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The Description of Coloured Noise in Dynamical Systems†

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ABSTRACT

The paper considers the description via covariance matrices of noise which can appear at measurement points of lumped physical systems, themselves excited by sources of white gaussian noise, with the possibility of direct input to output coupling allowed.

The relevance is examined of the assumptions made in modelling the problem, and necessary and sufficient conditions are presented which mathematically characterize the covariance matrix of the measured noise.

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§ 1. INTRODUCTION AND PHYSICAL MOTIVATION

IN the past 20- or so years, the physical phenomenon of noise has developed from being a subject of somewhat academic interest to statisticians and physicists to being a subject of great interest to engineers as an important factor in many engineering design problems, especially those involving communications or control systems. The mathematical description of noise in engineering systems has accordingly become particularly important.

It is necessary to distinguish sharply between, on the one hand, the physical mechanisms by which noise originates and the mathematical description of the associated noise sources, and on the other hand the mathematical description of the resultant noise appearing at some part of a system other than a noise source.

The former topic will not concern us so much here, being by its nature more a problem for the physicists. It is however pertinent to mention two of the principal sources of noise in systems. The first is thermodynamic in origin, being exemplified by thermal noise in a resistor; provided the frequencies considered are not too high, this noise is white, i.e. uncorrelated from instant to instant. Such noise is of course not merely confined to resistors in electrical circuits; it will in fact arise in any dissipative element in a physical system. A further source of noise commonly occurring in electric circuits, but less commonly elsewhere, is shot noise, which arises as a statistical phenomenon associated with the discrete rather than continuous nature of current flow. This noise, again with certain frequency limitations, is also white.

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The presence of energy storage elements or dynamic elements in a physical system serves to colour the noise observable at parts of the system other than the noise sources, i.e. the noise is no longer uncorrelated between any two instants.

While the description of *noise source* statistics remains essentially a problem of physics, the problem of describing *coloured noise* statistics where the coloured noise arises from white noise is essentially mathematical. One needs to specify the statistics of the noise sources, and the mathematical model of the physical system, together with the means by which the noise is introduced and the means by which the coloured noise is measured.

This paper discusses the description of coloured noise under the conditions where the noise sources produce *white gaussian noise*, and the physical system in which the noise is present is modelled by a *linear, finite-dimensional dynamical system* (Kalman 1963).

The assumption of gaussian noise is appropriate for several reasons. First, experience shows that it is generally the best first model when there is no more specific information, while at the same time some physical processes do appear to be correctly described by this model. Second, gaussian noise is characterized by two parameters, viz. its mean and covariance, rather than an infinite number of parameters, as for example in a distribution function characterization. Third, when gaussian inputs are used with our linear mathematical model of the physical system, gaussian outputs result, and all random variables within the system can be characterized by only two parameters, viz. their mean and covariance. In certain situations all means may be zero, thus allowing the description of variables by one parameter only.

The mathematical model assumed for the physical system is specific in some ways, and general in others. It is specific because it considers only those systems which are linear, and have a finite number of lumped energy storage elements, and a finite number of dissipative elements. It is general because even with these restrictions, an enormous number of physical systems can be so described, perhaps after some suitable transformation. Elementary examples are provided by lumped electrical circuits composed of inductors, resistors, capacitors and transformers, and by lumped mechanical circuits containing for example springs, masses, dashpots and gears.

In certain situations, including situations requiring noise description, it can be both convenient and appropriate to approximate a distributed physical system by a lumped one, thus allowing application of the lumped theory to these distributed systems.

The systems we consider will in general be time-varying. In passing we note that the theory we present has been more or less worked out for the time-invariant situation, while at the same time the advent of time-varying control systems and the inherent time-varying nature of some physical processes suggests that consideration of time-varying problems is pertinent.

The question arises as to what extent the assumption of time-variation of the system affects the modelling of the noise sources. The answer is little or not at all, provided the time-variations occur at frequencies that are not too high, that is, at frequencies which are in the range of those for which the noise models in the time-invariant case can be considered to apply.

A less general treatment than the one presented here has been given by Kalman (1965); this reference is more concerned with problems of filtering than the description of physical processes.

§ 2. Process Description

We consider systems which can be described by equations of the form:

$$\dot{x}=F(t)x+G(t)u, \tag{1 a}$$

$$y=H'(t)x+J(t)u. \tag{1 b}$$

Here x, y and u are respectively n, m and p-vector functions of time, the state output and input of the system, and $F(t)$, $G(t)$, $H(t)$, $J(t)$ are matrices of appropriate dimension. The superscript prime describes matrix transposition. For a discussion of the properties of (1) when $J=0$, see Kalman (1963); many of these properties carry over to the situation where J is non-zero.

The state vector in essence summarizes the past history of the system (Zadeh and Desoer 1963), and may for example have for its elements variables corresponding to the excitation of the energy storage elements of the system. The vector x is an intermediate variable of the system, that is, a variable which is neither input nor output, and it is possible to give a description of (1) which does not include x. Such a description will be of the form:

$$L(p, t)y=M(p, t)u, \tag{2}$$

together with appropriate initial conditions; here $L(p, t)$ and $M(p, t)$ are matrices with elements which are polynomial in $p \equiv d/dt$, with time-variable coefficients. Provided certain restrictions are placed on the order of the derivatives in (2), then (2) can always be replaced by a set of equations of the form (1). If desired (2) rather than (1) can be regarded as the prototype of the systems discussed.

When u in (1) is deterministic, one can readily write down the resulting y, which is conveniently described with the assistance of two further entities, the inital state $x(t_0)$, (assuming an inital time t_0), and the transition matrix $\Phi(t, \tau)$ which is defined by:

$$\frac{d}{dt}\Phi(t, \tau)=F(t)\Phi(t, \tau), \tag{3 a}$$

$$\Phi(\tau, \tau)=I, \tag{3 b}$$

where I is the identity matrix. Then, see Zadeh and Desoer (1963), y is related to u through:

$$y(t)=J(t)u(t)+H'(t)\Phi(t,t_0)x(t_0)+H'(t)\int_{t_0}^{t}\Phi(t,\tau)G(\tau)u(\tau)\,d\tau. \tag{4}$$

We now examine the stochastic case, and assume that the variable u is no longer deterministic but is associated with a source of white noise; for convenience we shall assume u to have zero mean. We shall also assume it is gaussian, described by a covariance matrix:

$$E[u(t)u'(\tau)]=Q(t)\delta(t-\tau). \tag{5}$$

Here δ is the Dirac delta function, and Q is non-negative definite for all t. Thus u is uncorrelated from one instant of time to another. The variable y now becomes a random variable; for its description we require knowledge of the initial conditions on x. For convenience, but without any loss of generality, we shall assume $x(t_0)$ is a gaussian random variable of mean zero, and covariance $X(t_0, t_0)$. Since the assumption of zero mean can very readily be abrogated, we note that these assumptions on $x(t_0)$ include the deterministic case used in (4).

It is immediately evident from (4) by taking expectations that $y(t)$ will have zero mean. Also by use of (4), we may write down $y(t)y'(\tau)$ and take the expectation to yield the covariance of y in terms of the known noise statistics and the parameters of the system. This covariance sums up all known statistical information about y. Direct calculation, using properties of distributions outlined in, for example, Schwartz (1957), yields:

$$\begin{aligned}E[y(t)y'(\tau)]=&H'(t)\Phi(t,t_0)\left[X(t_0,t_0)+\int_{t_0}^{\tau}\Phi(t_0,\sigma)G(\sigma)Q(\sigma)G'(\sigma)\Phi'(t_0,\sigma)\,d\sigma\right]\\&\times\Phi'(\tau,t_0)H(\tau)1(t-\tau)\\&+H'(t)\Phi(t,t_0)\left[X(t_0,t_0)+\int_{t_0}^{t}\Phi(t_0,\sigma)G(\sigma)Q(\sigma)G'(\sigma)\Phi'(t_0,\sigma)\,d\sigma\right]\\&\times\Phi'(\tau,t_0)H(\tau)1(\tau-t)\\&+H'(t)\Phi(t,\tau)G(\tau)Q(\tau)J'(\tau)1(t-\tau)\\&+J(t)Q(t)G'(t)\Phi'(t,\tau)H(\tau)1(\tau-t)\\&+J(t)Q(t)J'(t)\delta(t-\tau),\end{aligned} \tag{6}$$

where $1(t)$ is the unit step function. This is of the form:

$$E[y(t)y'(\tau)]=A(t)\delta(t-\tau)+B(t)C(\tau)1(t-\tau)+C'(t)B'(\tau)1(\tau-t), \tag{7}$$

with

$$A(t)=J(t)Q(t)J'(t), \tag{8 a}$$

$$B(t)=H'(t)\Phi(t,t_0), \tag{8 b}$$

$$C(\tau)=\left[X(t_0,t_0)+\int_{t_0}^{\tau}\Phi(t_0,\sigma)G(\sigma)Q(\sigma)G'(\sigma)\Phi'(t_0,\sigma)\,d\sigma\right]\Phi'(\tau,t_0)H(\tau)+\Phi(t_0,\tau)G(\tau)Q(\tau)J'(\tau). \tag{8 c}$$

Whereas the noise source described by the variable u was assumed to be white gaussian noise (corresponding to the $\delta(t-\tau)$ in (5)), the noise described by y is gaussian (because $x(t_0)$ and $u(t)$ are gaussian) but coloured, corresponding to the $1(t-\tau)$ and $1(\tau-t)$ in (6) or (7).

§ 3. Main Result

In this section we characterize the class of operators of the form:

$$R(t,\tau)=A(t)\delta(t-\tau)+B(t)C(\tau)1(t-\tau)+C'(t)B'(\tau)1(\tau-t), \tag{9}$$

which arise as the covariance of the output of a linear finite-dimensional physical system excited by white noise. The characterization is via necessary and sufficient conditions on the matrices A, B and C.

Theorem. The operator $R(t,\tau)$ of the form (9) is the covariance of the output of a linear finite-dimensional physical system of the type (1) if and only if there exist matrices $K(t)$, $V(t)$, $M(t)$ and a symmetric non-negative definite matrix $N(t_0)$, used to define:

$$N(t)=N(t_0)+\int_{t_0}^{t} M(\sigma)M'(\sigma)\,d\sigma, \tag{10}$$

such that:

$$\begin{aligned}R(t,\tau)=&V(t)(N(\tau)V'(\tau)+M(\tau)K'(\tau))1(t-\tau)\\&+(V(t)N(t)+K(t)M'(t))V'(\tau)1(\tau-t)+K(t)K'(t)\delta(t-\tau).\end{aligned} \tag{11}$$

Proof of sufficiency

Consider the system:

$$\dot{x}=M(t)u, \tag{12 a}$$

$$y=V(t)x+K(t)u, \tag{12 b}$$

excited by white noise such that:

$$E[u(t)u'(\tau)]=I\delta(t-\tau). \tag{13}$$

Then $\Phi(t,\tau)=I$ for all t and τ, and (6) yields:

$$\begin{aligned}E[y(t)y'(\tau)]=&V(t)\left[X(t_0,t_0)+\int_{t_0}^{\tau} M(\sigma)M'(\sigma)\,d\sigma\right]V'(\tau)1(t-\tau)\\&+V(t)\left[X(t_0,t_0)+\int_{t_0}^{t} M(\sigma)M'(\sigma)\,d\sigma\right]V'(\tau)1(\tau-t)\\&+V(t)M(\tau)K'(\tau)1(t-\tau)+K(t)M'(t)V(\tau)1(\tau-t)+K(t)K'(t)\delta(t-\tau).\end{aligned}$$

By making the identifications:

$$X(t_0,t_0)=N(t_0), \tag{14}$$

$$N(t)=N(t_0)+\int_{t_0}^{t} M(\sigma)M'(\sigma)\,d\sigma, \tag{10}$$

eqn. (11) follows.

Proof of necessity

We are given the system (1), which yields $R(t, \tau)$ equal to the right-hand side of (6), and we must establish the existence of $K(t)$, $V(t)$, $M(t)$ and $N(t_0)$ such that (11) holds.

The matrix Q, being symmetric and non-negative definite, possesses a not necessarily unique square root which is also symmetric, i.e. there exists some symmetric matrix, call it $Q_{1/2}$, such that $Q_{1/2}^2 = Q$ (Gantmacher 1959). Then define:

$$K(t) = J(t)Q_{1/2}(t), \tag{15 a}$$

$$V(t) = H'(t)\Phi(t, t_0), \tag{15 b}$$

$$M(t) = \Phi(t_0, t)G(t)Q_{1/2}(t), \tag{15 c}$$

$$N(t_0) = X(t_0, t_0). \tag{15 d}$$

Equation (6) now becomes:

$$\begin{aligned} E[y(t)y'(\tau)] = {} & V(t)\left[N(t_0) + \int_{t_0}^{\tau} M(\sigma)M'(\sigma)\, d\sigma\right] V'(\tau)1(t-\tau) \\ & + V(t)\left[N(t_0) + \int_{t_0}^{t} M(\sigma)M'(\sigma)\, d\sigma\right] V'(\tau)1(\tau - t) \\ & + V(t)M(\tau)K'(\tau)1(t-\tau) + K(t)M(t)V'(\tau)1(\tau-t) + K(t)K'(t)\delta(t-\tau), \end{aligned} \tag{16}$$

whence, by using (10), (11) is recovered.

This completes the proof.

§ 4. Discussion

Operators $R(t, \tau)$ of the form (9) which satisfy:

$$\int_{t_1}^{t_2}\int_{t_1}^{t_2} w'(t)R(t, \tau)w(\tau)\, dt\, d\tau \geqslant 0, \tag{17}$$

for all integrable functions $w(\cdot)$ are termed non-negative definite, (Davenport and Root 1958). It is worth noting that such functions arise naturally in other ways than as covariances, for instance as energy functions in time-varying circuits (Anderson and Newcomb 1966 b). Non-negative definiteness over arbitrary intervals and symmetry of the form $R(t, \tau) = R'(\tau, t)$, termed self-adjointness, are two conditions which a covariance must satisfy. A hitherto unanswered question which then arises is: given $R(t, \tau)$ of the form (9), with $A(t) = A'(t)$ and such that (17) holds, is it the covariance of the output of a linear finite-dimensional dynamical system, excited by white noise, or equivalently, can $R(t, \tau)$ be re-written in the form (11)?

If this re-writing can be done, then there results a solution of the time-varying spectral factorization problem of Zadeh (1961), where it is required to find the impulse response of a system driven by white noise when the covariance of the output is known. From the form (11), one can pass to a system of the form (1) whose impulse response can then be evaluated as the solution of the spectral factorization problem.

In any case however the material in this paper establishes that if a covariance matrix $R(t, \tau)$ is indeed the result of putting white noise through a dynamical system, then $R(t, \tau)$ can be re-written in such a way that the description of the system itself can be recovered from this representation of $R(t, \tau)$. The mechanics of carrying out this re-writing have yet to be explained in detail.

The application of the results of § 3 to the study of properties of time-varying linear networks is currently under investigation. It appears to be the case that under broad condition a covariance $R(t, \tau)$ can be broken into the sum of a causal and an anti-causal term, the causal term being given by:

$$z(t, \tau) = \tfrac{1}{2}A(t)\delta(t-\tau) + B(t)C(\tau)l(t-\tau), \tag{18}$$

with this operator representing the port impedance of a passive network. Results of a similar nature for time-invariant networks have recently led to a new network synthesis technique (Anderson and Newcomb 1966 a).

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