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On the convergence of a model reference adaptive control algorithm with unknown high frequency gain

Brian D.O. ANDERSON,
Soura DASGUPTA, A.C. TSOI

Dept. of Systems Engineering, Australian National University, GPO Box 4, Canberra A.C.T. 2601, Australia

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A model reference adaptive control algorithm is shown not to be exponentially convergent when the high frequency gain of the plant is unknown. The implication is that such an algorithm will be lacking in some way in robustness.

Keywords: Adaptive control, Model reference adaptive control, Robust adaptive control.

1. Introduction

In this paper we examine a model reference adaptive control scheme, proposed by Morse [1], for exponential convergence of parameter estimates. The need for exponential convergence, we believe, is important from the standpoint of robustness. Adaptive algorithms without such convergence have been known to exhibit unacceptable behaviour in the presence of noise and modelling inadequacies [2,3]; in particular parameter estimates can drift to the point where the closed loop becomes instantaneously unstable. The resulting unstable behaviour, containing additional modes, may allow correction of the parameter estimates to the point where the closed loop becomes instantaneously stable. Thus bursts of oscillation are observed. Exponentially convergent algorithms on the other hand are totally stable [4, pp. 107–108], a property which allows them to retain stability in face of modest departures from ideality.

In [1] the algorithm in question has been shown to be globally stable. Yet, while the tracking error converges to zero, the parameter estimates do not necessarily converge to their desired values. However, when the high frequency gain is known *a priori*, the exponential convergence of parameter estimates results whenever the reference input is persistently exciting (p.e.) [5]. In this note we consider the case where the high frequency gain is unknown. Rather surprisingly we discover that in such a case the structure of the algorithm precludes exponential convergence *even with sufficiently rich reference inputs*. The problem appears to lie in the identification of the gain parameter. The signal central to its identification converges asymptotically to zero and consequently loses persistence of excitation.

The structure of this note is the following. Section 2 recounts the essentials of Morse's algorithm. Section 3 derives a key stability lemma, which establishes a necessary condition for exponential convergence and Section 4 demonstrates how Morse's algorithm violates this condition.

2. The control algorithm

In this section we briefly outline the philosophy and nature of Morse's algorithm, adhering closely to the terminology employed in [1], but omitting details irrelevant to the course of our development.

We note that the algorithm ideas of [1] obviously drew on the earlier paper [6,7]; the ideas of [7] were modified and developed with a part-stability proof in [8], published simultaneously with [1], and with the stability proof completed in [9]. Reference [10] also completes this stability proof. Because our concern is with the stability problem, we shall stay with the algorithm of [1], for which a stability proof was simultaneously published.

Consider a single-input single-output plant, modelled by a strictly proper transfer function

$$T_p(s) = \frac{g_p \alpha_p(s)}{\beta_p(s)} \tag{1}$$

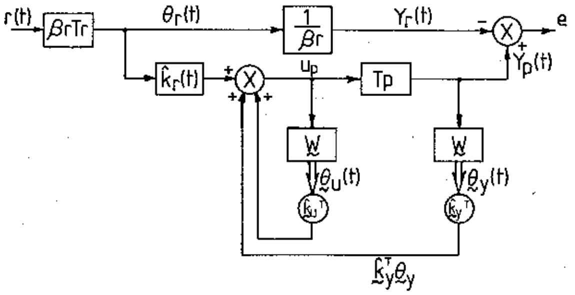

Fig. 1. A block diagram representation of Morse's algorithm.

having degree and relative degree of n and n^* respectively; g_p is a nonzero constant with known sign (assumed positive) and α_p and β_p are monic, coprime polynomials and $\alpha_p(s)$ is strictly stable as well. It is assumed that the output $y(t)$ is required to track a reference trajectory $y_r(t)$, itself the output of a known, stable transfer function T_r, having relative degree no smaller than n^* (as otherwise explicit differentiation would be required) and a reference input $r(t)$.

Consider next the scheme depicted in Figure 1. Here $1/\beta_r$ is an arbitrary, known, stable, all pole transfer function of degree n^*, $\boldsymbol{W}(s)$ is a stable transfer function, $\boldsymbol{\theta}^T = (\boldsymbol{q}_u^T, \boldsymbol{\theta}_y^T, \boldsymbol{\theta}_r)$ is an auxiliary signal vector and $\hat{\boldsymbol{k}} = (\hat{\boldsymbol{k}}_u^T, \hat{\boldsymbol{k}}_y^T, \hat{\boldsymbol{k}}_r^T)$ is a parameter estimate vector. When $\hat{\boldsymbol{k}} = \boldsymbol{k}_p$ the transfer function relating θ_r to y_p equals $1/\beta_r$, with $\boldsymbol{k}_{up}$, $\boldsymbol{k}_{yp}$ and $k_{rp} = 1/g_p$ serving to assign the zeros, poles and gain of the plant respectively.

Defining $\boldsymbol{k} = \hat{\boldsymbol{k}} - \boldsymbol{k}_p$ we can redraw Figure 1 as Figure 2 with the transfer function representing the system within the dotted box equalling $1/\beta_r(s)$. Accordingly we have the error model of Figure 3

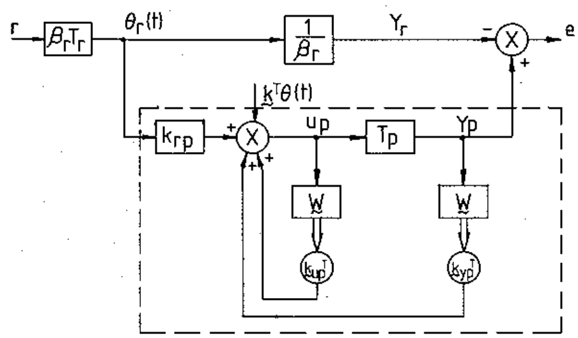

Fig. 2. An equivalent representation of Morse's algorithm.

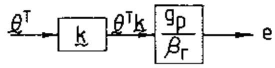

Fig. 3. An error model for Morse's algorithm.

as

$$e = y - y_r$$

$$= \frac{1}{\beta_r}\left(\theta_r + g_p \boldsymbol{k}^T\boldsymbol{\theta}\right) - \frac{1}{\beta_r}\theta_r = \frac{g_p}{\beta_r}\boldsymbol{k}^T\boldsymbol{\theta}.$$

Now if we are to identify k by the usual gradient algorithm, existing results tell us [11,12] that $1/\beta_r$ will need to be strictly positive real, a condition clearly unattainable for $n^* > 1$. In [1] this problem is circumvented by adding an auxiliary signal so that the error model for this augmented signal e'' approaches that in Figure 4 asymptotically. This auxiliary signal is $\psi(t)\hat{g}(t)$ in Figure 5 and the augmented error equals e''. Clearly

$$\psi = -\frac{1}{\beta_r}\left(\boldsymbol{\theta}^T(\boldsymbol{k}_p + \boldsymbol{k})\right) + \boldsymbol{\phi}^T(\boldsymbol{k}_p + \boldsymbol{k})$$

$$= \boldsymbol{\phi}^T\boldsymbol{k}(t) - \frac{1}{\beta_r}\boldsymbol{\theta}^T(t)\boldsymbol{k}(t).$$

Hence, $\boldsymbol{\phi} = \mathrm{diag}(\beta_r^{-1}I)\boldsymbol{q}$, as is evident from Figure 3. Thus if $\hat{g}(t) \equiv g_p$, then $e'' = g_p\boldsymbol{\phi}^T\boldsymbol{k}(t)$. However in general for the unknown gain algorithm, this will not hold.

Given below are the equations which define the signals in Figure 5. Let $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be a minimal realization with

$$\boldsymbol{c}^T(sI - \boldsymbol{A})^{-1}\boldsymbol{b} = 1/\beta_r, \tag{2}$$

$$\boldsymbol{\phi}^T = \boldsymbol{c}^T\boldsymbol{H}, \tag{3}$$

$$\dot{\boldsymbol{H}} = \boldsymbol{A}\boldsymbol{H} + \boldsymbol{b}\boldsymbol{\theta}^T, \tag{4}$$

$$\dot{z} = \boldsymbol{A}z + \boldsymbol{b}\hat{\boldsymbol{k}}^T\boldsymbol{\theta}, \tag{5}$$

$$\psi = \hat{\boldsymbol{k}}^T\boldsymbol{\theta} - \boldsymbol{c}^T z, \tag{6}$$

$$\dot{\hat{g}} = -q\psi\bar{e}, \quad q > 0, \tag{7}$$

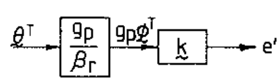

Fig. 4. Error model *not requiring strictly positive real condition* for convergence.

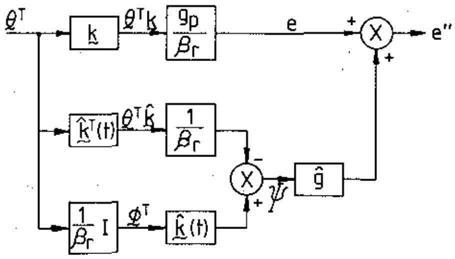

Fig. 5. Augmented error model.

$$\dot{k} = -Q\theta\bar{e}, \quad Q > 0, \tag{8}$$

$$\bar{e} = \frac{1}{\lambda_0 + \phi^T Q \phi}(\hat{g}\psi + e). \tag{9}$$

Remark. The signal $\bar{e}$ in (9) is actually e'' (see Figure 5) divided by $(\lambda_0 + \phi^T Q \phi)$.

3. A stability lemma

The following lemma proves crucial in the derivation of the main result we are after. A discrete time version of the lemma appeared in [13].

Lemma. *Consider the differential equation*

$$\dot{\omega} = G(\omega, t)\omega \tag{10}$$

with $G(\omega, t)$ continuous in ω with a Lipschitz condition uniform in t and $\omega(\cdot) \in R^n$. Assume that initial conditions on ω lie in some arbitrarily large but bounded region. Then the linearization of (10) *about the zero trajectory is exponentially stable whenever* (10) *is exponentially stable.*

Proof. Exponential stability of (10) implies the existence of a Lyapunov function $V(\omega, t)$ and $c_1, \ldots, c_4 > 0$ [14, p. 86] such that

$$c_1\|\omega\|^2 \leq V(\omega, t) \leq c_2\|\omega\|^2, \tag{11}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=0}^{n} \frac{\partial V}{\partial \omega_i} f_i(\omega, t) \leq -c_3\|\omega\|^2, \tag{12}$$

$$\left|\frac{\partial V}{\partial \omega}\right| \leq c_4\|\omega\|. \tag{13}$$

Consider any two possible initial conditions of (10), $\omega_1(0)$ and $\omega(0)$, both bounded in magnitude by K. Now, there exists a $c_5 > 0$ such that

$$\|G(\omega_1(t), t) - G(\omega(t), t)\|$$
$$\leq c_5\|\omega_1(t) - \omega(t)\| \leq c_5\{\|\omega_1(t)\| + \|\omega(t)\|\}.$$

Since (10) is exponentially stable both $\|\omega_1(t)\|$ and $\|\omega(t)\|$ decay exponentially to zero. Thus there is a T, depending on K, such that for all $t \geq T$,

$$\|\omega_1(t)\| + \|\omega(t)\| \leq (1 - q')c_3/(nc_4c_5)$$

with c_3, c_4 and c_5 defined as above and $0 < q' < 1$.

Thus for all $t \geq T$,

$$\|G(\omega_1(t), t) - G(\omega(t), t)\| \leq \frac{(1 - q')c_3}{nc_4}, \tag{14}$$

Now, as shown in [14, p. 86] the perturbed system

$$\dot{\omega}(t) = G(\omega, t)\omega + R(\omega, t)$$

will be exponentially stable if

$$|R_i(\omega(t), t)| \leq \frac{(1 - q')c_3\|\omega\|}{nc_4}, \quad i = 1, 2, \ldots, n.$$

Thus using (14), for all sufficiently large t,

$$\begin{aligned}\dot{\omega} &= G(\omega(t), t)\omega \\ &\quad + (G(\omega_1(t), t)\omega - G(\omega(t), t)\omega) \\ &= G(\omega_1(t), t)\omega\end{aligned} \tag{15}$$

is exponentially stable by (14). Now, suppose $\omega_1(0) = 0$. Then clearly $\omega_1(t) = 0$ $\forall t \geqslant 0$. Thus by (15) $\dot{\omega} = G(0, t)\omega$ is exponentially stable as well. □

4. Stability analysis

We now demonstrate using the above lemma that Morse's algorithm with unknown g_p cannot be exponentially convergent. It has been shown in [1] that the error model reduces to

$$\begin{aligned}\dot{x} &= A_1 x + b_1(k^T\theta + r), \\ \theta &= c_1 x + d_1 r + \bar{\varepsilon}(t), \\ \dot{H} &= AH + b\theta^T, \\ \phi^T &= c^T H, \\ \dot{z} &= Az + bk^T\theta, \\ \psi &= k^T\phi - c^T z, \\ \dot{k} &= -Q\phi\bar{e}, \\ \dot{g} &= -q\psi\bar{e}, \\ \bar{e} &= \frac{1}{\lambda_0 + \phi^T Q\phi}\left[g_p k^T\phi + g\psi + \varepsilon(t)\right],\end{aligned} \tag{16}$$

where $[A_1, b_1, c_1, d_1]$ is a stable but unknown system and $\bar{\varepsilon}(t)$ and $\varepsilon(t)$ are linear combinations of exponentially decaying signals.

Theorem. *Consider equation* (16), *Define* $x_0(t)$, $H_0(t)$ *and* $\phi_0(t)$ *as the values obtained for* $x(\cdot)$, $H(\cdot)$ *and* $\phi(\cdot)$ *when* $k \equiv 0$, $g = \hat{g} - g_p \equiv 0$ *and* $\tilde{x}$, $\tilde{H}$ *and* $\tilde{\phi}$ *as* $x - x_0$, $H - H_0$ *and* $\phi - \phi_0$ *respectively. Then for a bounded and piecewise continuous reference input,*

$$\omega^T \triangleq \left[\tilde{x}^T, \tilde{\phi}^T, \tilde{\bar{H}}^T, z^T, k^T, g\right]$$

is not exponentially stable, with

$$\tilde{\bar{H}}^T \triangleq \left[\tilde{h}_1^T, \tilde{h}_2^T, \dots, h_{2n+1}^T\right],$$

$\tilde{h}_i$ *being the i-th column of* $\tilde{H}$.

Proof. The error model becomes

$$\dot{\tilde{x}} = A_1 \tilde{x} + b_1 k^T \theta,$$

$$\tilde{\theta} = c_1 \tilde{x},$$

$$\dot{\tilde{H}} = A\tilde{H} + b\tilde{\theta}^T,$$

$$\dot{z} = Az + bk^T\theta,$$

$$\psi = k^T\phi - c^T z,$$

$$\dot{k} = -\frac{Q\phi}{\alpha(t)}\left(g_p k^T \phi + g\psi\right),$$

$$\dot{g} = -\frac{q\psi}{\alpha(t)}\left(g_p k^T \phi + g\psi\right),$$

where $\alpha(t) = (\lambda_0 + \phi^T Q \phi)$. In this we have neglected the exponentially decaying signals $\varepsilon(t)$ and $\bar{\varepsilon}(t)$. Thus we have

$$\dot{\omega} = G(\omega, t)\omega \tag{17}$$

with

$$G(\omega, t) = \begin{bmatrix} A_1 & 0 & 0 & b_1\theta^T & 0 \\ f(b, c) & A & 0 & 0 & 0 \\ 0 & 0 & A & b\theta^T & 0 \\ 0 & 0 & 0 & -\frac{Q\phi}{\alpha(t)} g_p \phi^T & -\frac{Q\phi}{\alpha(t)}\psi(t) \\ 0 & 0 & 0 & -\frac{q\psi}{\alpha(t)} g_p\phi^T & -\frac{q}{\alpha(t)}\psi^2 \end{bmatrix}.$$

Now by [1] for piecewise continuous and bounded $r(t)$ all entries of $G(\cdot, \cdot)$ are bounded and the $G(\cdot, \cdot)$ can easily be shown to be Lipschitz. Now the last row corresponding to the update law for $\hat{g}$ is zero when $\omega = 0$, since $\psi = 0$ when $\omega = 0$. Thus $\dot{\omega} = G(0, t)\omega$ is not exponentially stable, whence by the lemma of Section 3, (17) cannot be exponentially stable. □

Remark. (1) The intuition behind the lack of exponential stability is the following. The equations governing the update laws for $\hat{k}$ and $\hat{g}$ can be rewritten as

$$\begin{bmatrix} \dot{k} \\ \dot{g} \end{bmatrix} = -\begin{bmatrix} Q/\alpha(t) & 0 \\ 0 & q/\alpha(t) \end{bmatrix}\begin{bmatrix} \phi \\ \psi \end{bmatrix}\bar{e}$$

$$= \frac{-1}{\alpha(t)}\begin{bmatrix} Q/g_p & 0 \\ 0 & q \end{bmatrix}\begin{bmatrix} \phi \\ \psi \end{bmatrix}\begin{bmatrix} \phi^T & \psi \end{bmatrix}\begin{bmatrix} k \\ g \end{bmatrix} + \varepsilon_2(t) \tag{18}$$

where $\varepsilon_2(t)$ is exponentially decaying. Thus by a result in [11], (18) will be exponentially stable when $[\phi^T, \psi]$ is persistently exciting. However, in [1], it is shown that

$$\lim_{t \to \infty} \dot{k}(t) = 0.$$

Thus from Figure 5 it is clear that $\lim_{t \to \infty}\psi(t) = 0$. Thus persistence of excitation is being asymptotically lost.

(2) It is clear from the foregoing that the lack of exponential stability is inherent in the very structure of this algorithm. More precisely the problem lies in the fact that g_p is estimated as both $1/\hat{k}_r$ and $\hat{g}$ and the update law of $\hat{g}$ is structurally deficient in the sense that the associated component of the regression vector approaches zero. The reason for updating $\hat{g}$ in the first place is to avoid the problem of inverting $\hat{k}_r$ when the latter crosses zero. Had a lower bound on the magnitude of g_p been known this problem is avoidable [15].

(3) The convergence in the $n^* \leq 1$ case is of course trivial as then $1/\beta_r$ can be chosen as positive real and the artifact of Figure 5 becomes redundant.

5. Conclusion

We have demonstrated the lack of exponential stability of a Model Reference Adaptive Control Algorithm when the plant high frequency gain is unknown. Whether or not this applies to other model reference adaptive control algorithms is in principle an open question. However, in view of the broad similarities in the algorithms of [1,7], for example, or more generally [16], we expect the conclusion to be general for plants with relative degree greater than one.

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