Use of Frequency Dependence in Linear Quadratic Control Problems to Frequency-Shape Robustness

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Linear quadratic control problems are considered where frequency-dependent weighting on the control is assumed. When high frequencies are weighted more heavily than low frequencies, two qualitative conclusions can be drawn: passband robustness is reduced and high-frequency robustness is improved. The result gives a theoretical underpinning to the use of frequency-dependent weighting when high-frequency uncertainty is present in the plant.

I. Introduction

IN several recent papers, frequency-dependent cost functions have been proposed as a way of dealing with plant uncertainties within the context of linear quadratic optimal control. The use of such weighting is described, e.g., in Refs. 1-4, with Refs. 3 and 4 giving a number of practical insights and discussion of specific examples. In this paper, we examine the use of a particular form of frequency-dependent weighting, one which emphasizes high frequencies in the penalty on control effort. This weighting is motivated by the assumption that the model may not accurately describe the actual system at high frequencies. Hence, we wish to attenuate high-frequency control activity.

The general conclusion developed in the paper is that, roughly, frequency-dependent weighting which penalizes high-frequency control activity will improve robustness outside the passband (relative to a nonfrequency-dependent weighting situation), but in the process, some robustness may be lost in the passband.

Throughout the analysis, the availability of the state is assumed. Should the state not be available, it is known that robustness results can be carried through in the case of a minimum-phase plant using an appropriately designed observer.[5] For nonminimum-phase plants, there is no simple solution using either classical or optimal methods.

II. Modified Linear Quadratic Problem

To illustrate the idea, we shall confine attention to a single-input system. The idea can then be extended subsequently to multiple-input systems. Instead of attempting to minimize for

$$\dot{x} = Ax + bu \tag{1}$$

the performance index

$$V = \int_0^\infty [u^2 + (c'x)^2]\,\mathrm{d}t \tag{2}$$

we shall work with

$$V = \left(\frac{1}{2}\pi\right)\int_{-\infty}^{\infty}\left[\left|\frac{j\omega\beta+1}{j\omega\alpha+1}u(j\omega)\right|^2 + |c'x(j\omega)|^2\right]\mathrm{d}\omega \tag{3}$$

where $\beta > \alpha > 0$, and it is assumed that the optimal control and associated state trajectory as time functions are square integrable on $(0,\infty)$. Clearly, the effect of $\beta > \alpha > 0$ is to give extra penalty to high-frequency u compared to low-frequency u. To work with Eq. (3), we shall use an equivalent time-domain index. Loosely, we can write

$$V = \int_0^\infty\left[\left|\frac{j\omega\beta+1}{j\omega\alpha+1}u(t)\right|^2 + x'(t)cc'x(t)\right]\mathrm{d}t \tag{4}$$

with obvious abuse of notation. What is meant is that the quantity u^2 in Eq. (2) is replaced by v^2 with $v(\cdot)$ defined by

$$\dot{z} = -\frac{1}{\alpha}z + \frac{\beta}{\alpha}u \qquad v = \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)z + \frac{\beta}{\alpha}u \tag{5}$$

[so that the transfer function linking u to v is $(j\omega\beta+1)/(j\omega\alpha+1)$]. This means that the optimal value of V depends both on $x(0)$ and $z(0)$. However, the dependence on $z(0)$ will not concern us, since our main interest is in the structure of the controller.

III. Reformulated Conventional Control Problem

With the above definition of $v(\cdot)$, we have from Eq. (5)

$$\dot{z} = -\frac{1}{\beta}z + v \qquad u = \left(\frac{1}{\beta} - \frac{\alpha}{\beta^2}\right)z + \frac{\alpha}{\beta}v \tag{6}$$

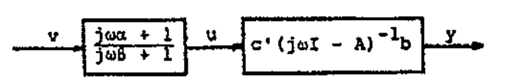

Fig. 1 Relationship between u, v, and y.

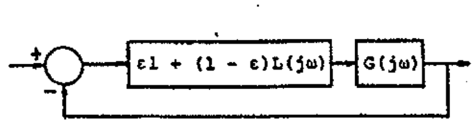

Fig. 2 System considered in first robustness result, $0 \le \epsilon \le 1$.

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and the control problem is then the conventional one of minimizing

$$V=\int_0^{\infty}[v^2+(c'x)^2]\,\mathrm{d}t \tag{7}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z \\ x \end{bmatrix}=\begin{bmatrix} -\beta^{-1} & 0 \\ b(\beta^{-1}-\alpha\beta^{-2}) & A \end{bmatrix}\begin{bmatrix} z \\ x \end{bmatrix}+\begin{bmatrix} 1 \\ b\alpha\beta^{-1} \end{bmatrix}v \tag{8}$$

(See Fig. 1.)

For this problem to have a well-defined solution with an asymptotically stable closed loop, it is necessary and sufficient that[6]

$$\begin{bmatrix} -\beta^{-1} & 0 \\ b(\beta^{-1}-\alpha\beta^{-2}) & A \end{bmatrix},\ \begin{bmatrix} 1 \\ b\alpha\beta^{-1} \end{bmatrix} \tag{9}$$

is completely stabilizable and

$$\begin{bmatrix} -\beta^{-1} & 0 \\ b(\beta^{-1}-\alpha\beta^{-2}) & A \end{bmatrix},\ [0 \quad c'] \tag{10}$$

is completely detectable.

It turns out that these properties are guaranteed if $[A,b]$ and $[A,c']$ are completely stabilizable and completely detectable (see Appendix A). However, it is logical to assume that in practice the triple $[A,b,c]$ is minimal, i.e, $[A,b]$ and $[A,c']$ are completely controllable and completely observable (which properties certainly imply the complete stabilizability and detectability of $[A,b]$ and $[A,c']$).

With the minimality assumption on $[A,b,c]$, there is a unique minimal solution P to the following equation, which defines a stabilizing feedback law and which is nonnegative definite [and positive definite given observability instead of detectability in Eq. (10)]:

$$P\begin{bmatrix} -\beta^{-1} & 0 \\ b(\beta^{-1}-\alpha\beta^{-2}) & A \end{bmatrix}+\begin{bmatrix} -\beta^{-1} & b'(\beta^{-1}-\alpha\beta^{-2}) \\ 0 & A' \end{bmatrix}P$$

$$-P\begin{bmatrix} 1 \\ \alpha\beta^{-1}b \end{bmatrix}[1 \quad \alpha\beta^{-1}b']P+\begin{bmatrix} 0 & 0 \\ 0 & cc' \end{bmatrix}=0 \tag{11}$$

Let

$$\begin{bmatrix} k_z \\ k_x \end{bmatrix}=-P\begin{bmatrix} 1 \\ \alpha\beta^{-1}b \end{bmatrix} \tag{12}$$

Then the optimal and stabilizing control law is given by

$$v(t)=k_z z(t)+k_x' x(t) \tag{13}$$

Assuming $z(0)=0$, Eqs. (13) and (6) yield

$$v(j\omega)=\frac{k_z}{j\omega+\beta^{-1}}v(j\omega)+k_x' x(j\omega)$$

or

$$v(j\omega)=\frac{j\omega+\beta^{-1}}{j\omega+\beta^{-1}-k_z}k_x' x(j\omega) \tag{14}$$

IV. Solution of Original Problem

Recognizing the connection between u and v (Fig. 1), we have

$$u(j\omega)=\frac{j\omega\alpha+1}{j\omega\beta+1}v(j\omega)=\frac{j\omega\alpha+1}{j\omega\beta+(1-\beta k_z)}k_x' x(j\omega) \tag{15}$$

The fact that the pole of the transfer function which weights $u(j\omega)$ in Eq. (3) appears as a zero of a transfer function describing the feedback controller is standard, see Ref. 3.

V. Bound on k_z

We shall establish that

$$k_z<0 \tag{16}$$

To do this, we shall first show that it is impossible to have $P_{11}=0$, where

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \tag{17}$$

For if $P_{11}=0$, the non-negative definite symmetric nature of P implies that $P_{12}=0$. Then the 2-2 entry of Eq. (11) becomes

$$P_{22}A+A'P_{22}-\alpha^2\beta^{-2}P_{22}bb'P_{22}+cc'=0$$

and the minimality of $[A,b,c]$ guarantees[7] that P_{22} is positive definite (we shall write $P_{22}>0$). Then the 2-1 entry gives

$$-P_{12}'\beta^{-1}+P_{22}b(\beta^{-1}-\alpha\beta^{-2})+A'P_{12}'$$

$$-(P_{12}'+P_{22}\alpha\beta^{-1}b)(P_{11}+\alpha\beta^{-1}b'P_{12}')=0$$

With $P_{11}=0$ and $P_{12}=0$, we would have a contradiction to the fact that $P_{22}>0$.

Now the 1-1 entry of Eq. (11) yields

$$-2P_{11}\beta^{-1}+2P_{12}b(\beta^{-1}-\alpha\beta^{-2})-(P_{12}+\alpha\beta^{-1}P_{12}b)^2=0 \tag{18}$$

whence

$$2P_{12}b(\beta^{-1}-\alpha\beta^{-2})=2P_{11}\beta^{-1}+(P_{11}+\alpha\beta^{-1}P_{12}b)^2$$

Now $P_{11}>0$ since P is nonnegative definite and $P_{11}\neq 0$. Also, $\beta>\alpha$. Hence $P_{12}b>0$. Then

$$k_z=-P_{11}-\alpha\beta^{-1}P_{12}b<0 \tag{19}$$

The fact that k_z is negative will be useful in evaluating the robustness of controllers based on frequency-dependent weightings. This is discussed in Sec. VIII.

VI. The Return Difference Equality

The conventional control problem described in Sec. III gives rise to a version of the return difference equality as described in Ref. 7. Let $T_1(j\omega)$ = transfer function from $v(j\omega)$ to $y(j\omega)=c'x(j\omega)$ and $T_2(j\omega)$ = loop gain. Then the return difference equality becomes

$$1+|T_1(j\omega)|^2=|1-T_2(j\omega)|^2 \tag{20}$$

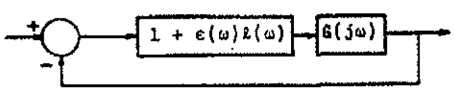

Fig. 3 System considered in second robustness result, $|\epsilon(\omega)|\leq 1$.

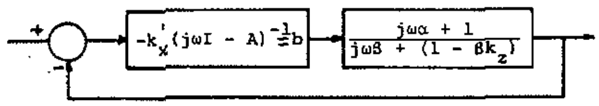

Fig. 4 Closed loop for the frequency-weighted problem.

negativity of k_z, we then obtain for large ω,

$$|G(j\omega)| < |\hat{G}(j\omega)|$$

It is in fact permissible to take $\alpha=0$, and then $|G(j\omega)|$ falls off as ω^{-2}.

In the light of the robustness result associated with Eq. (26), this means that there is improved robustness at high frequencies.

Thus we are led to the key conclusion: for the problem defined in Sec. II, improved robustness may be obtained at high frequencies, but this improvement is achieved at the expense of some reduction in robustness at low-to-mid frequencies.

IX. Example

Consider the system $\dot{x}=u$. With the performance index

$$V=\int_0^\infty (u^2+x^2)\,dt$$

the optimum control law is $u=-x$. With the performance index

$$V=\int_0^\infty \left[\left|\frac{10j\omega+1}{j\omega+1}\right|^2 u^2+x^2\right]dt$$

we obtain $k_x=-1$, $k_z=-0.26904$ and so

$$u=-\frac{j\omega+1}{10j\omega+3.69}x$$

The loop gains are

$$\hat{G}(j\omega)=\frac{1}{j\omega}, \qquad G(j\omega)=\frac{1}{j\omega}\,\frac{j\omega+1}{10j\omega+3.69}$$

The phase margin associated with $\hat{G}$ is 90 deg. The phase margin associated with G is 71 deg, which is less. On the other hand, if we move outside the passband we see that, for example,

$$|\hat{G}(j10)|=0.1, \qquad |G(j10)|=0.01$$

illustrating that greater high-frequency uncertainty can be accommodated with the frequency-weighted design.

X. Conclusion

In this paper, we have examined one way in which classical and optimal control ideas can be blended. In classical control, it is well known that a phase lag compensation nework can be used to increase the attenuation at high frequencies, but at the expense of increasing the phase lag in low or middle frequencies, which carries the possibility of worsening the phase margin. By inserting a frequency weighting in the loss function of a linear quadratic problem with the same amplitude characteristic as the usual phase lag compensation, we are able to get a form of lag compensation in the optimal system also. The general effect is to diminish robustness at low frequencies and improve it at high frequencies.

The advantage of using the linear quadratic approach is that stability is automatically taken care of, and also the extension to multivariable systems is straightforward.

Appendix A: Verification of Detectability and Stabilizability

It is a standard result that $[F,H']$ is detectable if and only if[10]

$$Fw=\lambda w, \quad H'w=0, \quad w\neq 0$$

implies $Re[\lambda]<0$.

Suppose then that

$$\begin{bmatrix} -\beta^{-1} & 0 \\ b(\beta^{-1}-\alpha\beta^{-2}) & A \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}=\begin{bmatrix} \lambda w_1 \\ \lambda w_2 \end{bmatrix}$$

$$[0 \;\; c']\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}=0$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\neq 0$$

If $w_1\neq 0$, $\lambda=-\beta^{-1}<0$. If $w_1=0$, then

$$Aw_2=\lambda w_2, \qquad c'w_2=0, \qquad w_2\neq 0$$

and detectability of $[A,c']$ implies $Re[\lambda]<0$.

The stabilizability property is proved in the same way.

Appendix B: The Asymptotic Behavior of the Loop Gain

Consider the Kalman-Bucy filtering problem for the system $\dot{x}=Fx+gu$, $y=h'x+v$, where u and v are unit intensity independent white noise processes and $Re\lambda_i(F)<0$ for all i. The mean square error in estimating y is $h'P_fh$, where P_f is the solution of the steady-state Riccati equation. The mean square error is also given by the following formula (usually associated with Wiener filtering, but still applicable[11]):

$$h'P_fh=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln\left[1+|h(j\omega I-F)^{-1}g|^2\right]d\omega \qquad \text{(B1)}$$

Accordingly, by duality, if A, B, c define a conventional regulator problem with P as the solution of the steady-state Riccati equation, then so long as $Re\lambda_i(A)<0$, there holds

$$b'Pb=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln[1+|c'(j\omega I-A)^{-1}b|^2]d\omega \qquad \text{(B2)}$$

Let us now consider the variation to this formula when $Re\lambda_i(A)<0$ fails. Recall that we are assuming $[A,b]$ and $[A,c]$ are controllable. Temporarily, suppose that A has no purely imaginary eigenvalues. Let $\bar{A}$ be such that $Re\lambda_i(\bar{A})<0$ and

$$|c'(j\omega I-\bar{A})^{-1}b=|c'(j\omega I-A)^{-1}b| \qquad \text{(B3)}$$

We could construct $\bar{A}$ in the following way. Without loss of generality, suppose that $[A,b]$ is in the controllable canonical form, thus

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & \cdot & \\ & & & \cdot & \\ & & & & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n \end{bmatrix}$$

$$b=\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \qquad c'=[c_1 \; c_2 \ldots c_n] \qquad \text{(B4)}$$

so that

$$c'(sI-A)^{-1}b=(c_n s^{n-1}+c_{n-1}s^{n-2}+\ldots+c_1)$$
$$\times(s^n+\alpha_n s^{n-1}+\ldots+\alpha_1)^{-1}$$

Define a polynomial $s^n+\bar{\alpha}_n s^{n-1}+\ldots+\bar{\alpha}_1$ as one with zeros consisting of the zeros in $Re[s]<0$ of $s^n+\alpha_n s^{n-1}+\ldots+\alpha_1$ together with the negatives of the zeros in $Re[s]>0$. Let $\bar{A}$ be the associated companion matrix. The return difference equality yields

$$[1+b'(-sI-\bar{A}')^{-1}\bar{P}b][1+b'\bar{P}(sI-\bar{A})^{-1}b]$$
$$=1+b'(-sI-\bar{A}')^{-1}cc'(sI-\bar{A})^{-1}b \qquad \text{(B5)}$$

If

$$1+b'\bar{P}(sI-\bar{A})^{-1}b=(s^n+\bar{\beta}_n s^{n-1}+\ldots+\bar{\beta}_1)$$
$$\times(s^n+\bar{\alpha}_n s^{n-1}+\ldots+\bar{\alpha}_1)^{-1}$$

then $s^n+\bar{\beta}_n s^{n-1}+\ldots+\bar{\beta}_1$ must have all zeros in $Re[s]<0$, see Ref. 7. Also Eq. (B5) implies

$$[(-s)^n+\bar{\beta}_n(-s)^{n-1}+\ldots+\bar{\beta}_1][s^n+\bar{\beta}_n s^{n-1}+\ldots+\bar{\beta}_1]$$
$$=[(-s)^n+\bar{\alpha}_n(-s)^{n-1}+\ldots+\bar{\alpha}_1][s^n+\bar{\alpha}_n s^{n-1}+\ldots+\bar{\alpha}_1]$$
$$+[c_n(-s)^{n-1}+c_{n-1}(-s)^{n-2}+\ldots+c_1]$$
$$\times[c_n s^{n-1}+c_{n-1}s^{n-2}+\ldots+c_1]$$

Writing a similar equality based on $1+b'P(sI-A)^{-1}b$, using the relation between $s^n+\bar{\alpha}_n s^{n-1}+\ldots+\bar{\alpha}_1$ and $s^n+\alpha_n s^{n-1}+\ldots+\alpha_1$, and using the zero restriction property of $s^n+\bar{\beta}_n s^{n-1}+\ldots+\bar{\beta}_1$ and $s^n+\beta_n s^{n-1}+\ldots+\beta_1$ yields

$$s^n+\bar{\beta}_n s^{n-1}+\ldots+\bar{\beta}_1=s^n+\beta_n s^{n-1}+\ldots+\beta_1 \qquad \text{(B6)}$$

Now

$$b'\bar{P}b=\lim_{s\to\infty} sb'\bar{P}(sI-\bar{A})^{-1}b=\bar{\beta}_1-\bar{\alpha}_1=\beta_1-\bar{\alpha}_1$$

and

$$b'Pb=\beta_1-\alpha_1$$

Suppose now that the eigenvalues of $\bar{A},A$ are $\bar{\lambda}_1, \bar{\lambda}_2,\ldots\bar{\lambda}_n$ and $\lambda_1=-\bar{\lambda}_1,\lambda_2=-\bar{\lambda}_2,\ldots,\lambda_i=-\bar{\lambda}_i,\lambda_{i+1}=\bar{\lambda}_{i+1},\ldots,\lambda_n=\bar{\lambda}_n$. Then

$$-\bar{\alpha}_1=\sum_{j=1}^{n}\bar{\lambda}_j=-\sum_{j=1}^{i}\lambda_j+\sum_{i=1}^{n}\lambda_j$$

while

$$-\alpha_1=\sum_{1}^{n}\lambda_j$$

and so

$$b'Pb=b'\bar{P}b+\bar{\alpha}_1-\alpha_1=b'\bar{P}b+2\sum_{j=1}^{i}\lambda_j$$
$$=b'\bar{P}b+2\sum_{j=1}^{i}Re(\lambda_j)$$

(Note that $Re(\bar{\lambda}_j)<0, 1\leq j\leq i$; hence $b'Pb>b'\bar{P}b$.) The overall conclusion is now

$$b'Pb=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln[1+|c'(j\omega I-A)^{-1}b|^2]\,d\omega+2\sum_{j=1}^{i}Re\lambda_j \qquad \text{(B7)}$$

where $\lambda_1,\ldots,\lambda_i$ are the unstable eigenvalues of A.

Now let A have purely imaginary eigenvalues. We shall show that Eq. (B7) remains valid. First, note that it is not hard to show that the integral in Eq. (B7) is still well defined. Let $A(\epsilon)$ depend continuously on ϵ with $A(0)=A$ and for $0<\epsilon<\epsilon_0$, $A(\epsilon)$ possessing no purely imaginary eigenvalues. Then

$$b'P_\epsilon b=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln[1+|c'(j\omega I-A_\epsilon)^{-1}b|^2]\,d\omega$$
$$+2\sum_{j=1}^{i}Re\lambda_j[A(\epsilon)]$$

(the summation occurring over those λ_j having a positive real part). Letting $\epsilon\to 0$ and using the continuity with ϵ of P_ϵ and the integral, Eq. (B7) is established.

Note that in the regulator problem, the loop gain is $-b'P\times(j\omega I-A)^{-1}b$ and so the asymptotic behavior of the loop gain magnitude is $b'Pb\omega^{-1}$.

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