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Identification of Scalar Errors-in-Variables Models with Dynamics*

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When all measured variables include noise, it is not possible to uniquely identify a system, but a class of candidates for the system transfer function can be found.

Key Words—Identification; errors-in-variables.

Abstract—The paper considers the task of identifying a causal linear dynamic system excited by stationary Gaussian zero mean noise of unknown spectrum, and given measurements of the system input and output contaminated by independent additive stationary noise signals of unknown spectra. While the solution is normally not unique, finite-dimensional parameterizations of the solution set are given (in some cases involving just one parameter), even though the various spectra may not be rational.

1. INTRODUCTION

WE ARE CONCERNED with the problem of identifying a linear dynamic system given noisy measurements of it. These measurements are of both the input and output of the system, which will be assumed to be a scalar system in this paper, and in contrast to the common situation, the input as well as the output is contaminated with noise.

More specifically, we postulate the existence of three real random sequences $\{x_k^*\}$, $\{u_k\}$, $\{v_k\}$, mutually independent and stationary, together with a time-invariant linear system defined by a real bounded linear causal convolution operator $(w_k, k \geq 0\}$ mapping $\{x_k^*\}$ into a sequence $\{y_k^*\}$ according to

$$y_k^* = \sum_{-\infty}^{k} w_{k-l} x_l^*. \tag{1.1}$$

The processes $\{x_k^*\}$, $\{y_k^*\}$ are not available for measurement, but rather we can measure for $k \in (-\infty, \infty)$

$$x_k = x_k^* + u_k \tag{1.2a}$$

$$y_k = y_k^* + v_k \tag{1.2b}$$

(see Fig. 1).

If the processes are not Gaussian, cumulants beyond second order can frequently be used to identify $\{w_k\}$ (see Deistler, 1984). We shall confine attention here to Gaussian processes. Also we shall assume that the concern is not to identify a unique $\{w_k\}$, but to pin down, preferably in a tidy way, the class of $\{w_k\}$ which fit the data. This approach in the nondynamic case goes back to Gini (1921) and Frisch (1934) and has been reevaluated by Kalman (1981). Results applying to the dynamical system case can be found in Anderson and Deistler (1984).

In this paper, our aim is to develop further this second class of results.

2. REVIEW

Let us recall first the following static result, see e.g. Madansky (1959) and Moran (1971). Suppose that:

$$y_k^* = w x_k^* \tag{2.1a}$$

$$x_k = x_k^* + u_k \tag{2.1b}$$

$$y_k = y_k^* + v_k \tag{2.1c}$$

where w is a real scalar to be identified and $\{x_k^*\}$, $\{u_k\}$ and $\{v_k\}$ are discrete-time, zero mean, white

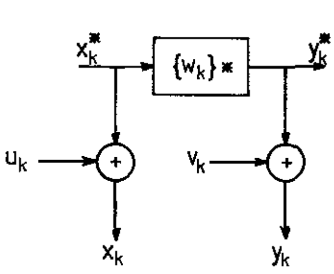

FIG. 1. Structure for identification.

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noise Gaussian processes. We assume known the matrix

$$\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix} = E\left\{\begin{bmatrix}x_k\\ y_k\end{bmatrix}[x_k \quad y_k]\right\} \tag{2.2}$$

and we assume that $E[x_k^{*2}] > 0$. Then the range of w compatible with (2.2) is

$$\begin{aligned} &\left[\frac{\sigma_{xy}}{\sigma_{xx}}, \frac{\sigma_{yy}}{\sigma_{xy}}\right] \quad \text{if } \sigma_{xy} > 0\\ &\left[\frac{\sigma_{yy}}{\sigma_{xy}}, \frac{\sigma_{xy}}{\sigma_{xx}}\right] \quad \text{if } \sigma_{xy} < 0\\ &0 \text{ if } \sigma_{xy} = 0.\end{aligned} \tag{2.3}$$

If $\sigma_{xy} > 0$, the case $w = \sigma_{xy}\sigma_{xx}^{-1}$ corresponds to $\sigma_u = E(u_k^2) = 0$ and $w = \sigma_{yy}\sigma_{xy}^{-1}$ corresponds to $\sigma_v = E(v_k^2) = 0$. Interior values of w correspond to $\sigma_u\sigma_v > 0$, since one can show that

$$\begin{aligned}\sigma_u &= \sigma_{xx} - \frac{\sigma_{xy}}{w}\\ \sigma_v &= \sigma_{yy} - w\sigma_{xy}.\end{aligned} \tag{2.4}$$

Of course, if $\sigma_u = 0$ is part of the *a priori* information, the conclusion that $w = \sigma_{xy}\sigma_{xx}^{-1}$ is immediate and well known. We also note that the physical origins of σ_u may rest in the properties of some physical instrument, for which one may know certain bounds on σ_u. These bounds will serve to pin down the interval of admissible w more precisely.

Let us now turn to the dynamic problem described in Section 1. First, it would clearly be reasonable to demand that $\{x_k^*\}$, $\{u_k\}$ and $\{v_k\}$ are regular processes, so that each possesses a bounded spectral density $\sigma_{x^*x^*}(\omega)$, $\sigma_{uu}(\omega)$, $\omega \in [-\pi, \pi]$, either identically zero, or zero only on a set of measure zero and meeting the Paley–Weiner condition, viz:

$$\int_{-\pi}^{+\pi} ln\sigma_{x^*x^*}(\omega)\mathrm{d}\omega > -\infty. \tag{2.5}$$

We shall however assume a little more: the x_k^* process is generated by

$$x_k^* = \sum_{-\infty}^{k} w_{k-l}^x \varepsilon_l \tag{2.6}$$

where $\{\varepsilon_k\}$ is a zero mean, stationary, white noise sequence and $\{w_k^x, k \geq 0\}$ is a causal impulse response satisfying

$$\sum_{k=0}^{\infty} \rho^k |w_k^x| < \infty \quad \text{for some } \rho > 1. \tag{2.7}$$

We assume also that the impulse response $\{w_k, k \geq 0\}$ satisfies the strengthened stability requirement

$$\sum_{k=0}^{\infty} \rho^k |w_k| < \infty \quad \text{for some } \rho > 1. \tag{2.8}$$

The assumptions in the paragraph above, together with the assumptions that $\{x_k^*\}$, $\{u_k\}$ and $\{v_k\}$ are independent, zero mean processes of which measurements are available according to (1.2), will be termed the *standing assumptions*. The power spectrum matrix of $[x \quad y]'$ will be termed the *standard data*.

Suppose that $W(z)$ is the transfer function associated with the sequence $\{w_k\}$, defined by

$$W(z) = \sum_0^{\infty} w_k z^k. \tag{2.9}$$

Here, z is the backward shift operator, and the standing assumptions ensure that $W(z)$ is analytic inside $|z| < \rho$; in particular then, $W(z)$ will have no poles in $|z| \leq 1$, and can only have a finite number of zeros there. Now it is easily checked that

$$\begin{aligned}\sum(\omega) &= \begin{bmatrix}\sigma_{xx}(\omega) & \sigma_{xy}(\omega)\\ \sigma_{yx}(\omega) & \sigma_{yy}(\omega)\end{bmatrix}\\ &= \begin{bmatrix}\sigma_{x^*x^*}(\omega) + \sigma_{uu}(\omega) & W(e^{j\omega})\sigma_{x^*x^*}(\omega)\\ W(\mathrm{e}^{-j\omega})\sigma_{x^*x^*}(\omega) & |W(e^{j\omega})|^2\sigma_{x^*x^*}(\omega) + \sigma_{vv}(\omega)\end{bmatrix}.\end{aligned} \tag{2.10}$$

Our problem is to define, to the fullest extent possible, what W(z) *is, under the standing assumptions and given the standard data* (2.10).

We have first, see Anderson and Deistler (1984), *Lemma* 2.1. Under the standing assumptions, and with the standard data

$$\frac{\sigma_{xy}(\omega)}{\sigma_{xx}(\omega)} \leq W(e^{j\omega}) \leq \frac{\sigma_{yy}(\omega)}{\sigma_{yx}(\omega)} \tag{2.11}$$

where $a \leq b$ for complex a, b means $\arg a = \arg b$, $|a| \leq |b|$.

Lemma 2.1 is easily proved. [We remark that in case ω is a zero of $\sigma_{xx}(\omega)$ of multiplicity v, one can show that it is an isolated zero, and is also a zero of $\sigma_{xy}(\omega)$ with multiplicity at least as great as v. Hence the left member of the inequality can be defined at zeros of $\sigma_{xx}(\omega)$ by a limiting process.] What is of more interest from the point of view of the identification is the following almost-converse of Lemma 2.1.

Lemma 2.2. Adopt the standing assumptions, and assume available the standard data. Suppose that $\hat{W}(z)$ is analytic in $|z| < \rho$ for some $\rho > 1$ and satisfies

$$\frac{\sigma_{xy}(\omega)}{\sigma_{xx}(\omega)} \leq \hat{W}(e^{j\omega}) \leq \frac{\sigma_{yy}(\omega)}{\sigma_{yx}(\omega)}. \qquad (2.12)$$

Then there are computable $\hat{\sigma}_{x^*x^*}(\omega)$, $\hat{\sigma}_{uu}(\omega)$, $\hat{\sigma}_{vv}(\omega)$, all nonnegative and bounded, such that

$$\sum(\omega) = \begin{bmatrix} \hat{\sigma}_{x^*x^*}(\omega) + \hat{\sigma}_{uu}(\omega) & \hat{W}(e^{j\omega})\hat{\sigma}_{x^*x^*}(\omega) \\ \hat{W}(e^{-j\omega})\hat{\sigma}_{x^*x^*}(\omega) & |W(\hat{e}^{j\omega})|^2\sigma_{\hat{x}^*x^*}(\omega) + \sigma_{\hat{v}v}(\omega) \end{bmatrix}. \qquad (2.13)$$

Proof. Define

$$\hat{\sigma}_{x^*x^*}(\omega) = \frac{|\sigma_{xy}(\omega)|}{|\hat{W}(e^{j\omega})|}$$

$$\hat{\sigma}_{uu}(\omega) = \sigma_{xx}(\omega) - \frac{|\sigma_{xy}(\omega)|}{|\hat{W}(e^{j\omega})|} \qquad (2.14)$$

$$\hat{\sigma}_{vv}(\omega) = \sigma_{yy}(\omega) - |\sigma_{xy}(\omega)|\,|\hat{W}(ej\omega)|.$$

It is trivial to verify that $\hat{\sigma}_{x^*x^*}(\omega)$, $\hat{\sigma}_{uu}(\omega)$ and $\hat{\sigma}_{vv}(\omega)$ are nonnegative and bounded, and that (2.13) holds.

The idea of the Lemma is that if $\hat{W}(e^{j\omega})$ "solves" (2.12), then a set-up of the form of Fig. 1 with $\hat{W}(z)$ as the transfer function associated with the x^* to y^* mapping could have generated the standard data. Note that it is not being asserted that the full range of standing assumptions is necessarily met.

Our focus in the next section is on the task of solving (2.12). Let us note here that any solution of (2.12) necessarily has the property that $\hat{W}(e^{j\omega}) = \hat{W}^*(e^{-j\omega})$, and thus corresponds to a real (not necessarily causal) impulse response.

3. CONDITIONS FOR CAUSALITY

In this section, we are interested in knowing to what extent a requirement of causality (and stability) of the operator $\{w_k\}$ coheres with the problem data. The main result is simple, and is as follows:

Theorem 3.1. Let the standing assumptions hold and assume available the standard data. Suppose further that $\sigma_{xy}(\omega) \neq 0$ for any ω. Then the change in argument of $\sigma_{xy}(\omega)$ moving from $\omega = 0$ to 2π is $2\pi N$ for some integer $N \geq 0$. The number of zeros of $\hat{W}(z)$ in any solution of (2.12) satisfying the standing assumptions located in $|z| \leq 1$ is N.

Proof. The fact that $W(e^{j\omega})\sigma_{x^*x^*}(\omega) = \sigma_{xy}(\omega) \neq 0$ for any real ω [see (2.10)] while $W(z)$ is free of poles on $|z| = 1$ guarantees that $W(e^{j\omega}) \neq 0$ for any real ω. Because $W(z)$ is analytic in $|z| \leq 1$, it has a finite number of zeros there, N say. The principle of the argument then guarantees that the change in argument of $\sigma_{xy}(\omega)$ moving from $\omega = 0$ to 2π is $2\pi N$. The second part of the theorem is immediate.

The assumption that $\sigma_{xy}(\omega) \neq 0$ in the theorem statement allows ready application of the principle of the argument. Let us now observe the variation possible when $\sigma_{xy}(\omega) = 0$ for some ω, ω_1 say.

Now the standing assumptions guarantee that $\sigma_{xy}(\omega)$ is the evaluation on the unit circle of a function analytic in an annulus containing $|z| = 1$ in its interior. Hence either $\sigma_{xy}(\omega) \equiv 0$, or ω_1 is an isolated zero of finite order. The former case is trivial. So assume that ω_1 has order m as a zero of $\sigma_{xy}(\omega)$. Given that $\sigma_{xy}(\omega) = W(e^{j\omega})\sigma_{x^*x^*}(\omega)$ and that $\sigma_{x^*x^*}(\omega) \geq 0$ for all real ω, it is clear that, *a priori*, for any integer r with $2r \leq m$ we could have $\sigma_{x^*x^*}(\omega)$ possessing a zero of order $2r$ and $W(e^{j\omega})$ possessing a zero of order $m - 2r$ at ω_1. There are $\lceil m/2 \rceil$ ways the mth-order zero can split between $W(e^{j\omega})$ and $\sigma_{x^*x^*}(\omega)$. Now consider the change in arg $W(z)$ in moving counter-clockwise around the contour of Fig. 2, which consists of the unit circle except for a small indentation of radius ε, with the indentation approximately semicircular and becoming exactly so as $\varepsilon \to 0$. Then, assuming there are no other zeros on the unit circle of $\sigma_{xy}(\omega)$,

[Change in $\arg\sigma_{xy}(\omega)$ from $\omega_1 + \varepsilon$ to $\omega_1 - \varepsilon$ moving counter-clockwise] $- \pi(m - 2r) =$ [Change in $\arg W(e^{j\omega})$ moving around the closed contour] $= 2\pi N$. (3.1)

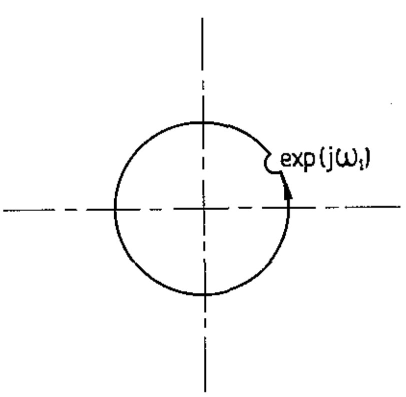

FIG. 2. Contour used for proof of Lemma 4.1.

In seeking solutions of the inequality (2.12) one can consider the separate problem of solving

$$\frac{\sigma_{xy}(\omega)}{\sigma_{xx}(\omega)(e^{j\omega}-e^{j\omega_1})^{m-2r}} \leq \tilde{W}(e^{j\omega}) \leq \frac{\sigma_{yy}(\omega)}{\sigma_{yx}(\omega)(e^{j\omega}-e^{j\omega_1})^{m-2r}} \tag{3.2}$$

for each r with $2r \leq m$, and with the additional constraint that $\arg \tilde{W}(e^{j\omega})$ is continuous at $\omega = \omega_1$, since $\tilde{W}(e^{j\omega})$ is not zero there. Then $\arg \tilde{W}(e^{j\omega})$ is known for all $\omega \in [0, 2\pi]$ and the problem is equivalent to one when there is no unit circle zero.

When there is more than one zero of $\sigma_{xy}(\omega)$ on the unit circle, obvious modifications are made to the above calculations; there is a potential increase in the number of special cases.

Henceforth in the interests of simplicity, we shall avoid all consideration of such special cases, by simply assuming that $\sigma_{xy}(\omega) \neq 0$ for all ω. This ensures that $\hat{W}(e^{j\omega}) \neq 0$ for any real ω also.

Evidently, the information in (2.12) coupled with an assumption that $\hat{W}(e^{j\omega})$ is causal determines the number of zeros of $\hat{W}(e^{j\omega})$ inside the unit circle. In the following sections, we shall show how one can check for the existence of causal solutions to (2.12) with a specified number of zeros inside the unit circle, and construct such solutions. The simplest case is that of minimum phase $\hat{W}(z)$—those where there are no zeros inside the unit circle. Such $\hat{W}(z)$ have the additional property that $\hat{W}^{-1}(z)$ is causal, and serve to put the x^*, y^* processes on the same footing: one can no more regard x^* as "causing" y^* by passage through a causal $\hat{W}(z)$ than one can regard y^* by passage through a causal $\hat{W}^{-1}(z)$.

The information that $N > 0$ on the other hand precludes the possibility of the standard data having arisen from a situation where the roles of $\{y_k^*\}$ and $\{x_k^*\}$ are reversed, but otherwise the standing assumptions are in force. For if $\{y_k^*\}$ served as input to a causal system with transfer function $L(z)$ analytic inside $|z| < \rho$ for some $\rho > 1$, the analog of (2.12) is

$$\frac{\sigma_{yx}(\omega)}{\sigma_{yy}(\omega)} \leq L(e^{j\omega}) \leq \frac{\sigma_{xx}(\omega)}{\sigma_{xy}(\omega)}. \tag{3.3}$$

Since $L(z)$ is analytic in $|z| < \rho$, the number of zeros of $L(z)$ inside $|z| = 1$ is $(1/2\pi)$ times the change in argument of $\sigma_{yx}(\omega)$ moving from 0 to 2π, or $(-1/2\pi)$ times the change in argument of $\sigma_{xy}(\omega)$ moving from 0 to 2π, i.e. $-N$. With $N > 0$, this is an impossibility.

Thus the standard data allows one always to exclude two of the following three alternatives: $\{y_k^*\}$ is generated by passing $\{x_k^*\}$ into a minimum phase causal system (equivalently, $\{x_k^*\}$ is generated by passing $\{y_k^*\}$ into a minimum phase causal system); $\{y_k^*\}$ is generated by passing $\{x_k^*\}$ into a nonminimum phase causal system; and $\{x_k^*\}$ is generated by passing $\{y_k^*\}$ into a nonminimum phase causal system. Of course, the possibility is not excluded by the standard data itself that the standard data could actually be generated by passing some $\{z_k^*\}$ into two linear causal systems with outputs $\{x_k^*\}$ and $\{y_k^*\}$.

An arrangement where, in loose notation

$$x_k^* = W_1(e^{j\omega})z_k^* \qquad y_k^* = W_2(e^{j\omega})z_k^* \tag{3.4}$$

with $W_1(e^{j\omega})$ and $W_2(e^{j\omega})$ both causal is, formally, indistinguishable from one where

$$y_k^* = W_2(e^{j\omega})W_1^{-1}(e^{j\omega})x_k^* \tag{3.5}$$

or

$$x_k^* = W_1(e^{j\omega})W_2^{-1}(e^{j\omega})y_k^*. \tag{3.6}$$

If (3.4) actually represents the true situation, there is no way this fact can be deduced from the standard data, let alone W_1, W_2 found, irrespective of the causality of $W_2W_1^{-1}$ or $W_1W_2^{-1}$.

4. MINIMUM PHASE SOLUTION

In order to study the possibility of finding a minimum phase $\hat{W}(z)$, we shall use the following result relating the real part on the unit circle of a function analytic inside the unit circle to the imaginary part of that function. The result with unit circle replaced by left half plane is an old one, see e.g. Bode (1945). The proof of the result may be found in the Appendix.

Lemma 4.1. Let $T(z)$ be analytic in $|z| \leq 1$, and denote by $R(e^{j\omega})$ and $I(e^{j\omega})$ the real and imaginary part of $T(e^{j\omega})$, for ω real:

$$T(e^{j\omega}) = R(e^{j\omega}) + jI(e^{j\omega}). \tag{4.1}$$

Suppose that R and I are even and odd functions of ω. Then

$$R(e^{j\omega_1}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} R(e^{j\omega})d\omega + \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{I(e^{j\omega_1})\sin\omega_1 - I(e^{j\omega})\sin\omega}{\cos\omega - \cos\omega_1} d\omega. \tag{4.2}$$

Note that the second integrand is well behaved at $\omega = \omega_1$. Note also that the formula only allows recovery of the real part of $T(z)$ from the imaginary part to within an additive constant, a fact which is entirely in accord with intuition.

Now consider the task of finding a $\hat{W}(e^{j\omega})$ to satisfy (2.12), given the standard data, and given that the standing assumption holds as well as $\sigma_{xy}(\omega) \neq 0$. Suppose further that $\hat{W}(z)$ is known to be minimum phase, as a result of studying $\arg \sigma_{xy}(\omega)$. Then define

$$T(z) = \ln \hat{W}(z). \tag{4.3}$$

Observe that $T(z)$ is analytic in $|z| \leq 1$, and we can use Lemma 4.1. Now (2.12) yields

$$I(e^{j\omega}) = \arg \sigma_{xy}(\omega) \tag{4.4}$$

and then (4.2) can be used to recover $R(e^{j\omega})$ to within an additive constant. Since $R(e^{j\omega}) = \ln|\hat{W}(e^{j\omega})|$, this means that from $\arg \hat{W}(e^{j\omega})$, we can recover the amplitude response of $\hat{W}(e^{j\omega})$ to within a multiplicative constant, i.e.

$$\hat{W}(e^{j\omega}) = \mu \bar{W}(e^{j\omega}) \tag{4.5}$$

for some fixed $\bar{W}(e^{j\omega})$; now in order to meet the amplitude constraints implicit in μ, we must have $\mu \in [\mu_{\min}, \mu_{\max}]$ when

$$\mu_{\min} = \max_{\omega} \left| \frac{\sigma_{xy}(\omega)}{\sigma_{xx}(\omega)\bar{W}(e^{j\omega})} \right|$$
$$\mu_{\max} = \min_{\omega} \left| \frac{\sigma_{yy}(\omega)}{\sigma_{yx}(\omega)\bar{W}(e^{j\omega})} \right|. \tag{4.6}$$

In summary we have proved the following result.

Theorem 4.2. Assume available the standard data, and suppose all standing assumptions hold, together with $\sigma_{xy}(\omega) \neq 0$ for all ω. Suppose there exists a causal solution $\bar{W}(e^{j\omega})$ of (2.12) and that the change in argument of $\sigma_{xy}(\omega)$ over $[0, 2\pi]$ is zero. Then solutions of (2.12) are minimum phase and are of the form

$$\hat{W}(e^{j\omega}) = \mu \bar{W}(e^{j\omega}) \tag{4.5}$$

where $\bar{W}(e^{j\omega})$ is computable to within a scaling constant, and $\mu \in [\mu_{\min}, \mu_{\max}]$, with the quantities $\mu_{\min}$ and $\mu_{\max}$ defined in (4.6).

Not only is this result striking in its similarity to that applying for the static (white noise) case, but the static case can be recovered from it, in the following way. Adopting the standing assumptions, suppose that the standard data Σ is a constant matrix (this would be so if Σ arises from a static problem). Then $\arg \sigma_{xy}(\omega) = 0$ for all ω, and so Theorem 3.1 guarantees that any solution $\hat{W}(z)$ of (2.12) is minimum phase. Use of the procedure of Theorem 4.2 then establishes that $\hat{W}(z)$ *must* be constant, and the bounds $[\mu_{\min}, \mu_{\max}]$ coincide with those defined for ω in the static problem. Then even if there is no *a priori* assumption that the data arose from a static problem, if they are consistent with a static problem, *and* if the standing assumptions hold, then they necessarily come from a static problem.

The causality of $\hat{W}$ (which is part of the standing assumption) is crucial. Without it, it is easy to define a $\hat{W}(z)$, by

$$\hat{W}(e^{j\omega}) = \mu(\omega)\bar{W}(e^{j\omega})$$

where $\mu(\omega)$ is any real smooth function of ω satisfying

$$\mu_{\min} \leq \mu(\omega) \leq \mu_{\max}.$$

Last, we note that if the spectrum matrix $\Sigma(\omega)$ of the standing data is rational, then $\hat{W}(e^{j\omega})$ is also rational.

5. CAUSAL SOLUTION WITH A PRESCRIBED NUMBER OF UNIT CIRCLE ZEROS

In the previous section, we showed that the class of minimum phase solutions was in general a one-parameter family, with the parameter being a scaling constant. Here we shall show that the class of solutions $\hat{W}(z)$ where $\hat{W}(z)$ has no poles in $|z| \leq 1$ and N zeros in $|z| < 1$ is a $(1 + N)$-parameter family.

Consider a class of real rational functions $U_A(z)$ parametized by a set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ where the α_i are complex $|\alpha_i| < 1$, and nonreal α_i occur in complex conjugate pairs. Then $U_A(z)$ is defined by

$$U_A(z) = \frac{\prod_{i=1}^{N} (z - \alpha_i)}{\prod_{i=1}^{N} (\alpha_i z - 1)}. \tag{5.1}$$

Notice that on the unit circle, $|z| = 1$, there holds $|U_\alpha(z)| = 1$; also $U_\alpha(z)$ is causal and real rational.

We can now state the following counterpart to Theorem 4.1. A constructive procedure for generating the solutions defined in the theorem statement can be found in the proof of the theorem.

Theorem 5.1. Assume available the standard data, and suppose all standing assumptions hold, together with $\sigma_{xy}(\omega) \neq 0$ for all ω. Suppose there exists a causal solution to (2.12) and that the change in argument of $\sigma_{xy}(\omega)$ as ω varies from 0 to 2π is $2\pi N$. Let A be a set $\{\alpha_i, \alpha_2, \ldots, \alpha_N\}$, $\alpha_i \in C$, $|\alpha_i| < 1$, nonreal α_i occurring in complex conjugate pairs, and let $U_A(z)$ be defined as in (5.1). Then there exists a minimum phase $W_A(z)$ unique to within a scaling constant, such that all $W(z)$ which are causal, meet the phase constraint imposed by (2.12) and have precisely N zeros inside $|z| < 1$ at $z = \alpha_i, i = 1, \ldots, N$ can be described as

$$\hat{W}(z) = \mu U_A(z) \bar{W}_A(z) \quad (5.2)$$

where μ is an arbitrary scaling constraint. Further, the amplitude constraint imposed by (2.12) can be met if and only if the interval $[\mu_{\min}, \mu_{\max}]$ defined below is nonempty and it contains μ:

$$\mu_{\min} = \max_{\omega} \left| \frac{\sigma_{xy}(\omega)}{\sigma_{xx}(\omega) \bar{W}_A(e^{j\omega})} \right|$$
$$\mu_{\max} = \min_{\omega} \left| \frac{\sigma_{yy}(\omega)}{\sigma_{yx}(\omega) \bar{W}_A(e^{j\omega})} \right|. \quad (5.3)$$

Proof. If $\hat{W}(z)$ is causal and has precisely N zeros inside the unit circle at $\alpha_1, \ldots, \alpha_N$, we can certainly write

$$\hat{W}(z) = U_A(z) V(z) \quad (5.4)$$

for some minimum phase $V(z)$. If $\arg \hat{W}(e^{j\omega})$ is known, $\arg V(e^{j\omega})$ is known (and depends on A), even if $|\hat{W}(e^{j\omega})|$ is unknown. Then one can use the constructive procedure of Theorem 4.2 based on Lemma 4.1, to obtain $|V(e^{j\omega})|$ from $\arg V(e^{j\omega})$ to within an arbitrary multiplicative constant. This leads to a description of $\hat{W}(z)$ in the form of (5.2), where μ is an arbitrary constant and $\bar{W}_A(z)$ is a (real) minimum phase transfer function. At this stage, $\hat{W}(z)$ is guaranteed to meet the phase restriction of (2.12). Choice of $\mu \in [\mu_{\min}, \mu_{\max}]$ then ensures that the amplitude restrictions are also met.

We remark that for arbitrary A, there is no guarantee that $\mu_{\min} \leq \mu_{\max}$, and thus no guarantee that a $\hat{W}(z)$ can be found with zeros at $\alpha_1, \ldots, \alpha_N$. If there is a true $W(z)$ producing standard data (and there is when the standing assumptions are in force), then naturally there is one set A for which $\mu_{\min} \leq \mu_{\max}$. Let us observe that if $\mu_{A\min} < \mu_{A\max}$ there will be sets $\hat{A} = \{\hat{\alpha}_1, \ldots, \hat{\alpha}_N\}$ where $\max_i |\alpha_i - \hat{\alpha}_i|$ is small such that $\mu_{A\min} < \mu_{A\max}$. For suppose

$$W(z) = \mu \frac{\Pi(z - \alpha_i)}{\Pi(\alpha_i z - 1)} \bar{W}_A(z). \quad (5.5)$$

It is not hard to see that if

$$U_A(z) = \Pi \frac{(z - \alpha_i)}{(\alpha_i z - 1)} \qquad U_{\hat{A}}(z) = \frac{\Pi(z - \hat{\alpha}_i)}{\Pi(\hat{\alpha}_i z - 1)} \quad (5.6)$$

then the arguments of $U_A^{-1}(z)W(z)$ and $U_{\hat{A}}^{-1}(z)W(z)$ have a difference, the maximum value of which around the unit circle depends continuously on $\max_i |\alpha_i - \hat{\alpha}_i|$. In fact, as is easily seen, the maximum value of the derivative of the difference in arguments also depends continuously on $\max_i |\alpha_i - \hat{\alpha}_i|$. By an argument set out in greater detail below, this allows us to conclude that the minimum phase transfer functions with the two argument characteristics and with gain adjusted to be 1 at $\omega = 0$ have gains such that the difference in the logarithm of the gains has a maximum value on the unit circle depending continuously on $\max_i |\alpha_i - \hat{\alpha}_i|$. It follows that $|\mu_{\hat{A}\max} - \mu_{A\max}| \to 0$ and $|\mu_{\hat{A}\min} - \mu_{A\min}| \to 0$ as $\max_i |\alpha_i - \hat{\alpha}_i| \to 0$, and then that for A near $\hat{A}$, $\mu_{\hat{A}\min} < \mu_{\hat{A}\max}$.

To explain the introduction of the derivative of the phase, let us note first that (4.2) does *not* allow the drawing of the conclusion that a small $L_\infty[0, 2\pi]$ change in $I(e^{j\omega})$ produces a small L_∞ change in $R(e^{j\omega})$, assuming the latter is normalized so that, for example $R(1) = 0$. The underlying explanation is that

$$\int_{-\pi}^{+\pi} \frac{d\omega}{|\cos\omega - \cos\omega_1|} < \infty \quad (5.7)$$

is not true. Now straightforward calculation shows that for $0 \leq \omega < \omega_1 \leq \pi$ and $0 \leq \omega_1 < \omega \leq \pi$.

$$\frac{d}{d\omega} \ln \left| \frac{\sin\left(\frac{\omega_1 + \omega}{2}\right)}{\sin\left(\frac{\omega_1 - \omega}{2}\right)} \right| = \frac{\sin\omega_1}{\cos\omega - \cos\omega_1} \quad (5.8)$$

and an integration by parts of (4.2) yields (after careful attention to limits)

$$R_0(e^{j\omega_1}) \triangleq \int_{-\pi}^{\pi} \frac{I(e^{j\omega_1})\sin\omega_1 - I(e^{j\omega})\sin\omega}{\cos\omega - \cos\omega_1} d\omega$$
$$= \frac{2}{\sin\omega_1} \int_0^{\pi} \frac{d}{d\omega} [I(e^{j\omega})\sin\omega] \ln \left| \frac{\sin\left(\frac{\omega_1 + \omega}{2}\right)}{\sin\left(\frac{\omega_1 - \omega}{2}\right)} \right| d\omega. \quad (5.9)$$

Some algebra will show also that

$$\int_0^{\pi} \frac{1}{\sin\omega_1} \left| \left[\ln \left| \frac{\sin\left(\frac{\omega_1 + \omega}{2}\right)}{\sin\left(\frac{\omega_1 - \omega}{2}\right)} \right| \right] \right| d\omega$$

is bounded for all $\omega_1 \in [0, 2\pi]$. This means that a small L_∞ variation in $\frac{d}{d\omega}[I(e^{j\omega})\sin\omega]$ produces a small L_∞ variation in $R_0(e^{j\omega})$.

As for the minimum phase case, one can check that if the spectrum matrix $\Sigma(\omega)$ of the standing data is rational, then, $\hat{W}(e^{j\omega})$ is also rational.

6. MULTIVARIABLE PROBLEMS

It is natural to pose a multivariable version of the problem considered previously. In the first instance, one can suppose that $\{x_k^*\}$, $\{u_k\}$ and $\{v_k\}$ are vector processes of the same dimension. The white noise case can be analysed fairly easily. Assume that

$$y_k^* = Wx_k^* \tag{6.1a}$$

$$x_k = x_k^* + u_k \tag{6.1b}$$

$$y_k = y_k^* + v_k \tag{6.1c}$$

where W is a real matrix to be identified, and $\{x_k^*\}$, $\{u_k\}$ and $\{v_k\}$ are independent zero mean white noise processes, and that the following matrix is known:

$$\sum = \begin{bmatrix} \sum_{xx} & \sum_{xy} \\ \sum_{yx} & \sum_{yy} \end{bmatrix} = E\left\{\begin{bmatrix} x_k \\ y_k \end{bmatrix} [x_k' \quad y_k']\right\}. \tag{6.2}$$

Suppose that

$$\sum_{xx} > 0. \tag{6.3}$$

It follows that the set of W consistent with Σ is defined by

$$W = \sum_{yx}\left(\sum_{xx} - A\right)^{-1} \tag{6.4}$$

for any symmetric A satisfying

$$\sum_{xx} - \sum_{xy}\sum_{yy}^{\#}\sum_{yx} \geq A \geq 0 \tag{6.5}$$

(consistent with W being finite). Here, $\sum_{yy}^{\#}$ denotes the Moore–Penrose pseudo-inverse.

Now when one allows nonwhite spectra, the formula (6.4) still remains valid where $\sum$ and A become nonnegative definite (or commonly positive definite) Hermitian matrices that are functions of ω, and $W = W(e^{j\omega})$.

A different procedure than that described in this paper is needed to obtain the solutions W of (6.4) and (6.5). The procedure of this paper is unworkable at more than one step; in the matrix case, the phase of W is not known; even if one knew the phase of every entry of W, one cannot, if W is known to be minimum phase, construct the entries of each entry of W using the procedure of Section 4, since individual entries of W will not necessarily be minimum phase. As it turns out, there is an analog with the task of spectral factorization, see for example Rozanov (1967). The first algorithm advanced to handle the spectral factorization problem was not capable of extension to the matrix case, and a second algorithm had to be found. This second spectral factorization algorithm as it turns out suggests a procedure for solving (6.4) and (6.5) when a minimum phase $W(z)$ exists; whether or not it extends to more general $W(z)$ is under examination.

7. CONCLUSION

By hypothesizing causality of a transfer function appearing in a dynamic errors-in-variable model and making certain other reasonable assumptions, it proves possible to parametrize the class of transfer functions consistent with the available data in a finite-dimensional way. Of particular interest are minimum phase transfer functions, where the parameter is a scalar.

In contrast to the approach in this paper, the ideas of Deistler (1984) are aimed at identifying situations where the transfer functions can be obtained uniquely. Broadly speaking, this is done by postulating rational data, and then imposing some sort of minimality of degree assumptions and genericity assumptions that narrow the class of transfer functions down to a single one. Close examination of the results of Anderson and Deistler (1984) illustrates that generally, nonminimum phase transfer functions are harder to identify, in the sense that more assumptions are needed to ensure uniqueness, than minimum phase transfer functions. It would be interesting to consider the situation where the data was not available within a certain frequency range, or known to be inaccurate, on account perhaps of sampling error. At the very least, this would lead to a study of the robustness of the procedure presented.

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APPENDIX: PROOF OF LEMMA 5.1

Let $C_{+\varepsilon}$ denote the closed oriented contour shown in Fig. 2. The indentation into the unit circle as $\varepsilon \to 0$ becomes semi-circular, with radius ε. Let $c_{+\varepsilon}$ denote the same contour as $C_{+\varepsilon}$ except that the semi-circular indentation is excluded, and let $\gamma_{+\varepsilon}$ denote the semi-circular indentation. Then

$$\int_{C_{+\varepsilon}} \frac{T(z) - jI(e^{j\omega_1})}{z - e^{j\omega_1}} \mathrm{d}z = 0$$

whence

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{c_{+\varepsilon}} \frac{T(z) - jI(e^{j\omega_1})}{z - e^{j\omega_1}} \mathrm{d}z &= -\lim_{\varepsilon\to 0} \int_{\gamma_{+\varepsilon}} \frac{T(z) - jI(e^{j\omega_1})}{z - e^{j\omega_1}} \mathrm{d}z \\ &= -\lim_{\varepsilon\to 0} \int_{\gamma_{+\varepsilon}} \frac{R(e^{j\omega_1})}{z - e^{j\omega_1}} \mathrm{d}z \\ &= j\pi R(e^{j\omega_1}). \qquad \text{(A.1)}\end{aligned}$$

With $C_{-\varepsilon}$, $c_{-\varepsilon}$ and $\gamma_{-\varepsilon}$ denoting corresponding contours where $j\omega_1$ is replaced by $-j\omega_1$, we have also

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{c_{-\varepsilon}} \frac{T(z) - jI(e^{-j\omega_1})}{z - e^{-j\omega_1}} \mathrm{d}z &= j\pi R(e^{-j\omega_1}) \\ &= j\pi R(e^{j\omega_1}). \qquad \text{(A.2)}\end{aligned}$$

Now observe also that

$$\begin{aligned}&\int_{c_{+\varepsilon}\cap c_{-\varepsilon}} R(z)\left[\frac{1}{z - e^{-j\omega_1}} + \frac{1}{z - e^{j\omega_1}}\right]\mathrm{d}z \\ &= \int_{-\pi}^{-\pi-\omega_1-\varepsilon} + \int_{-\omega_1+\varepsilon}^{\omega_1-\varepsilon} \\ &\quad + \int_{\omega_1+\varepsilon}^{\pi} \left\{R(e^{j\omega})\left[\frac{1}{e^{j\omega} - e^{j\omega_1}} + \frac{1}{e^{j\omega} - e^{-j\omega_1}}\right] je^{j\omega}\mathrm{d}\omega\right\} \\ &= j\int_{I_\varepsilon} R(e^{j\omega}) \frac{2(e^{j\omega} - \cos\omega_1)}{e^{2j\omega} - 2e^{j\omega}\cos\omega_1 + 1} e^{j\omega}\mathrm{d}\omega\end{aligned}$$

$I_\varepsilon = [-\pi, -\pi - \omega_1 - \varepsilon] \cup [-\omega_1 + \varepsilon, \omega_1 - \varepsilon] \cup [\omega_1 + \varepsilon, \pi]$

$$\begin{aligned}&= j\int_{I_\varepsilon} R(e^{j\omega}) \frac{e^{j\omega} - \cos\omega_1}{\cos\omega - \cos\omega_1}\mathrm{d}\omega \\ &= j\int_{I_\varepsilon} R(e^{j\omega})\mathrm{d}\omega \qquad \text{using the evenness of } R(e^{j\omega}). \qquad \text{(A.3)}\end{aligned}$$

Next,

$$\begin{aligned}&\int_{C_{+\varepsilon}\cap C_{-\varepsilon}} jI(z)\left[\frac{1}{z - e^{-j\omega_1}} + \frac{1}{z - e^{j\omega_1}}\right]\mathrm{d}z \\ &= -\int_{I_\varepsilon} I(e^{j\omega}) \frac{e^{j\omega} - \cos\omega_1}{\cos\omega - \cos\omega_1}\mathrm{d}\omega \\ &= -j\int_{I_\varepsilon} I(e^{j\omega}) \frac{\sin\omega}{\cos\omega - \cos\omega_1}\mathrm{d}\omega\end{aligned}$$

using the oddness of $I(e^{j\omega})$. (A.4)

Also

$$\begin{aligned}&\int_{C_{+\varepsilon}\cap C_{-\varepsilon}} \left[\frac{-jI(e^{j\omega_1})}{z - e^{j\omega_1}} - \frac{jI(e^{-j\omega_1})}{z + e^{-j\omega_1}}\right]\mathrm{d}z \\ &= \int_{I_\varepsilon} I(e^{j\omega_1})\left[\frac{1}{e^{j\omega} - e^{j\omega_1}} - \frac{1}{e^{j\omega} - e^{-j\omega_1}}\right]e^{j\omega}\mathrm{d}\omega \\ &= \int_{I_\varepsilon} I(e^{j\omega_1}) \frac{e^{j\omega_1} - e^{-j\omega_1}}{e^{j\omega} - 2\cos\omega_1 + e^{-j\omega}}\mathrm{d}\omega \\ &= j\int_{I_\varepsilon} I(e^{j\omega_1}) \frac{\sin\omega_1}{\cos\omega - \cos\omega_1}\mathrm{d}\omega. \qquad \text{(A.5)}\end{aligned}$$

Now add (A.1) and (A.2) to obtain

$$2j\pi R(e^{j\omega_1}) = \lim_{\varepsilon\to 0} \int_{C_{+\varepsilon}\cap C_{-\varepsilon}} \left[\frac{T(z) - jI(e^{j\omega_1})}{z - e^{j\omega_1}} + \frac{T(z) - jI(e^{-j\omega_1})}{z - e^{-j\omega_1}}\right]\mathrm{d}z$$

and using (A.3), (A.4) and (A.5), we have

$$2j\pi R(e^{j\omega_1}) = \lim_{\varepsilon\to 0} j\int_{I_\varepsilon} R(e^{j\omega})\mathrm{d}\omega + \lim_{\varepsilon\to 0} j\int_{I_\varepsilon} \frac{I(e^{j\omega_1})\sin\omega_1 - I(e^{j\omega})\sin\omega}{\cos\omega - \cos\omega_1}\mathrm{d}\omega.$$

The result of the lemma statement is then immediate.