Lumped approximation of distributed systems and controllability questions

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Indexing terms: *Mathematical techniques, Control theory*

Abstract: The approximation of a distributed system by a lumped system using both exact analysis and qualitative reasoning is studied in some detail for the lossless *LC* transmission line, and with much less detail for an *RC* line, a towed cable, and a deep-sea mining hauling pipe. We delimit, at least partly, the region of validity of the approximation, especially in analysing controllability questions. We also suggest why a form of nonuniform lumping may be more appropriate for the towed cable and the deep-sea mining hauling pipe.

1 Introduction

In Reference 1 the following observations are made. Suppose a uniform string under tension has a nonzero initial displacement of velocity, and that one end of the string can be shaken arbitrarily; then the string can be brought to rest in a time equal to twice the one-way propagation time of waves down the string, and, for generic initial conditions, it cannot be brought to rest in a shorter time. Suppose next that the string is approximated by a discrete lumped mass model. Controllability of the resulting linear state-variable equation can be verified, which permits the conclusion that the lumped mass model is controllable from an arbitrary initial state to the origin in an arbitrarily short time. Since the controllability property is evidently very different, the question arises: 'what goes wrong?'

In this paper, we answer this question. The short answer is that the lumped model is only a satisfactory approximation in a frequency band extending from zero (or DC) to an upper limit which is dependent on the dimension of the lumped model. If one attempts to control the lumped model using controls which are in some way frequency limited, then the delay behaviour of the distributed model is approximated, but otherwise the lumped model behaviour will not approximate the distributed model behaviour.

Section 2 discusses this question using a uniform lossless *LC* transmission line, rather than a string. The equations of both systems are essentially the same. (There is, however, electrical-engineering literature on approximations of lines; see, for example, References 2 and 3, which we discuss in Section 3.) The key is to compare transfer functions of the lumped and distributed systems, and the associated frequency-dependent group delays [see page 230 of Reference 4].

In Section 3, we discuss several aspects of these results. We regard the construction of the lumped model as involving spatial sampling of the line, and the creation in the process of a spatial aliasing frequency. We relate this to a temporal aliasing frequency and argue that calculations on the lumped model using excitations at frequencies above the aliasing frequency will yield misleading information about the distributed system. We also relate the ideas to constant-*k* filter theory [2, 3] and finally examine in more detail the controllability properties of the lumped model by using frequency-limited signals and by studying the behaviour of the controllability gramian.

In Section 4, we discuss other distributed systems: an *RC* line (which represents the heat equation), a towed cable [5, 6], and a deep-sea mine hauling pipe [7]. We argue why nonuniform spatial sampling should be advantageous, thus providing a theoretical justification for an observation of Reference 7.

2 Transfer function analysis of a lumped line approximation

Let us consider the arrangement of Fig. 1, in which there are present m inductors and m capacitors. This arrangement is an approximation of a uniform transmission line, excited at one end with a current generator, and terminated at the other end with a resistor. If the transmission line has inductance L and capacitance C per unit length, and is of length l, then

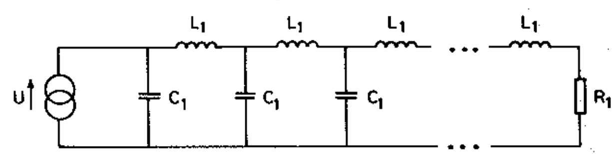

Fig. 1 *Lumped approximation to uniform LC line*

$$L_1 = \frac{lL}{m} \quad C_1 = \frac{lC}{m} \tag{1}$$

On the transmission line, waves travel with a velocity $(LC)^{-1/2}$, and the characteristic impedance of the line is $(L/C)^{1/2}$. If the resistive termination takes this value, no reflections of waves can occur at the resistor.

Our aim here will be to compare a transfer function associated with the lumped model of Fig. 1 with the corresponding transfer function for the transmission line.

We define a state-variable realisation as follows: the entries of the state vector in positions 1, 3, 5, ..., are the

Paper 3776D (C8), first received 1st May 1984 and in revised form 3rd January 1985

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capacitor voltages multiplied by $\sqrt{C_1}$ and the entries in positions 2, 4, 6, ..., are the inductor currents multiplied by $\sqrt{L_1}$. Then the state-variable equations are

$$\dot{x}=\begin{bmatrix} 0 & -\frac{1}{\sqrt{L_1C_1}} & 0 & & & \\ \frac{1}{\sqrt{L_1C_1}} & 0 & -\frac{1}{\sqrt{L_1C_1}} & & & \\ 0 & \frac{1}{\sqrt{L_1C_1}} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & -\frac{1}{\sqrt{L_1C_1}} \\ & & & & \frac{1}{\sqrt{L_1C_1}} & -\frac{R}{L_1} \end{bmatrix} x + \begin{bmatrix} \frac{1}{\sqrt{C_1}} \\ 0 \\ 0 \\ \\ \\ 0 \end{bmatrix} u \tag{2}$$

Suppose we are interested in the current flowing in the qth inductor from the left, namely $(1/\sqrt{L})x_p$, where $p = 2q$. Thus the output equation is

$$y=\left[0, \ldots, 0 \; \frac{1}{\sqrt{L_1}} \; 0, \ldots, 0\right]x \tag{3}$$

To simplify the notation, let

$$\alpha=\frac{1}{\sqrt{L_1C_1}}, \; \beta = R_1/L_1$$

Lemma 1: Let F, G, H define the realisation in the preceding text and let $\Delta_0 = 1$ and

$$\Delta_r(s) = \det\begin{bmatrix} s & \alpha & & & \\ -\alpha & s & & & \\ & & \ddots & & \\ & & & s & \alpha \\ & & & -\alpha & s+\beta \end{bmatrix} \tag{4}$$

where the matrix is of size $r \times r$ for $r \geqslant 1$. Then, with $n = 2m$,

$$W_l(s) = H'(sI-F)^{-1}G = \frac{\alpha^p \, \Delta_{(n-p)}(s)}{\Delta_n(s)} \tag{5}$$

(The subscript l is to emphasise the lumped character of the model)

Proof: By Cramer's rule, the $p-1$ entry of $(sI-F)^{-1}$ is the (l, p) cofactor of the matrix on the right in eqn. 4. It is easily checked that this is $\alpha^{p-1}\,\Delta_{n-p}$. The forms of H and G, the fact that $\det(sI-F)=\Delta_n$, and the definition of α then yield eqn. 5.

Lemma 2: For real ω, define $\theta(\omega)$ by $\cos\theta(\omega) = \sqrt{1-\omega^2/4\alpha^2}$, $\sin\theta(\omega) = \omega/2\alpha$ so that when $0 \leqslant \omega/2\alpha < 1$, $\theta(\omega)$ is in the range $[0, 2\pi)$. Then, for r even and $j=\sqrt{-1}$,

$$\alpha^{-r}\,\Delta_r(j\omega) = \frac{\cos(r+1)\theta(\omega)}{\cos\theta(\omega)} + j\frac{\beta}{\alpha}\frac{\sin r\theta(\omega)}{\cos\theta(\omega)}$$

Proof: A simple calculation based on eqn. 4 shows that

$$\Delta_n(s) = s\,\Delta_{n-1}(s) + \alpha^2\,\Delta_{n-2}(s)$$

Suppose that $u(s)$, $v(s)$ are the roots of $x^2 - sx - \alpha^2 = 0$. Then, for some $\gamma(s)$, $\delta(s)$,

$$\begin{aligned} \Delta_n(s) &= \gamma(s)u^n + \delta(s)v^n \\ &= \gamma(s)\left[\frac{s}{2}+\sqrt{\alpha^2+\frac{s^2}{4}}\right]^n + \delta(s)\left[\frac{s}{2}-\sqrt{\alpha^2+\frac{s^2}{4}}\right]^n \end{aligned}$$

Noting from eqn. 4 that $\Delta_1(s) = s+\beta$, $\Delta_2(s) = s^2+\beta s + \alpha^2$ leads to the identification

$$\gamma(s) = \frac{1}{2}\left[1+\frac{\frac{s}{2\alpha}+\beta/\alpha}{\sqrt{1+s^2/4\alpha^2}}\right]$$

$$\delta(s) = \frac{1}{2}\left[1-\frac{\frac{s}{2\alpha}+\frac{\beta}{\alpha}}{\sqrt{1+s^2/4\alpha^2}}\right]$$

Now set $s = j\omega$. There follows, using the definition of $\cos\theta(\omega)$ and $\sin\theta(\omega)$, and recalling that r is even,

$$\begin{aligned} \alpha^{-r}\,\Delta_r(j\omega) &= \gamma(j\omega)\exp[jr\theta(\omega)] + \delta(j\omega)\exp[-jr\theta(\omega)] \\ &= [\gamma(j\omega)+\delta(j\omega)]\cos r\theta(\omega) \\ &\quad + j[\gamma(j\omega)-\delta(j\omega)]\sin r\theta(\omega) \\ &= \cos r\theta(\omega) + j\,\frac{j\sin\theta(\omega)+\frac{\beta}{\alpha}}{\cos\theta(\omega)}\sin r\theta(\omega) \\ &= \frac{\cos(r+1)\theta(\omega)}{\cos\theta(\omega)} + j\frac{\beta}{\alpha}\frac{\sin r\theta(\omega)}{\cos\theta(\omega)} \end{aligned}$$

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Now recall that we are evaluating the transfer function of eqn. 5 in which n is even by construction, and p is even by assumption. We obtain

$$H'(j\omega I - F)^{-1}G = \frac{\cos(n-p+1)\theta(\omega) + j\frac{\beta}{\alpha}\sin(n-p)\theta(\omega)}{\cos(n+1)\theta(\omega) + j\frac{\beta}{\alpha}\sin n\theta(\omega)} \tag{6}$$

Now notice

Observation: R_1 is a matching resistance $\Leftrightarrow R_1 = \sqrt{L/C} \Leftrightarrow R_1 = \sqrt{L_1/C_1} \Leftrightarrow R_1/L_1 = \sqrt{1/L_1C_1} \Leftrightarrow \alpha = \beta$.

To get a feel for the difference between the lumped and discrete models, suppose temporarily that

$$\alpha = \beta \quad \text{(matching condition holds)} \tag{7}$$

Then the transfer function is

$$\begin{aligned} W_l(j\omega) &= H'(j\omega I - F)^{-1}G = \exp[-jp\theta(\omega)]K(j\omega) \\ &= \exp\left[-jp\sin^{-1}\frac{l\sqrt{LC}\omega}{2m}\right]K(j\omega) \end{aligned} \tag{8}$$

where

$$K(j\omega) = \frac{1 + \exp\left[-j(n-p+1)\theta\right] \times j[\sin(n-p)\theta - \sin(n-p+1)\theta]}{1 + \exp\left[-j(n+1)\theta\right] j[\sin n\theta - \sin(n+1)\theta]} \tag{9}$$

Let us compare this with the corresponding expression for the ideal line. A current input simply propagates along the line with velocity $(LC)^{-1/2}$. In the lumped model, we are considering an inductor effectively at distance $pl/2m$ along the line. So the transfer function will be

$$W_d(j\omega) = \exp\left[-j\frac{pl}{2m}\sqrt{LC}\omega\right] \tag{10}$$

(The subscript d emphasises the distributed character of the model)

Eqns. 8 and 10 show then that the lumped model will be a good approximation of the distributed model provided that we can write

$$p \sin^{-1}\frac{l\sqrt{LC}}{2m}\omega \simeq p\frac{l\sqrt{LC}}{2m}\omega \tag{11a}$$

and

$$K(j\omega) \simeq 1 \tag{11b}$$

Now observe that for $0 \leqslant y \leqslant \pi/2$

$$\begin{aligned} 0 \leqslant y - \sin y &= \frac{y^3}{3!} - \frac{y^5}{5!} + \cdots \\ &\leqslant \frac{y^3}{3!} \\ &\leqslant \frac{\pi^3}{48}(\sin y)^3 \quad \left(\text{since } \frac{2y}{\pi} \leqslant \sin y\right) \end{aligned}$$

whence for $0 \leqslant (l/2m)\sqrt{LC}\omega \leqslant \pi/2$,

$$0 \leqslant p\sin^{-1}\frac{\omega l}{2m}\sqrt{LC} - \frac{pl}{2m}\sqrt{LC}\omega \leqslant \frac{p\pi^3}{48}\left(\frac{\omega l}{2m}\sqrt{LC}\right)^3 \tag{12}$$

and a measure of the error involved in the approximation of eqn. 11a is available. Clearly, for

$$\omega \ll \frac{2m}{l\sqrt{LC}} \tag{13}$$

eqn. 11a holds in the sense that the error term $p\pi^3/48((\omega l/2m)\sqrt{LC})^3$ is small as a percentage of $(pl/2m)\sqrt{LC}\omega$. For a small absolute error, we must recognise the multiplying effect of p. Notice that the maximum value of p is m, and the right side of eqn. 12 will be guaranteed to be much less than 1 for any p provided that

$$\omega \ll \frac{2m^{2/3}}{l\sqrt{LC}} \tag{14}$$

Now eqn. 13 is the condition also that $\theta(\omega) \ll 1$, and when this is so, it is not hard to check using eqn. 9 that $K(j\omega) \simeq 1$, as required by eqn. 11b, for all n, p. *A fortiori*, eqn. 14 implies $K(j\omega) \simeq 1$.

Consequently eqn. 13 is the condition for the phase error between W_l and W_d to be small as a percentage of the total phase of either, and also for the amplitude error to be small. Of course, even if the phase error is small in a percentage sense, it may not be small as a fraction of 2π. Eqn. 14 gives a strengthened condition, guaranteeing closeness of phase in an absolute sense.

It is also interesting to compare the time delay for the two transfer functions $W_l(j\omega)$ and $W_d(j\omega)$. The group or envelope delay $T(\omega)$ at frequency ω is [2] $-d\phi(j\omega)/d\omega$, where $\phi(j\omega)$ is the phase of the relevant transfer function. In this case, we get

$$T_d(\omega) = \frac{pl}{2m}\sqrt{LC} \tag{15a}$$

For the lumped circuit, the phase shift comes from $\exp\left[-jp\sin^{-1}\omega/2\alpha\right]$ and $K(j\omega)$ in eqn. 8. We obtain

$$T_l(\omega) = \frac{\frac{pl}{2m}\sqrt{LC}}{\left[1 - \frac{\omega^2 l^2}{4m^2}LC\right]^{1/2}} - \frac{d}{d\omega}\phi_K(j\omega) \tag{15b}$$

We do not evaluate the second quantity on the right, but note simply that it will be small when eqn. 13 holds; the first quantity will, of course, also approximate to $T_d(\omega)$ when eqn. 13 holds. Consequently, eqn. 13, rather than the stronger eqn. 14, serves to define the region of good approximation of delay behaviour.

Now let us consider briefly what happens when the matching condition of eqn. 7 fails. Set

$$\eta = \frac{1 - \beta/\alpha}{1 + \beta/\alpha} \tag{16}$$

(Notice that $|\eta|$ is a measure of mismatch.) Then eqn. 6 yields

$$\begin{aligned} W_l(j\omega) &= \frac{\exp\left[-jp\theta(\omega)\right] - \eta\exp\left[-j\overline{2n+1-p}\theta(\omega)\right]}{1 - \eta\exp\left[-2j\overline{n+1}\theta(\omega)\right]}K_\eta(j\omega) \\ &= \{\exp\left[-jp\theta(\omega)\right] - \eta\exp\left[-j\overline{2n+1-p}\theta(\omega)\right]\} \\ &\quad\times\{1 + \eta\exp\left[-2j\overline{n+1}\theta(\omega)\right] \\ &\quad+ \eta^2\exp\left[-4j\overline{n+1}\theta(\omega)\right] + \cdots\}K_\eta(j\omega) \end{aligned}$$

The explicit form of $K_\eta(j\omega)$ is not important: what is relevant is that, when eqn. 13 holds, $K_\eta(j\omega) \simeq 1$.

One can also verify that

$$\begin{aligned} W_d(j\omega) = \Big\{&\exp\left[-j\frac{pl}{2m}\sqrt{LC}\omega\right] \\ &- \eta\exp\left[-j\frac{(2m+p)l}{2m}\sqrt{LC}\omega\right]\Big\} \\ &\times\{1 + \eta\exp\left[-jl\theta(\omega)\right] + \eta^2\exp\left[-2jl\theta(\omega)\right] + \cdots\} \end{aligned}$$

Both transfer functions have the following interpretation: the response is due to a wave undergoing no reflection, a wave undergoing one reflection, two reflections, etc., each reflection causing, in general, an attenuation of the waves. These conclusions, with respect to phase approximation, remain valid. Those with regard to delay approximation remain valid if we look at the delays associated with successive identifiable reflections, subject to one qualification. The more reflections there are, the greater the delay involved, and so the greater the error in the delays of the two systems. Provided though that $|\eta| < 1$, so that $R = 0$, ∞ are disallowed, the amplitude of the successive reflections gets smaller with each reflection.

3 Discussion of transmission-line results

3.1 Sampling theorem viewpoint

There are at least two ways in which we would like the lumped model to approximate the distributed model. First, the lumped model should reflect the same properties in the response of each state variable component as the variable on the line located spatially at the nominal sampling point to which the state variable corresponds. Secondly, the lumped model should exhibit the wave-like property of the line, wherein a sinusoidal spatial variation goes together with a sinusoidal temporal variation, the relation between the two frequencies being governed by the velocity of propagation in the medium.

Let us now observe why simultaneous imposition of both these requirements creates difficulties. In the lumped model, there are m spatial samples, or a spatial sampling frequency of m/l per unit length. Now it is well known that sampling, including spatial sampling, has the effect of making frequencies higher than the aliasing frequency (here $m/2l$ per unit length) look like, or become, indistinguishable from a certain frequency lying within an interval determined by the aliasing frequency, the interval here being $(-m/2l, m/2l)$. Thus a spatial variation at a rate 0.75 m/l per unit length, for example, is indistinguishable from a spatial variation at a rate of 0.25 m/l per unit length. Now if the wave-like property is preserved by the lumping, and if the velocity of propagation of the waves is $c = (LC)^{-1/2}$, it then follows that temporal excitation of the lumped model at a frequency of $0.25c\ m/l$ Hz should give the same variation of amplitudes of the state vector components as would result from excitation at a frequency of $0.75c\ m/l$ Hz. For this variation of amplitudes in the lumped model will then, and only then, correspond to the variation of amplitudes of (normalised) voltage and currents at sampling points along the line.

Hence a nice model (in the sense of one meeting the two requirements just nominated) would have the property that the amplitude (if not the phase also) of the transfer function from the input to any entry of the state variable would vary periodically with temporal frequency, exhibiting the same value at $\omega + 2\pi c(nm/l)$ for all n. Such a variation can never be exhibited by a lumped model. The very best we could hope for then would be for the lumped model to behave satisfactorily, in the sense of approximating the distributed line, over a limited (temporal) frequency range. A pointer to this frequency range can be found by taking the spatial aliasing frequency and scaling it by the wave velocity, to get a temporal aliasing frequency. For the line, it is

$$\omega_a = 2\pi \frac{m}{2l\sqrt{LC}} = \frac{\pi m}{l\sqrt{LC}} \tag{17}$$

This should be compared with eqn. 13, the condition for the model to be a good approximation.

3.2 Constant-k filter argument [3, 4]

Delay lines are often constructed by cascading T sections of the form depicted in Fig. 2. An analysis in Reference 4, involving assumptions on impedance matching, shows that m such sections in cascade, excited on the left and 'appropriately' terminated on the right, will produce a time delay of $m\sqrt{L_1C_1}$, provided that the excitation frequency obeys

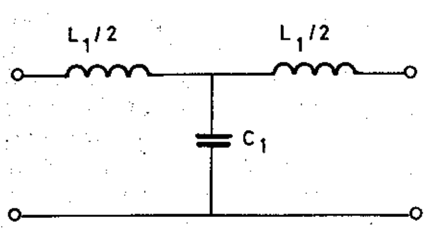

Fig. 2 *Prototype section of constant-k filter*

$$\omega \ll \frac{2}{\sqrt{L_1C_1}} = \frac{2m}{l\sqrt{LC}} \tag{18}$$

Thus we recover, by a different argument, the condition of eqn. 13. The constant-k filter theory also notes that, for excitation frequencies ω higher than $2m/l\sqrt{LC}$, the lumped network transfer function from input to termination will attenuate, but have frequency independent phase, so that no group delay should be expected.

3.3 Controllability and the controllability Gramian

We now see why it is that the lumped model can be instantaneously controllable while the distributed model is not. The lumped model is not a good approximation of the distributed model for excitation frequencies ω of the order of $2m/l\sqrt{LC}$ or higher. Instantaneous control of the lumped model requires application of those frequencies.

If we consider controlling the lumped model with signals which are artificially restricted in frequency, we could demand that we use piecewise constant signals changing value no more often than $m/\pi l\sqrt{LC}$ times per second. Noting that the lumped model has a $2m$-dimensional state vector, this will imply that, for a generic nonzero initial state, it will take least time $2m \div [m/\pi l\sqrt{LC}] = 2\pi l\sqrt{LC}$ to bring it to zero. This is in crude agreement with the least time taken to bring the distributed system to zero, which is $l\sqrt{LC}$ or $2l\sqrt{LC}$ according as to whether the line is or is not terminated in its characteristic impedance [1].

Finally, it is instructive to look at the controllability Gramian [8]. Consider the lumped case only. Then the controllability Gramian associated with $\dot{x} = Fx + Gu$ over the interval $[0, T]$ is

$$W(T) = \int_0^T e^{Ft}GG'e^{F't}\,dt$$

and the minimum energy required to move the system from state x_0 at time 0 to state zero at time T is [8] $x_0'W^{-1}(T)x_0$. Accordingly, $\lambda_{min}W(T)$ provides a measure of the worst-case control problem. Our particular interest is to study the variation with T of this quantity using the matrices F and G in eqn. 2. We also consider the variation of $n = 2m$, the dimension of the lumped model of eqn. 2.

Fig. 3 illustrates the results for several different values of n. The calculation may be intrinsically numerically difficult, but the general form of the curves is quite certain. One unit of normalised time is the time taken for transmission one way along the ideal line. The eigenvalue normalisation is obtained as

$$\frac{\lambda_{min}W(T)}{\lambda_{min}W(\infty)}$$

The clear conclusion is that, if one wishes to control in a time less than twice the one-way delay, this will be much more costly than controlling in a time greater than twice the one-way delay. One would expect that, for very large n,

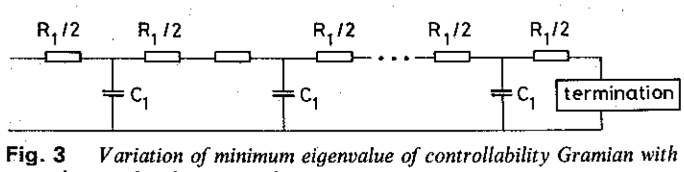

Fig. 3 *Variation of minimum eigenvalue of controllability Gramian with control interval and system order*

the plot would be approximately zero for 0 to 2 and unity thereafter.

3.4 Discretisation in time and space

It may be that, for simulation or other purposes, a discrete-time, lumped (= discretised in space) model is desired. One can obtain such a model by passing first to a continuous-time lumped model, and then time-discretising, or one can discretise the original partial differential equation both in time and space. This latter procedure is the one usually suggested in the literature, and it is pertinent to record several standard ideas [9, 10]. Let k, h be the discretisation intervals for time and space, respectively, and let c be the wave velocity. Then $k/h > c$ is unacceptable: the solution for the difference equation within a (discretised) region is defined by a set of initial conditions over a smaller region of discretised space than for the partial differential equation, and there is instability in the face of round-off errors. If $k/h = c$, the difference equation system is particularly simple. There is a matching of initial condition requirements, but one is on the verge of round-off error instability. If $k/h < c$, then initial data over a bigger region are required for the difference equation than for the differential equation to establish the solution at a certain point, where the difference equation is stable in the face of round-off errors. If $k \to 0$ and $h \to 0$ while k/h remains constant and $k/h < c$, then the solution of the difference equation approaches that of the differential equation and the domains of dependence approach one another.

In terms of the earlier discussion invoking sampling-theorem ideas, the remarks in the preceding text have the following interpretation. Let the temporal frequency induced by the spatial sampling frequency be ω_a, as in eqn. 17, and let the temporal aliasing frequency associated with the time sampling be $\bar{\omega}_a$. Then $k/h > c$ is equivalent to $\bar{\omega}_a > c^{-1} w_a$.

4 Other distributed systems

4.1 The heat equation or RC line equation

The heat equation applicable for a uniform rod is essentially the same equation as that for the voltage on a uniform RC transmission line, with resistance R and capacitance C per unit length, namely

$$\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t} \tag{19}$$

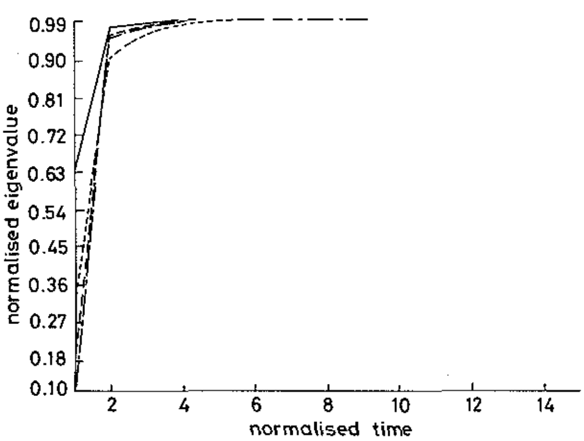

Fig. 4 *Lumped approximation to uniform RC line made from cascade of prototype constant-k sections*

curve 1 ——— $N = 2$ curve 3 — — $N = 6$
curve 2 - - - - $N = 4$ curve 4 —·—·— $N = 8$

Such a line can be approximated by an nth order lumped system as depicted in Fig. 4.

Using constant-k filter theory, as in References 3 and 4, one can establish that if

$$\omega \ll \frac{2}{R_1 C_1} \tag{20a}$$

or

$$\omega \ll \frac{n^2}{l^2} \frac{2}{RC} \tag{20b}$$

then the transfer function of p sections is approximately

$$\exp\left[-\frac{pl}{n}\sqrt{\frac{\omega RC}{2}}(1+j)\right] \tag{21}$$

with an associated group delay of

$$\frac{pl}{n}\sqrt{\frac{RC}{2}}\frac{1}{2\sqrt{\omega}} \tag{22}$$

The sinusoidal solutions of eqn. 19 are all of the form

$$v(x, t) = \exp[\gamma x + j\omega t]$$

$$\gamma = \mp\sqrt{\frac{\omega RC}{2}}(1+j) \tag{23}$$

The negative sign is clearly needed given an excitation at $x = 0$ with a line extending into $x > 0$. This result agrees with the analysis leading to eqn. 21 and eqn. 22. So, under eqn. 20, the lumped model is a good approximation. Once we allow high-frequency excitation, however, we draw misleading conclusions; for example, we could not expect that the group delay associated with the lumped system and with the line were comparable.

4.2 Towed cable

Let us now consider the transverse motion of a uniform towed cable [5, 6]. With certain simplifying assumptions, the equation becomes

$$\frac{\partial^2 y}{\partial t^2} + \alpha(x)\frac{\partial^2 y}{\partial x^2} + 2\beta\frac{\partial^2 y}{\partial x\,\partial t} + b\frac{\partial y}{\partial x} + c\frac{\partial y}{\partial t} = 0 \tag{24}$$

Here, β, b and c are constants, and $\alpha(x)$ is spatially dependent; because of drag on the cable, the tension in the cable varies linearly from a maximum at the towpoint, to approximately zero at the trailing end, and this accounts for the spatial dependence of $\alpha(\cdot)$. A lumped approximation for the equation is also given in Reference 5. Now, as described in Reference 4, $\beta^2 - \alpha(x)$ is normally positive for all values of x of interest. This means that the equation defined is hyperbolic, and wavelike motion is to be expected. Because of the dependence of $\alpha(\cdot)$ on x, it turns out that waves propagate one way only [5].

Now suppose the cable has length L and is approximated using n uniformly spaced discrete masses. The spatial aliasing frequency is then n/L. [Because the waves travel only one way, the sign of the spatial frequency is automatically determined, and we can choose the interval of unique determination of frequency to be $[0, N/L)$ rather than $(-N/2L, N/2L)$.] The transverse motions of the lumped approximation can be expected to mimic those of the distributed system only for frequencies less than $[n/L \min_{0 \leqslant x \leqslant L} \sqrt{\beta^2 - \alpha(x)}]$ and we would expect that accurate representation of the distributed system by the lumped system would require significantly lower frequencies again.

Since $\alpha(\cdot)$ is x-dependent, one could contemplate uneven spatial sampling which would compensate in the following sense. Pick points $x_1, x_2, \ldots, x_{n-1}, x_n$ along the cable at which mass will be presumed to be concentrated. Choose the x_i so that the time for a wave on the cable to travel from x_i to x_{i+1} is independent of i. Thus the greater is $\alpha(x)$, the wider is the spacing. For the towed cable, the functional form of $\alpha(\cdot)$ then dictates that the x_i be farther apart near the towing point. Near the towing point, the tension, and therefore the wave velocity, is greater.

4.3 Deep-sea mining hauling pipe

Reference 7 discusses the approximation via a lumped model (for the purposes of developing a controller) of a deep-sea mining hauling pipe. The distributed parameter equation is of the type

$$\frac{\partial^2 w}{\partial t^2} = \gamma_1 \frac{\partial^2 w}{\partial x\, \partial t} + \gamma_2 \frac{\partial w}{\partial t} + \gamma_3 \frac{\partial^4 w}{\partial x^4} + \gamma_4(x) \frac{\partial^2 w}{\partial x^2} + \gamma_5 \frac{\partial w}{\partial x} + \gamma_6 u \tag{25}$$

[Here, $u(\cdot)$ is the control, and $\gamma_1, \gamma_2, \gamma_3, \gamma_5$ and γ_6 are constant, while γ_4 is spatially dependent.] The only difference from the towed cable, apart from the explicit inclusion of u (which can also be included in the cable equation if desired), is the fourth-order spatial derivative, which accounts for the bending moment in the pipe; this term can be, and usually is, neglected in the towed cable. The dependence of $\gamma_4(\cdot)$ on x is like that of $\alpha(\cdot)$ in eqn. 24.

The author of Reference 7 compared the possibility of uniform spatial sampling with nonuniform sampling where each gap is a fixed multiple of the previous one, and with a mixture of the two strategies. By simulation, he found that at the upper end of the pipe (roughly in the upper half), spacing should be uniform, and in the lower part of the pipe, the interval between sampling points should get shorter, the deeper one moved. The upper end of the pipe is analogous to the towing point of the cable, so that, in qualitative terms, his empirical observations are seen to agree with the conclusions of our argument.

5 Conclusions

The key ideas can be summed up as follows. Lumped approximations of distributed systems are only approximations over a limited, low-pass, temporal frequency band. The size of this band can be determined by relating a spatial aliasing frequency, readily determined from knowledge of the distributed system and of the procedure for obtaining the lumped model, to a temporal aliasing frequency. The velocity of waves in the medium provides the relationship between the two aliasing frequencies. In the event that the wave velocity is not uniform, nonuniform spatial sampling may be required. This has proved advantageous in the analysis of a towed cable.

In considering the specific question of controllability, it follows that any control actions which rely on fast acting controls will give rise to responses which will almost certainly be different for related lumped and distributed models.

In considering the question of closed-loop control, the spectrum of the external inputs and the desired closed-loop bandwidth for the closed-loop system (which between them determine the spectrum of the input to the plant) will be the crucial quantities that determine the order of the lumped approximation model.

6 References

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