IDENTIFICATION OF SCALAR ERRORS-IN-VARIABLES MODELS WITH DYNAMICS

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Abstract

The paper considers the task of identifying a causal linear dynamic system excited by stationary gaussian zero mean noise of unknown spectrum, and given measurements of the system input and output contaminated by independent additive stationary noise signals of unknown spectra. While the solution is normally not unique, finite-dimensional parameterizations of the solution set are given (in some cases involving just one parameter), even though the various spectra may not be rational.

Keywords. Errors-in-variables; identification; linear systems; modelling.

1. INTRODUCTION

We are concerned with the problem of identifying a linear dynamic system given noisy measurements of it. These measurements are of both the input and output of the system, which will be assumed to be a scalar system in this paper, and in contrast to the common situation, the input as well as the output is contaminated with noise.

More specifically, we postulate the existence of three real random sequences $\{x_k^*\}$, $\{u_k\}$, $\{v_k\}$, mutually independent and stationary, together with a time-invariant linear system defined by a real bounded linear causal convolution operator $\{w_k, k \geq 0\}$ mapping $\{x_k^*\}$ into a sequence $\{y_k\}$ according to

$$y_k^* = \sum_{-\infty}^{k} w_{k-\ell}\, x_\ell^* \qquad (1.1)$$

The processes $\{x_k^*\}$, $\{y_k^*\}$ are not available for measurement, but rather we can measure for $k \in (-\infty, \infty)$

$$x_k = x_k^* + u_k \qquad (1.2a)$$

$$y_k = y_k^* + v_k \qquad (1.2b)$$

(See figure 1).

If the processes are not gaussian, cumulants beyond second order can frequently be used to identify $\{w_k\}$, (see Deistler (1984)). We shall confine attention here to gaussian processes. Also we shall assume that the the concern is not to identify a unique $\{w_k\}$, but to pin down, preferably in a tidy way, the class of $\{w_k\}$ which fit the data. This approach in the nondynamic case goes back to Gini (1921) and Frisch (1934) and has been reevaluated by Kalman (1981). Results applying to the dynamical system case can be found in Anderson and Deistler (1984).

In this paper, our aim is to develop further this second class of results.

2. REVIEW

Let us recall first the following static result, see eg. Madansky (1959) and Moran (1971). Suppose that:

$$y_k^* = w x_k^* \qquad (2.1a)$$

$$x_k = x_k^* \; P \; u_k \qquad (2.1b)$$

$$y_k = y_k^* + v_k \qquad (2.1c)$$

where w is a real scalar to be identified, and $\{x_k^*\}$, $\{u_k\}$ and $\{v_k\}$ are discrete-time, zero mean, white-noise gaussian processes. We assume known the matrix

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = E\{\begin{bmatrix} x_k \\ y_k \end{bmatrix} [x_k \; y_k]\}. \qquad (2.2)$$

and we assume that $E[x_k^{*2}] > 0$. Then the range of w compatible with (2.2) is

$$\left[\frac{\sigma_{xy}}{\sigma_{xx}}, \frac{\sigma_{yy}}{\sigma_{xy}}\right] \text{ if } \sigma_{xy} > 0$$

$$\left[\frac{\sigma_{yy}}{\sigma_{xy}}, \frac{\sigma_{xy}}{\sigma_{xx}}\right] \text{ if } \sigma_{xy} < 0$$

$$0 \text{ if } \sigma_{xy} = 0 \qquad (2.3)$$

If $\sigma_{xy} > 0$, the case $w = \sigma_{xy}\sigma_{xx}^{-1}$ corresponds to $\sigma_u = E(u_k^2) = 0$ and $w = \sigma_{yy}\sigma_{xy}^{-1}$ corresponds to $\sigma_v = E(v_k^2) = 0$. Interior values of w correspond to $\sigma_u\sigma_v > 0$, since one can show that

$$\sigma_u = \sigma_{xx} - \frac{\sigma_{xy}}{w} \qquad (2.4)$$

$$\sigma_v = \sigma_{yy} - w\sigma_{xy}.$$

Of course, if $\sigma_u = 0$ is part of the a priori information, the conclusion that $w = \sigma_{xy}\sigma_{xx}^{-1}$ is immediate and well known. We also note that the physical origins of σ_u may rest in the properties

$$R(e^{j\omega_1}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} R(e^{j\omega})d\omega + \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{I(e^{j\omega_1})\sin\omega_1 - I(e^{j\omega})\sin\omega}{\cos\omega - \cos\omega_1} d\omega \tag{4.2}$$

Note that the second integrand is well behaved at $\omega = \omega_1$. Note also that the formula only allows recovery of the real part of T(z) from the imaginary part to within an additive constant, a fact which is entirely in accord with intuition.

Now consider the task of finding a $\hat{W}(e^{j\omega})$ to satisfy (2.12), given the standard data, and given that the standing assumption holds as well as $\sigma_{xy}(\omega) \neq 0$. Suppose further that $\hat{W}(z)$ is known to be minimum phase, as a result of studying $\arg \sigma_{xy}(\omega)$. Then define

$$T(z) = \ln \hat{W}(z) \tag{4.3}$$

Observe that T(z) is analytic in $|z| \leq 1$, and we can use Lemma 4.1. Now (2.12) yields

$$I(e^{j\omega}) = \arg \sigma_{xy}(\omega) \tag{4.4}$$

and then (4.2) can be used to recover $R(e^{j\omega})$ to within an additive constant. Since $R(e^{j\omega}) = \ln|\hat{W}(e^{j\omega})|$, this means that from $\arg \hat{W}(e^{j\omega})$, we can recover the amplitude response of $\hat{W}(e^{j\omega})$ to within a multiplicative constant, ie

$$\hat{W}(e^{j\omega}) = \mu \bar{W}(e^{j\omega}) \tag{4.5}$$

for some fixed $\bar{W}(e^{j\omega})$; now in order to meet the amplitude constraints implicit in μ, we must have $\mu \epsilon [\mu_{min}, \mu_{max}]$ when

$$\mu_{min} = \max_{\omega} \left| \frac{\sigma_{xy}(\omega)}{\sigma_{xx}(\omega)\, \bar{W}(e^{j\omega})} \right| \qquad \mu_{max} = \min_{\omega} \left| \frac{\sigma_{yy}(\omega)}{\sigma_{yx}(\omega)\, \bar{W}(e^{j\omega})} \right| \tag{4.6}$$

In summary we have proved the following result.

<u>Theorem 4.2</u> Assume available the standard data, and suppose all standing assumptions hold, together with $\sigma_{xy}(\omega) \neq 0$ for all ω. Suppose there exists a causal solution $\hat{W}e^{j\omega})$ of (2.12) and that the change in argument of $\sigma_{xy}(\omega)$ over $[0,2\pi]$ is zero. Then solutions of (2.12) are minimum phase and are of the form

$$\hat{W}(e^{j\omega}) = \mu\bar{W}(e^{j\omega}) \tag{4.5}$$

where $\bar{W}(e^{j\omega})$ is computable to within a scaling constant, and $\mu \epsilon [\mu_{min}, \mu_{max}]$, with the quantities μ_{min} and μ_{max} defined in (4.6).

Not only is this result striking in its similarity to that applying for the static (white noise) case, but the static case can be recovered from it, in the following way. Adopting the standing assumptions, suppose that the standard data Σ is a constant matrix (this would be so if Σ arises from a static problem). Then $\arg \sigma_{xy}(\omega) = 0$ for all ω, and so Theorem 3.1 guarantees that any solution $\hat{W}(z)$ of (2.12) is minimum phase. Use of the procedure of Theorem 4.2 then establishes that $\hat{W}(z)$ <u>must</u> be constant, and the bounds $[\mu_{min}, \mu_{max}]$ coincide with those defined for ω in the static problem. Then even if there is no a priori assumption that the data arose from a static problem, if they are consistent with a static problem, <u>and</u> if the standing assumptions hold, then they necessarily come from a static problem.

The causality of $\hat{W}$ (which is part of the standing assumption) is crucial. Without it, it is easy to define a $\hat{W}(z)$, by

$$\hat{W}(e^{j\omega}) = \mu(\omega)\bar{W}(e^{j\omega})$$

where $\mu(\omega)$ is any real smooth function of ω satisfying

$$\mu_{min} \leq \mu(\omega) \leq \mu_{max}$$

Last, we note that if the spectrum matrix $\Sigma(\omega)$ of the standing data is rational, then $\hat{W}(e^{j\omega})$ is also rational.

5. CAUSAL SOLUTION WITH A PRESCRIBED NUMBER NUMBER OF UNIT CIRCLE ZEROS

In the previous section, we showed that the class of minimum phase solutions was in general a one parameter family, with the parameter being a scaling constant. Here we shall show that the class of solutions $\hat{W}(z)$ where $\hat{W}(z)$ has no poles in $|z| \leq 1$ and N zeros in $|z| < 1$ is a (1 + N) - parameter family.

Consider a class of real rational functions $U_A(z)$ parametised by a set $A = \{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ where the α_i are complex, $|\alpha_i| < 1$, and nonreal α_i occur in complex conjugate pairs. Then $U_A(z)$ is defined by

$$U_A(z) = \frac{\prod_{i=1}^{N} (z - \alpha_i)}{\prod_{i=1}^{N} (\alpha_i z - 1)} \tag{5.1}$$

Notice that on the unit circle, $|z| = 1$, there holds $|U_\alpha(z)| = 1$; also, $U_\alpha(z)$ is causal and real rational.

We can now state the following counterpart to Theorem 4.1. A constructive procedure for generating the solutions defined in the theorem statement can be found in the proof of the theorem.

<u>Theorem 5.1</u> Assume available the standard data, and suppose all standing assumptions hold, together with $\sigma_{xy}(\omega) \neq 0$ for all ω. Suppose there exists a causal solution to (2.12) and that the change in argument of $\sigma_{xy}(\omega)$ as ω varies from 0 to 2π is $2\pi N$. Let A be a set $\{\alpha_1, \alpha_2, \cdots, \alpha_N\}, \alpha_i \epsilon C$, $|\alpha_i| < 1$, nonreal α_i occurring in complex conjugate pairs, and let $U_A(z)$ be defined as in (5.1). Then there exists a minimum phase $\bar{W}_A(z)$ unique to within a scaling constant, such that all W(z) which are causal, meet the phase constraint imposed by (2.12) and have precisely N zeros inside $|z| < 1$ at $z = \alpha_i$, $i = 1, \cdots, N$ can be described as

$$\hat{W}(z) = \mu\, U_A(z)\, \bar{W}_A(z) \tag{5.2}$$

where μ is an arbitrary scaling constraint. Further, the amplitude constraint imposed by (2.12) can be met if and only if the interval $[\mu_{min}, \mu_{max}]$ defined below is nonempty and it contains μ:

$$\mu_{min} = \max_{\omega} \left| \frac{\sigma_{xy}(\omega)}{\sigma_{xx}(\omega)\, \bar{W}_A(e^{j\omega})} \right| \qquad \mu_{max} = \min_{\omega} \left| \frac{\sigma_{yy}(\omega)}{\sigma_{yy}(\omega)\, \bar{W}_A(e^{j\omega})} \right| \tag{5.3}$$

Proof If $\hat{W}(z)$ is causal and has precisely N zeros inside the unit circle at $\alpha_1,\cdots,\alpha_N$, we can certainly write

$$\hat{W}(z) = U_A(z)\ V(z) \tag{5.4}$$

for some minimum phase $V(z)$. If arg $\hat{W}(e^{j\omega})$ is known, arg $V(e^{j\omega})$ is known (and depends on A), even if $|\hat{W}(e^{j\omega})|$ is unknown. Then one can use the constructive procedure of Theorem 4.2 based on Lemma 4.1, to obtain $|V(e^{j\omega})|$ from arg $V(e^{j\omega})$ to within an arbitrary multiplicative constant. This leads to a description of $\hat{W}(z)$ in the form of (5.2), where μ is an arbitrary constant and $\overline{W}_A(z)$ is a (real) minimum phase transfer function. At this stage, $\hat{W}(z)$ is guaranteed to meet the phase restriction of (2.12). Choice of $\mu \in [\mu_{min},\mu_{max}]$ then ensures that the amplitude restrictions are also met.

We remark that for arbitrary A, there is no guarantee that $\mu_{min} \le \mu_{max}$, and thus no guarantee that a $\hat{W}(z)$ can be found with zeros at $\alpha_1,\cdots,\alpha_N$. If there is a true $W(z)$ producing standard data (and there is when the standing assumptions are in force), then naturally there is one set A for which $\mu_{min} \le \mu_{max}$. Let us observe that if $\mu_{Amin} < \mu_{Amax}$ there will be sets $\hat{A} = \{\hat{\alpha}_1,\cdots,\hat{\alpha}_N\}$ where $\max|\alpha_i - \hat{\alpha}_i|$ is small such that $\mu_{\hat{A}min} < \mu_{\hat{A}max}$. For suppose

$$W(z) = \mu \frac{\Pi\ (z - \alpha_i)}{\Pi\ (\alpha_i z - 1)}\ \overline{W}_A(z) \tag{5.5}$$

It is not hard to see that if

$$U_A(z) = \Pi \frac{(z - \alpha_i)}{(\alpha_i\ z - 1)} \quad U_{\hat{A}}(z) = \frac{\Pi(z - \hat{\alpha}_i)}{\Pi(\hat{\alpha}_i z - 1)} \tag{5.6}$$

then the arguments of $U_A^{-1}(z)W(z)$ and $U_{\hat{A}}^{-1}(z)W(z)$ have a difference, the maximum value of which around the unit circle depends continuously on $\max_i |\alpha_i - \hat{\alpha}_i|$. In fact, as is easily seen, the maximum value of the derivative of the difference is arguments also depends continously on $\max_i |\alpha_i - \hat{\alpha}_i|$. By an argument set out in greater detail below, this allows us to conclude that the minimum phase transfer functions with the two argument characteristics and with gain adjusted to be 1 at $\omega = 0$ have gains such that the difference in the logarithm of the gains has a maximum value on the unit circle depending continuously on $\max_i |\alpha_i - \hat{\alpha}_i|$. It follows that $|\mu_{\hat{A}max} - \mu_{Amax}| \to 0$ and $|\mu_{\hat{A}min} - \mu_{Amin}| \to 0$ as $\max_i |\alpha_i - \hat{\alpha}_i| \to 0$, and then that for A near $\hat{A}$, $\mu_{\hat{A}min} < \mu_{\hat{A}max}$.

To explain the introduction of the derivative of the phase, let us note first that (4.2) does not allow the drawing of the conclusion that a small $L_\infty[0,2\pi]$ change in $I(e^{j\omega})$ produces a small L_∞ change in $R(e^{j\omega})$, assuming the latter is normalized so that, for example $R(1) = 0$. The underlying explanation is that

$$\int_{-\pi}^{+\pi} \left|\frac{d\omega}{\cos\omega - \cos\omega_1}\right| < \infty \tag{5.7}$$

is not true. Now straightforward calculation shows that for $0 \le \omega < \omega \le \pi$ and $0 \le \omega_1 < \omega \le \pi$,

$$\frac{d}{d\omega} \ln \left| \frac{\ln\left(\frac{\omega_1 + \omega}{2}\right)}{\ln\left(\frac{\omega_1 - \omega}{2}\right)} \right| = \frac{\sin\omega_1}{\cos\omega - \cos\omega_1} \tag{5.8}$$

and an integration by parts of (4.2) yields (after careful attention to limits)

$$R_0(e^{j\omega_1}) \triangleq \int_{-\pi}^{\pi} \frac{I(e^{j\omega_1})\sin\omega_1 - I(e^{j\omega})\sin\omega}{\cos\omega - \cos\omega_1} d\omega$$

$$= \frac{2}{\sin\omega_1} \int_0^{\pi} \frac{d}{d\omega}[I(e^{j\omega})\sin\omega]\ \ln \left| \frac{\sin\left(\frac{\omega_1 + \omega}{2}\right)}{\sin\left(\frac{\omega_1 - \omega}{2}\right)} \right| d\omega \tag{5.9}$$

Some algebra will show also that

$$\int_0^{\pi} \frac{1}{\sin\omega_1} \left| \left[\ln \left| \frac{\sin\left(\frac{\omega_1 + \omega}{2}\right)}{\sin\left(\frac{\omega_1 - \omega}{2}\right)} \right| \right] \right| d\omega$$

is bounded for all $\omega_1 \in [0,2\pi]$. This means that a small L_∞ variation in $\frac{d}{d\omega}[I(e^{j\omega})\sin\omega]$ produces a small L_∞ variation in $R_0(e^{j\omega})$.

As for the minimum phase case, one can check that if the spectrum matrix $\Sigma(\omega)$ of the standing data is rational, then, $\hat{W}(e^{j\omega})$ is also rational.

MULTIVARIABLE PROBLEMS

It is natural to pose a multivariable version of the problem considered previously. In the first instance, one can suppose that $\{x_k^*\}$, $\{u_k\}$ and $\{v_k\}$ are vector processes of the same dimension. The white noise case can be analysed fairly easily. Assume that

$$y_x^* = Wx_k^* \tag{6.1a}$$

$$x_k = x_k^* + u_k \tag{6.1b}$$

$$y_k = y_k^* + v_k \tag{6.1c}$$

where W is a real matrix to be identified, and $\{x_k^*\}$, $\{u_k\}$ and $\{v_k\}$ are independent zero mean white noise processes, and that the following matrix is known:

$$\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} = E\{\begin{bmatrix} x_k \\ y_k \end{bmatrix}[x_k'\ y_k']\} \tag{6.2}$$

Suppose that

$$\Sigma_{xx} > 0 \tag{6.3}$$

It follows that the set of W consistent with Σ is defined by

$$W = \Sigma_{yx}(\Sigma_{xx} - A)^{-1} \tag{6.4}$$

for any symmetric A satisfying

$$\Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{\#}\Sigma_{yx} \ge A \ge 0 \tag{6.5}$$

(consistent with W being finite). Here, $\Sigma_{yy}^{\#}$ denotes the Moore-Penrose pseudo-inverse.

Now when one allows nonwhite spectra, the formula (6.4) still remains valid where Σ and A become nonnegative definite (or commonly positive definite) Hermitian matrices that are functions of ω, and $W = W(e^{j\omega})$.

A different procedure than that described in this paper is needed to obtain the solutions W of (6.4) and (6.5). The procedure of this paper is unworkable at more than one step: in the matrix case, the phase of W is not known; even if one knew the phase of every entry of W, one cannot, if W is known to be minimum phase, construct the entries of each entry of W using the procedure of Section 4, since individual entries of W will not necessarily be minimum phase. As it turns out, there is an analog with the task of spectral factorization, see for example Rozanov (1967). The first algorithm advanced to handle the spectral factorization problem was not capable of extension to the matrix case, and a second algorithm had to be found. This second spectral factorization algorithm as it turns out suggests a procedure for solving (6.4) and (6.5).

CONCLUSION

By hypothesising causality of a transfer function appearing in a dynamic errors-in-variable model and making certain other reasonable assumptions, it proves possible to parametrise the class of transfer functions consistent with the available data in a finite-dimensional way. Of particular interest are minimum phase transfer functions, where the parameter is a scalar.

In contrast to the approach in this paper, the ideas of Deistler (1984) are aimed at identifying situations where the transfer functions can be obtained uniquely. Broadly speaking, this is done by postulating rational data, and them imposing some sort of minimality of degree assumptions and genericity assumptions that narrow the class of transfer functions down to a single one. Close examination of the results of Anderson and Deistler (1984) illustrates that generally, nonminimium phase transfer functions are harder to identify, in the sense that more assumptions are needed to ensure uniqueness, than minimum phase transfer functions.

It would be interesting to consider the situation where the data was not available within a certain frequency range, or known to be inaccurate, on account perhaps of sampling error. At the very least, this would lead to a study of the robustness of the procedure presented.

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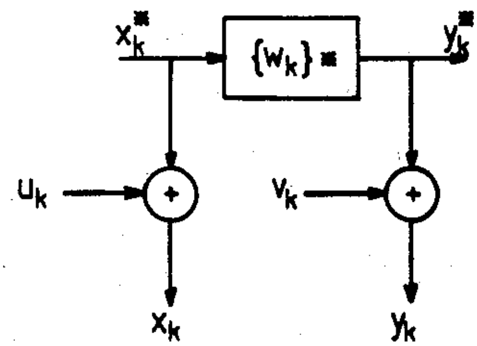

Fig. 1

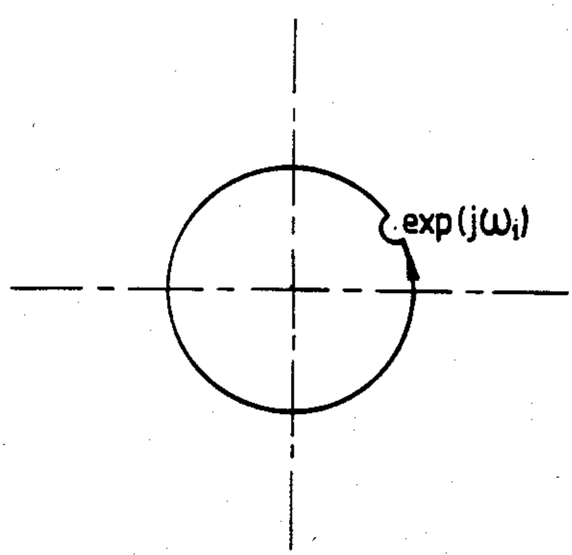

Fig. 2