with $\beta > \alpha$ shows that because of the bound in (3.11), the dynamic part of the compensator is stable, in fact, it has one stable pole and one stable zero. But it will only be a lag compensator if

$$\beta^{-1} - \alpha^{-1} < k_z < \beta^{-1} \tag{3.13}$$

It is a stable lead compensator if

$$k_z < \beta^{-1} - \alpha^{-1} \tag{3.14}$$

and the lower bound in (3.11) apparently does admit this possibility, despite this being contrary to the intuitive reasoning in Section 2. The precise condition for the optimal compensator to be indeed a lag is now investigated.

The return difference equality for the problem at hand is[6]:

$$T(s)T(-s) = 1 + \varrho^2\left[\frac{\alpha\beta^{-1}s + \beta^{-1}}{s+\beta^{-1}}\right]C(s)C(-s)\left[\frac{\beta^{-1} - \alpha\beta^{-1}s}{\beta^{-1} - s}\right] \tag{3.15}$$

where the return difference is

$$T(s) = 1 - \frac{k_z}{s+\beta^{-1}} - k_x'(sI-A)^{-1}b\left[\frac{\alpha\beta^{-1}s + \beta^{-1}}{s+\beta^{-1}}\right] \tag{3.16}$$

and $C(s) \triangleq c'\,(sI-A)^{-1}b$. Hence $T(s)$, the minimum-phase spectral factor of the right-hand side of (3.15), must be of the form

$$T(s) = \frac{n(s)}{(s+\beta^{-1})\ \det\ (sI-A)} \tag{3.17}$$

where $n(s)$ is Hurwitz, and also monic. [Letting $s \leftarrow \infty$ in (3.16) shows $T(\infty) = 1$, and using this in (3.17) shows $n(s)$ is monic.]

Assume temporarily that neither α^{-1} nor $-\alpha^{-1}$ is an eigenvalue of A. Then setting $s = \alpha^{-1}$ in (3.15) yields

$$T(\alpha^{-1})\ T(-\alpha^{-1}) = 1 \tag{3.18}$$

Also, from (3.16),

$$T(-\alpha^{-1}) = 1 - \frac{k_z}{\beta^{-1} - \alpha^{-1}} \tag{3.19}$$

and from (3.17)

$$T(\alpha^{-1}) = \frac{n(\alpha^{-1})}{(\alpha^{-1} + \beta^{-1})\det(\alpha^{-1}I - A)} \tag{3.20}$$

substituting (3.19) and (3.20) into (3.18) gives

$$\frac{n(\alpha^{-1})}{(\alpha^{-1} + \beta^{-1})\det(\alpha^{-1}I - A)}\left[1 - \frac{k_z}{\beta^{-1} - \alpha^{-1}}\right] = 1$$

whence

$$k_z - \beta^{-1} + \alpha^{-1} = \frac{(\alpha^{-2} - \beta^{-2})\det(\alpha^{-1}I - A)}{n(\alpha^{-1})} \tag{3.21}$$

Now $n(\alpha^{-1}) > 0$ since $\alpha > 0$ and $n(s)$ is monic and Hurwitz. Also $\alpha^{-2} - \beta^{-2} > 0$ since $\beta > \alpha$. The conclusion from (3.21) is therefore that $k_z > \beta^{-1} - \alpha^{-1}$, that is, the optimal compensator is a lag, provided

$$\det\ (\alpha^{-1}\ I - A) > 0 \tag{3.22}$$

The condition (3.22) is satisfied if and only if the open-loop plant has zero, or an even number of, positive real poles greater than α^{-1}. (To see this, consider the variation in the sign of $\det(sI-A)$ as s moves from a very large positive number to $+\alpha^{-1}$; for very large s, the sign is positive. Each time s passes through a pole of odd multiplicity the sign changes.) Clearly, (3.22) holds for all open-loop stable plants.

Next, suppose that $-\alpha^{-1}$ is an eigenvalue of A. Then the augmented state variable system (3.2) has an uncontrollable mode at $-\alpha^{-1}$, but this does not affect stabilizability so that P, k_z and k_x are well defined. If α varies slightly, so that α^{-1} is no longer an eigenvalue of A, the quantities k_z, k_x change continuously. Since all quantities in (3.21) then depend continuously on α, and (3.21) holds when $-\alpha^{-1}$ is not an eigenvalue of A and $\alpha > 0$, (3.21) continues to hold when $-\alpha^{-1}$ is an eigenvalue of A. If α^{-1} is an eigenvalue of A, the arguments leading to (3.21) cannot be used; however, k_z again depends continuously on α, and so (3.21) must hold. Notice that, as a result, $k_z = \beta^{-1} - \alpha^{-1}$ if and only if α^{-1} is an eigenvalue of A.

Guaranteed robustness

A robustness result is now reviewed. Recall that

$$G_c(j\omega) = -k_x'(j\omega I - A)^{-1}b \frac{j\omega\alpha + 1}{j\omega\beta + (1-\beta k_z)} \tag{3.23}$$

is the loop gain associated with the control-weighted optimal control, and it was shown[3] that

$$|1 + G_c(j\omega)| \geqslant \left|\frac{j\omega\beta + 1}{j\omega\beta + (1-\beta k_z)}\right|^2 \tag{3.24}$$

By comparison, for the conventional problem with the same A, b, c but no frequency weighting[6],

$$|1 + G(j\omega)|^2 \geqslant 1 \tag{3.25}$$

where G is the loop gain for the conventional problem. Because $k_z \leqslant 0$, the right-hand member of (3.24) may be less than 1. Now recalling that insensitivity to unmodelled dynamics in the pass band depends on $|1 + G(j\omega)|$ being large[7], it is deduced that some of the robustness guaranteed with the conventional problem in the passband is potentially lost. The lower the frequency, the more pronounced is the loss of robustness.

On the other hand, at high frequencies, i.e. in the stopband, it was established[3] that

$$|G_c(j\omega)| < |G(j\omega)| \tag{3.26}$$

which, in view of another standard robustness result[8], implies that the frequency-weighted optimal controller has increased robustness (in the face of gain errors associated with arbitrary phase errors) at high frequencies. (If the plant is uncertain to within a multiplication factor $1 + \epsilon L(\omega)$, with $|L(\omega)| < \lambda(\omega)$, stability of the perturbed plant then requires $|\lambda(\omega)G(j\omega)| < 1$ or $|\lambda(\omega)G_c(j\omega)| < 1$)

Asymptotic problem

Let us now consider the effect of allowing ϱ to approach infinity in (2.4). Let us define $n(s)$, $d(s)$, $e_\varrho(s)$ by

$$\frac{n(s)}{d(s)} = c'(sI-A)^{-1}b \quad \frac{e_\varrho(s)}{d(s)} = k_{x_\varrho}'(sI-A)^{-1}b \tag{3.27}$$

Assume that $n(\alpha\beta^{-1}s + \beta^{-1})$ is monic, by absorbing any non unity leading coefficient into ϱ. It is well known, see reference 6, that if $n(s)$ has all its roots in $R_e[s] < 0$, then the characteristic polynominal of the closed-loop system for very large ϱ^2 is approximately

$$\bar{d}_\varrho(s) = B_{r_\varrho}(s)n(s)(\alpha\beta^{-1}s + \beta^{-1}) \quad (3.28)$$

where $r = \deg d - \deg n$, and $B_{r_\varrho}(s)$ is a polynomial with zeros lying in $R_e[s] < 0$ in a Butterworth configuration of radius $|\varrho|^{1/r}$. It follows that

$$1 - \frac{k_{z_\varrho}}{s+\beta-1} - \frac{e_\varrho(s)}{d(s)}\frac{\alpha\beta^{-1}s+\beta^{-1}}{s+\beta-1} - \frac{B_{r_\varrho}(s)n(s)(\alpha\beta^{-1}s+\beta^{-1})}{d(s)}$$

ie $\quad d(s)(s+\beta^{-1}-k_{z_\varrho}) - e_\varrho(s)(\alpha\beta^{-1}s+\beta^{-1}) - B_{r_\varrho}(s)n(s)(\alpha\beta^{-1}s+\beta^{-1})$

from which we see that

$$k_{z_\varrho} - \beta^{-1} - \alpha^{-1} \quad (3.29)$$

and

$$d(s) - e_\varrho(s) - B_{r\varrho}n(s) \quad (3.30)$$

Equation (3.29) together with (3.7) shows that the controller approaches

$$u = \alpha\beta^{-1}k'_{x\varrho}x \quad (3.31)$$

and (3.30) and this observation show that the controller becomes identical with that which is obtained with no frequency weighting on u^2 in the performance index, and with $\varrho \to \infty$. This is as expected from (2.4): as $\varrho \to \infty$, the contribution to the performance index from u^2 becomes less and less important, and so the question of whether u is or is not frequency weighted becomes irrelevant.

While one can also examine the effect of letting $\varrho \to 0$, there seems to be no general conclusion which can readily be drawn.

4. THE (NOMINALLY) LEAD COMPENSATOR CASE

Attention is now turned to the case where $\beta < \alpha$ in the performance index (2.4). Intuitively one expects the optimal compensator to be a lead in this case, but, as in the nominally lag case, it turns out that the contrary is true for a certain class of plants. The analysis of Section 3 remains valid for the $\beta < \alpha$ case, too, except (not surprisingly) for the inequations.

Bounds on k_z

Since P is nonnegative definite, $P_{11} \geqslant 0$. Hence (3.10) implies

$$k_z^2 - 2(\beta^{-1} - \alpha^{-1})k_z \leqslant 0 \quad (4.1)$$

Since $\beta < \alpha$, $\beta^{-1} - \alpha^{-1} > 0$, and so (4.1) implies

$$0 \leqslant k_z \leqslant 2(\beta^{-1} - \alpha^{-1}) \quad (4.2)$$

Condition for the optimal compensator to be a lead

As noted previously, the compensator is a stable lead if $k_z < \beta^{-1} - \alpha^{-1}$, and a stable lag if $\beta^{-1} - \alpha^{-1} < k_z < \beta^{-1}$. Morever the compensator is unstable if

$$k_z > \beta^{-1} \quad (4.3)$$

It is evident that the upper bound on k_z contained in (4.2) does not of itself exclude the possibility of an unstable compensator. This is in contrast to the $\beta > \alpha$ case where it was seen that the controller is always stable. The potential instability of the compensator will be examined in more detail in Section 5 via an example.

Now equation (3.21) remains true when $\beta < \alpha$, but then since $\alpha^{-2} - \beta^{-2} < 0$, it follows that $k_z < \beta^{-1} - \alpha^{-1}$, that is the optimal compensator is indeed a (stable) lead, whenever (3.22) holds. Thus the class of plants for which the optimal compensator is a lead in the $\beta < \alpha$ case is precisely the same class for which the optimal compensator is a lag in the $\beta > \alpha$ case. This class includes all open-loop stable plants.

Guaranteed robustness

The important inequality (3.24) holds irrespective of the relative magnitudes of α and β. Now inspection reveals that the right hand member of (3.24) is not less than 1 for all k_z in the solution region (4.2). Hence comparing (3.24) and (3.25), it is deduced that when $\beta < \alpha$ robustness is potentially enhanced in the passband, or at low frequencies.

On the other hand, using the method of reference 3, it can be shgown that as $\omega \to \infty$, $\beta < \alpha$ implies that

$$|G_c(j\omega)| > |G(j\omega)| \tag{4.4}$$

That is, robustness is decreased at high frequency.

5. FIRST-ORDER PLANT

In this section, a detailed analysis is performed on a first-order plant. In addition to illustrating the theory of the preceeding sections, the closed-form solutions obtained provide further insight to the effects of frequency-weioghting.

The plant studied is

$$\dot{x} = ax + u \tag{5.1}$$

with the performance index (2.4) where c (now a scalar) = 1. By setting $T(s)$ equal to the minimum phase factor of the right-hand side of the return difference equation (3.15), then substituting it into (3.16) and solving for k_z and k_x, the optimum gains are found to be

$$k_x = -\frac{a^2\beta + \sqrt{\varrho^2 + a^2} + a\sqrt{a^2\beta^2 + 1 + \varrho^2\alpha^2 + 2\beta\sqrt{\varrho^2 + a^2}}}{a\alpha + 1} \tag{5.2}$$

$$k_z = -\frac{a\sqrt{\varrho^2 + a^2} + (1 + a\alpha) - a\beta - \sqrt{a^2\beta^2 + 1 + \varrho^2\alpha^2 + 2\beta\sqrt{\varrho^2 + a^2}}}{\beta(a\alpha + 1} \tag{5.3}$$

provided $a\alpha + 1 \neq 0$

Asymptotic results

The following asymptotic results on k_z are easily deduced from (5.3)

$$\lim_{a \to -\infty} k_z = 0 \quad (\varrho \text{ finite}) \tag{5.4}$$

$$\lim_{a \to \infty} k_z = 2(\beta^{-1} - \alpha^{-1}) \quad (\varrho \text{ finite}) \tag{5.5}$$

$$\lim_{\varrho \to \infty} k_z = \beta^{-1} - \alpha^{-1} \quad (a \text{ finite}) \tag{5.6}$$

Equations (5.4) and (5.5) show that the upper and lower bounds in (3.11) and also (4.2) are both attained, at least in the limit, and hence these bounds are tight (for finite ϱ). Equation (5.6) shows that the compensator transfer function tends to unity at all frequencies as $\varrho \to \infty$. This is a special case of the result obtained in Section 3; of course, it applies whether or not $\beta > \alpha$.

Another result that is easily obtained is that

$$a = \alpha^{-1} => k_z = \beta^{-1} - \alpha^{-1}, \text{ all } \varrho \tag{5.7}$$

(5.7) is in accordance with the condition (3.22) for the crossover from lag to lead compensaiton, and vice versa.

Stability of the compensator

It has already been noted that there is a possibility of compensator instability when $\beta < \alpha$. To check whether k_z given by (5.3) can satisfy the instability condition (4.3), form the quantity

$$k_z - \beta^{-1} = -\frac{a\sqrt{\varrho^2 + a^2} - \alpha\beta - \sqrt{(\alpha\sqrt{\varrho^2 + a^2} - \alpha\beta)^2 + (1 + a\alpha)(1 + 2\beta\sqrt{\varrho^2 + a^2} - a\alpha)}}{\beta(a\alpha + 1}$$

Minor algebra shows that $k_z - \beta^{-1} < 0$ (corresponding to compensator stability) if and only if

$$1 - a\alpha + 2\beta\sqrt{\varrho^2 + a^2} > 0$$

Further analysis reveals that $k_z - \beta^{-1} < 0$ if either

$$\alpha \lesseqgtr 2\beta \tag{5.8}$$

or

$$\alpha > 2\beta \text{ and } a < \frac{\alpha + 2\beta\sqrt{\varrho^2(\alpha^2 - 4\beta^2) + 1}}{\alpha^2 - 4\beta^2} \tag{5.9}$$

Inequality (5.9) shows that instability of the compensator occurs only when $\alpha > 2\beta$ and then also only for highly unstable plants: the last term in (5.9) is greater than α^{-1}. It seems difficult to generalize (5.9) to arbitrary plants because of the apparent dependence on ϱ. This is in contrast to the simple general condition (3.22) on the lead/lag transition. It is possible that the stability condition (5.8) applies to general plants; however little frequency shaping occurs when this condition holds.

High-frequency gain reduction — $\beta > \alpha$ case

The main motivation for introducing frequency weighting with $\beta > \alpha$ is to achieve a relative reduction in the high-frequency loop gain compared to the unweighted case, thereby enhancing robustness to unmodelled dynamics in the stopband. Although the nominal high-frequency gain reduction ratio called for through the performance index is β/α, the actual ratio is

$$\gamma = \frac{\beta}{(1 - \beta k_z)\alpha} \tag{5.10}$$

Observe that $\gamma \leqslant \beta/\alpha$ because $k_z \leqslant 0$ when $\beta > \alpha$. We wish now to obtain asymptotic expressions for γ valid when $\beta/\alpha \gg 1$. Whether or not this is indicative of what happens for a plant of higher than first order is not clear.

Usually in phase lag compensation the upper break frequency of the compensator, viz α^{-1} would be approximately $0 \cdot 1\, \omega_c$, where ω_c is the closed-loop unity gain frequency, and

$\beta^{-1} = 0{\cdot}1\ \alpha^{-1}$ would be typical. The closed-loop unity gain frequency is of the order of ϱ, and is usually less than the open-loop gain frequency, which is of the order of a.

Let us assume simply that

$$\beta^{-2} \ll \alpha^{-2} \ll |a|^2,\ \varrho^2 \text{ and } a < 0. \tag{5.11}$$

Then

$$k_z = \frac{\alpha\sqrt{\varrho^2 + a^2} + (1 + a\alpha) - a\beta - \sqrt{a^2\beta^2 + 1 + \varrho^2\alpha^2 + 2\beta\sqrt{\varrho^2 + a^2}}}{\beta(1 + a\alpha)}$$

$$= \frac{\alpha\sqrt{\varrho^2 + a^2} + (1 + a\alpha) - a\beta - |a|\beta\sqrt{1 + a^{-2}\beta^{-2} + \varrho^2\alpha^2 a^{-2}\beta^{-2} + 2a^{-2}\beta^{-1}\sqrt{\varrho^2 + a^2}}}{\beta(1 + a\alpha)}$$

$$\doteqdot \frac{\alpha\sqrt{\varrho^2 + a^2} + (1 + a\alpha) - a\beta + a\beta\sqrt{1 + 2a^{-2}\beta^{-1}\sqrt{\varrho^2 + a^2} + a^{-4}\beta^{-2}(\varrho^2 + a^2)}}{\beta(1 + a\alpha)}$$

$$= \frac{1 + a^{-1}\sqrt{\varrho^2 + a^2}}{\beta}$$

There results

$$\gamma \doteqdot \frac{|a|\beta}{\sqrt{\varrho^2 + a^2}\,\alpha} \tag{5.12}$$

Since usually $\varrho < a$, we see that

$$0{\cdot}5\ \beta/\alpha < \gamma < \beta/\alpha \tag{5.13}$$

This inequality allows one to roughly set the high frequency gain reduction at the design stage, i.e. when formulating the performance index.

While one can establish results also for $a \geqslant 0$, their interpretation is not clear.

6. STATE-WEIGHTED PERFORMANCE INDEX

So far the analysis has been confined to the control-weighted index (2.4). Consider now the alternative state-weighted index (2.5). Defining

$$w = \frac{1 + s\alpha}{1 + s\beta}\, c'x \tag{6.1}$$

it is clear that the state-weighted problem is equivalent to minimizing the conventional performance index

$$Vs = \int_0^\infty (u^2 + \varrho^2 w^2)\, dt \tag{6.2}$$

for the augmented system described by (2.3) and (6.1). (see also Figure 3.)

Defining $G_s(s)$ to be the loop gain of the optimal state-weighted system referred to the plant input, and $T_s(s) = 1 + G_s(s)$ the corresponding return difference, it is easy to see that the return difference equality for this problem is

$$|T_s(j\omega)|^2 = |1 + G_s(j\omega)|^2 = 1 + \varrho^2 \left|\frac{j\omega\alpha + 1}{j\omega\beta + 1}\right|^2 |c'(j\omega I - A)^{-1}b|^2 \tag{6.3}$$

Figure 3. Structure for state-weighted problem

Comparing this with the return difference equality for the control-weighted problem[3]:

$$|T_c(j\omega)|^2 = |1 + G_c(j\omega)|^2 = \left|\frac{j\omega\beta + 1}{j\omega\beta + (1 - \beta k_z)}\right|^2 \left\{1 + \varrho^2 \left|\frac{j\omega\alpha + 1}{j\alpha\beta + 1}\right|^2 |c'(j\omega I - A)^{-1} b|^2\right\} \tag{6.4}$$

it is clear that the zeros of T_s and T_c are the same and hence the optimal systems for the two problems have identical closed-loop poles. This was a main result of Reference 5. However, the robustness properties of the two system are by no means identical and this issue is taken up now.

Guaranteed robustness: $\beta > \alpha$ case

Comparing (6.3) and (6.4), it is seen that

$$|1 + G_c(j\omega)| = \left|\frac{j\psi\beta + 1}{j\omega\beta + (1 - \beta k_z)}\right| |1 + G_s(j\omega)| \tag{6.5}$$

The conclusion thus is that potentially more passband robustness is lost with control weighting when $\beta > \alpha$. Next consider the high frequency robustness. From (3.23), for large ω,

$$|G_c(j\omega) \simeq \frac{1}{\omega} |k_x' b| \frac{\alpha}{\beta} \tag{6.7}$$

Now setting $s = j\omega$ in (3.15) and (3.16) and comparing with (6.3), it is evident that

$$|1 + G_s(j\omega)^2 = \left|1 - \frac{k_z}{j\omega + \beta - 1} - k_x'(j\omega I - A)^{-1} b \frac{j\omega\alpha + 1}{j\omega\beta + 1}\right|^2$$

Therefore, when ω is high,

$$|G_s(j\omega)| \simeq \frac{1}{\omega} |k_z + \frac{\alpha}{\beta} k_x' b| \tag{6.8}$$

Therefore, when ω is high,

$$|G_s(j\overset{\alpha}{\omega})| \simeq \frac{1}{\omega} |k_z + \frac{}{\beta} k_x' b| \tag{6.8}$$

Now $k_z \leqslant 0$ and from (3.12), $k_x' b \leqslant 0$. Hence it is obvious from (6.6) and (6.7) that

$$|G_c(j\omega)| \leqslant |G_s(j\omega)| \tag{6.9}$$

That is, the improvement in high-frequency robustness is greater with control weighting.

Guaranteed robustness: $\beta < \alpha$ case

In this case, the inequality $k_z \geqslant 0$ applies (see (4.2)). Then it follows from (6.5) and (6.3) that

$$|1 + G_c| > |1 + G_s| > 1 \tag{6.10}$$

Thus the improvement in passband robustness is greater with control weighting.

Next look at the stopband. For large ω, the asymptotic expressions (6.7) and (6.8) still apply.

Now consider

$$[1 \ \alpha\beta^{-1}b'] \begin{bmatrix} k_z \\ k_x \end{bmatrix} = -[1 \ \alpha\beta^{-1}b'] \ P\begin{bmatrix} 1 \\ \alpha\beta^{-1}b \end{bmatrix}$$

so that, in view of the positive definiteness of P,

$$k_z + \alpha\beta^{-1}b'k_x < 0$$

Since $k_z > 0$, it follows that

$$\alpha\beta^{-}b'k_x < k_z + \alpha\beta^{-1}b'k_x < 0$$

Consequently, (6.7) and (6.8) imply that at high frequencies

$$|G_s| < |G_c| \tag{6.11}$$

That is, state weighting gives better high-frequency robustness.
These ideas are summarized in Table I.

Table I. Summary of robustness properties for control and state weighting

	$\beta > \alpha$: Penalize high frequency	$\beta < \alpha$: Penalize low frequencies
Passband robustness	State weighting better (More loss of robustness with control weighting)	Control weighting better
Stopband robustness	Control weighting better	State weighting better

7. MULTI-INPUT SYSTEMS

For the multi-input plant (2.2), a modest generalization of part of the previous results is obtained by

$$R(j\omega) = \text{diag}\left[\left|\frac{j\omega\beta_i + 1}{j\omega\alpha + 1}\right|^2\right], Q(j\omega) = Q \tag{7.1}$$

in the performance index (2.1) where Q is a real constant nonnegative definite symmetric matrix. It is assumed that the triple $\{A, B, Q^{1/2}\}$ is minimai and that B has full column rank. Notice that all channels have the same lead break frequency α^{-1} but different lag break frequencies β_i^{-1}. Defining

$$M = \text{diag}(\alpha, \ldots, \alpha), N = \text{diag}(\beta_1, \ldots, \beta_r)$$

We can write the performance as

$$V_1 = \int_0^\infty (v'v + x'Qx) \, dt \tag{7.2}$$

where

$$u = (sM + I)(sN + I)^{-1}v \tag{7.3}$$

The compensator model is thus

$$\begin{aligned} \dot{z} &= -N^{-1}z + v \\ u &= MN^{-1}v + (N^{-1} - MN^{-2})z \end{aligned} \tag{7.4}$$

Combining with (2.2) gives the augmented plant equation

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -N^{-1} & 0 \\ B(N^{-1} - MN^{-2}) & A \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} + \begin{bmatrix} I \\ BMN^{-1} \end{bmatrix} v \tag{7.5}$$

The associated algebraic Riccati equation is

$$\begin{aligned} &P\begin{bmatrix} -N^{-1} & 0 \\ B(N^{-1} - MN^{-2}) & A \end{bmatrix} + \begin{bmatrix} -N^{-1} & (N^{-1} - N^{-2}M)B' \\ 0 & A' \end{bmatrix} P \\ &- P\begin{bmatrix} I \\ BMN^{-1} \end{bmatrix} [IN^{-1}MB']P + \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} = 0 \end{aligned} \tag{7.6}$$

and the optimal gains are

$$[K_z' K_x'] = -[IN^{-1}MB']P \tag{7.7}$$

The optimal control is

$$v = K_x' z + K_z' x = K_z' (j\omega I + N^{-1})^{-1} v + K_x' x$$

whence

$$v = [j\omega N + I]\,[j\omega N + I - K_z' N]^{-1} K_x' x$$

and so in view of (7.3), the optimal plant input is

$$u = (j\omega M + I)(j\omega N + I - K_z' N)^{-1} K_x' x \tag{7.8}$$

Stability of the compensator

The 1–1 block entry of (7.6) yields

$$\begin{aligned} &-P_{11}N^{-1} + P_{12}B(N^{-1} - MN^{-2}) - N^{-1}P_{11} + (N^{-1} - MN^{-2})B'P_{12}' \\ &-(P_{11} + P_{12}BMN^{-1})(P_{11} + N^{-1}MB'P_{12}') = 0 \end{aligned}$$

Noting also that from (7.7),

$$K_z = -P_{11} - P_{12}\,BMN^{-1}$$

we obtain after minor manipulation

$$K_z(M^{-1} - N^{-1}) + (M^{-1} - N^{-1})K_z' = -K_zK_z' - P_{11}M^{-1} - M^{-1}P_{11} \tag{7.9}$$

and so

$$\begin{aligned} (K_z - N^{-1})(M^{-1} - N^{-1}) + (M^{-1} - N^{-1})(K_z' - N^{-1}) = &-K_zK_z' - P_{11}M^{-1} \\ &- M^{-1}P_{11} - 2N^{-1}(M^{-1} - N^{-1}) \end{aligned}$$

Noting that $M = \alpha I$ and, if $\beta_i > \alpha$ for all i then $M^{-1} - N^{-1} > 0$, it follows from the basic Lyapunov lemma that Re $\lambda_i(K_z - N^{-1}) < 0$; that is, the transfer function matrix associated with the compensator is again stable.

We also note for later use that, similarly, (7.9) implies that *Re* $\lambda_i(K_z) \leqslant 0$.

Return difference inequality

The return difference referred to the compensator input v is

$$\begin{aligned} T(s) &= I - K_z^t N(sN+I)^{-1} - K_x^t(sI-A)^{-1}B(sN+I)^{-1} \\ &= [sN + I - K_z^t N - K_x^t(sI-A)^{-1}B(sM+I)](sN+I)^{-1} \\ &= [I - K_x^t(sI-A)^{-1}B(sM+I)(sN+I-K_z^t N)](sN+I-K_z^t N)(sN+I)^{-1} \\ &= [I + G_c(s)](sN+I-K_z^t N)(sN+I)^{-1} \end{aligned} \tag{7.10}$$

where $G_c(s)$ is the loop transfer function matrix referred to the compensator input. Now the return difference inequality is

$$T^*(s)\ T(s) \geqslant I$$

Substituting (7.9) and setting $s = j\omega$, we obtain after minor manipulation that

$$[I + G_c(j\omega)]^*\ [I + G(j\omega)] \geqslant [\quad]^*\ [(j\omega N + I)(j\omega N + I - K_z^t N)^{-1}] \tag{7.11}$$

where [ ]* in (7.11) means conjugate transpose of the adjacent quantity.

Guaranteed robustness in the passband

Inequality (7.11) is making a robustness statement with respect to a break in the feedback loop at the compensator input. This statement can be made more explicit in the case where $N = \beta I$. Then (7.11) reduces to

$$[I + G_c(j\omega)]^*\ [I + G_c(j\omega)] \geqslant (\omega^2\beta^2 + 1)[\quad]^*\ [j\omega\beta I + I - \beta K_z]^{-1} \tag{7.12}$$

Defining $\sigma_i + j\omega_i \triangleq \lambda_i(K_z)$, and taking determinants, (7.12) yields.

$$|\det\ [I + G_c(j\omega)]^2 \geqslant \prod_{i=1} \frac{\omega^2\beta^2 + 1}{\beta^2(\omega - \omega_i)^2 + (1 - \beta\sigma_i)^2} \tag{7.13}$$

Remembering that $\sigma_i \leqslant 0$, it is easy to see that terms in the product on the right-hand side of (7.13) arising from real eigenvalues of of K_z, i.e. having $\omega_i = 0$, are less than (or equal to) 1. Also, terms arising from complex eigenvalues of K_z occur in pairs, and it is not difficult to show that the product of each pair is less than 1 if $\beta\omega \leqslant 1$. Thus the right-hand side of (7.13) is certainly less than or equal to 1 in the frequency range $0 \leqslant \omega \leqslant \beta^{-1}$.

The equation corresponding to (7.13) for the standard LQ problem is

$$|\det [I + G(j\omega)]|^2 \geqslant 1 \tag{7.14}$$

Since $|\det\ [I + G_c(j\omega)]|$ is the product of the singular values of $[I + G_c(j\omega)]$, it is clear from a comparison of (7.13) and (7.14) that this product is potentially less than the product of the singular values of $[I + G(j\omega)]$ in the frequency range $0 \leqslant \omega \leqslant \beta^{-1}$. Thus there is a potential loss of robustness in this portion of the passband, at least.

We have been unable to establish formally a result exhibiting improved stop-band robustness.

8. CONCLUSIONS

This paper has mainly studied the properties of linear optimal control systems when a frequency–dependent weighting is inserted in the quadratic index with the intention of achieving a similar effect to that of classical phase-lag compensation while concurrently ensuring stability. The emphasis has been on frequency-weighting the control term, and several properties of the optimal compensator were derived including a precise condition for the optimal compensator to be indeed a lag, and not a lead.

The state-weighting case was briefly considered and it was shown that the robustness properties of the resulting optimal system were, roughtly speaking, the converse of those for the control-weighting case.

A detailed analysis was undertaken of a first-order system with control weighting. Several asymptotic properties were derived which could provide clues to the behaviour of more general systems.

Finally, a modest extension of a portion of the results to the case of multi-input systems was achieved.

One apparent restriction of the paper is that state feedback has been assumed, which may not always be practical. In case the full state is not available, then for a minimum phase plant the loop recovery procedure of Reference 9 assures that we can duplicate the robustness conclusion of the paper. This is exactly the same conclusion as applies when we seek to extend the robustness results valid with state feedback and conventional linear-quadratic design to the ouput feedback case.

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