Systems & Control Letters 5 (1985) 229–236
North-Holland

February 1985

Frequency-weighted optimal Hankel-norm approximation of stable transfer functions

Geoff A. LATHAM and
Brian D.O. ANDERSON

Department of Systems Engineering, Institute of Advanced Studies, Australian National University, GPO Box 4, Canberra, ACT 2601, Australia

Received 7 September 1984

This paper shows how to introduce arbitrary frequency weighting into the optimal Hankel-norm approximation problem for scalar, finite-dimensional, linear, time-invariant systems. The procedure has applications to the closed-form design of reduced-order controllers.

Keywords: Order reduction, System approximation, Hankel-norm approximation, Frequency-weighted approximation.

1. Introduction

The present paper presents a method of frequency shaping the error obtained by performing an optimal Hankel-norm approximation of a scalar, finite-dimensional, linear, time-invariant system.

The optimal Hankel-norm approximation procedure finds a transfer function (or transfer function matrix) of prescribed order which approximates a given transfer function (or transfer function matrix) of greater order [1–3].

One motivation for frequency weighting comes from the desire to implement a reduced-order approximating controller within a closed-loop control system. Suppose an LQG designed series compensator is to be used in a control system implementation. The compensator will have the same dimension as the plant model. For simplicity, it is desirable to approximate the compensator by one of the lowest order possible while maintaining an overall acceptable degradation in performance. The approximating compensator should be obtained in a way that takes account of the frequency characteristics of the plant model. For example in the plant stop band and at frequencies of high loop gain, the detailed shape of the approximating compensator is not so important, however around the unity gain crossover frequency, it is desirable to resume accurate approximation.

We show here how to modify the approximation method originally developed by Adamjan, Arov and Krein [1] to allow for frequency weighting. The means for introducing frequency weighting while preserving the closed form solvability of the optimal Hankel-norm approximation problem is not immediately apparent. A technique for doing this is described in the paper.

2. Discrete-time approximation

We first establish some preliminary notation and results.

Definition 1. Let $G(z)$ be a strictly minimum phase, strictly stable, real rational scalar transfer function, i.e. $G(z)$ has no poles or zeros in $|z| \geqslant 1$, save for a possible zero at $z=\infty$. Let $r \geqslant 0$ be the smallest integer such that $\lim_{z\to\infty} z^r G(z)$ is nonzero. Then define the tilde operation by

$$\tilde{G}(z) \triangleq z^{-r} G(z^{-1}). \tag{2.1}$$

It may be noted here that $\tilde{G}$ and $\tilde{G}^{-1}$ are analytic in the closed unit disc.

Definition 2. Let $K(z)$ be a rational transfer function with Laurent expansion

$$K(z) = \sum_{i=-\infty}^{\infty} k_i z^{-i} \tag{2.2}$$

which converges in some open region containing the unit circle, $|z| = 1$.

Then define the Hankel matrix associated with $K(z)$, denoted $\Gamma(K(z))$, as the matrix with elements

$$\Gamma(K(z))_{ij} = k_{i+j-1}, \quad i,\ j = 1, 2, \ldots . \tag{2.3}$$

Note that $\Gamma(K(z))=\Gamma([K(z)]_+)$ where $[K(z)]_+$ denotes the strictly stable part of $K(z)$, i.e. the sum of those terms in a partial fraction expansion of $K(z)$ which have poles in $|z|<1$.

The tilde operation (2.1) has the desirable property of preserving the rank of the Hankel matrix $\Gamma(K)$ under the change $K \to K\tilde{G}$. This property is crucial and is proved in the following lemma.

Lemma 1. *Let $H(z)$ be a strictly stable proper rational transfer function and let $G(z)$ satisfy the conditions of Definition* 1 *and have Laurent expansion*

$$G(z)=\sum_{i=r}^{\infty} g_i z^{-i}, \quad g_r \neq 0,$$

which converges in an open region containing the unit circle. Then

(i) $\operatorname{rank} \Gamma(H(z)) = \operatorname{rank} \Gamma(H(z)\tilde{G}(z))$, (2.4)

(ii) $\Gamma(H\tilde{G}) = \Gamma(H)T(\tilde{G}) = T'(\tilde{G})\Gamma(H)$, (2.5)

where $T(\tilde{G})$ is the infinite lower triangular Toeplitz matrix with elements

$$(T(\tilde{G}))_{ij}=\begin{cases} g_{i-j+r}, & i \geqslant j, \\ 0, & i<j, \end{cases} \quad i,j=1,2,\ldots,$$

and $\Gamma(H)$ is as defined by (2.3). *Here $T'(\tilde{G})$ denotes the transpose of $T(\tilde{G})$.*

Proof. (i) Let $\delta_+(F(z)) \triangleq \delta([F(z)]_+)$ be the McMillan degree of the strictly stable part of $F(z)$. Expanding by partial fractions, we divide $H\tilde{G}$ into its strictly stable and unstable parts. We then have

$$\begin{aligned}\delta_+(H\tilde{G}) &= \delta_+([H\tilde{G}]_+) + \delta_+([H\tilde{G}]_-) \\ &= \delta_+([H\tilde{G}]_+).\end{aligned}$$

Observe now that by Definition 1, G has no stable pole and no stable pole of H can be cancelled by a zero of $\tilde{G}$. Hence $\delta_+([H\tilde{G}]_+)=\delta_+(H)$ and so by a theorem of Kronecker [4],

$$\operatorname{rank} \Gamma(H\tilde{G}) = \operatorname{rank} \Gamma(H).$$

(ii) The proof of (2.5) is computational. Let $H(z)$ have Laurent expansion

$$H(z)=\sum_{j=0}^{\infty} h_j z^{-j}.$$

The expansion of $\tilde{G}(z)$ is

$$\tilde{G}(z)=\sum_{k=0}^{\infty} g_{k+r} z^k,$$

thus

$$[H\tilde{G}(z)]_+ = \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} g_{k+r} h_{k+j} z^{-j},$$

so that

$$(\Gamma(H\tilde{G}))_{ij} = \sum_{k=0}^{\infty} g_{k+r} h_{i+j+k-1}, \quad i,j=1,2,\ldots.$$

The centre member of (2.5) is

$$(\Gamma(H)T(\tilde{G}))_{ij} = \sum_{k=1}^{\infty} h_{i+k-1} g_{k-j+r},$$

but $g_{k-j+r}=0$ if $j>k$. Substituting, this gives

$$\begin{aligned}(\Gamma(H)T(\tilde{G}))_{ij} &= \sum_{k \geqslant j} h_{i+k-1} g_{k-j+r} \\ &= \sum_{k\geqslant 0} h_{i+j+k-1} g_{k+r},\end{aligned}$$

which proves the result.

To prove the second equality of (2.5) we take the tranpose of $\Gamma(H\tilde{G})=\Gamma(H)T(\tilde{G})$ and note that Hankel matrices are symmetric.

In Lemma 1(i) we used the strictly minimum phase nature of G to ensure that no stable pole-zero cancellations occur in $H\tilde{G}$. If we define $\tilde{G}$ for stable non-minimum-phase G exactly as in (2.1), then all that is required of G for (2.4) to hold is that $\tilde{G}$ have no zeros coinciding with with stable poles of H. The strictly minimum phase property of G is however essential as it will allow us to use the following lemma.

Lemma 2. *Let $G(z)$ be strictly stable. Then $T(\tilde{G})$ is invertible if and only if G is strictly minimum phase and then $T(\tilde{G})^{-1} = T(\tilde{G}^{-1})$.*

Proof. This is a standard result and the proof may be found for example in [5].

We now restate the main result of [1], recalling that if $F(z)$ is a proper stable rational transfer function of McMillan degree n, then its associated Hankel matrix $\Gamma(F)$ has rank n. The result states that for any positive integer $k < n$ there exists a unique bounded Hankel matrix $\bar{\Gamma}$ of rank k such that

$$\|\Gamma(F) - \bar{\Gamma}\| = \sigma_{k+1}(F) \tag{2.6}$$

where $\sigma_{k+1}(F)$ is the $(k+1)$-st singular value of $\Gamma(F)$ (the singular values being ordered in descending magnitude). Further the matrix $\bar{\Gamma}$ is the Hankel matrix of some rational $\chi(z)$ with

$$\chi(z) = F(z) - \sigma_{k+1}(F)\phi(z) \tag{2.7}$$

where $\phi(z)$ is an all-pass function with exactly k strictly stable poles. The function $\chi(z)$ is a unique best L^∞-approximation to $F(z)$ having exactly k strictly stable poles and possibly some unstable poles, that is,

$$\|F(z) - \chi(z)\|_\infty = \sigma_{k+1}(F) \tag{2.8}$$

and $\delta_+(\chi(z)) = k$.

Here the L^∞-norm is taken on the unit circle. The transfer function $[\chi(z)]_+$ can be regarded as an approximation to $F(z)$ which is stable and has precisely k poles.

To make use of the explicit solution afforded by (2.6) and (2.7) we set $F(z) = H\tilde{G}(z)$ where $G(z)$ and $H(z)$ are as in Definition 1 and Lemma 1 respectively. Using Lemmas 1 and 2, the difference $\Gamma(H\tilde{G}) - \Gamma(\chi)$ may be written as

$$\left(\Gamma(H) - \Gamma(\chi\tilde{G}^{-1})\right)T(\tilde{G}) \tag{2.9}$$

which satisfies

$$\left\|\left(\Gamma(H) - \Gamma(\chi\tilde{G}^{-1})\right)T(\tilde{G})\right\| = \sigma_{k+1}(H\tilde{G}). \tag{2.10}$$

The bracketed expression in (2.10) suggests identifying $\Gamma(\chi\tilde{G}^{-1})$ as a rank k approximation to $\Gamma(H)$, the function $\chi\tilde{G}^{-1}(z)$ as an L^∞-approximation to $H(z)$, and the function $[\chi\tilde{G}^{-1}(z)]_+$ as a stable reduced-order approximation to $H(z)$. Note now that $\Gamma(\chi G^{-1})$ although not an optimal Hankel-norm approximation of $\Gamma(H)$, has the appropriate rank and incorporates the weighting G in a manner that preserves the structure of the problem. Defining

$$\begin{aligned} W = \chi\tilde{G}^{-1} &= \left(H\tilde{G} - \sigma_{k+1}(H\tilde{G})\phi(z)\right)\tilde{G}^{-1} \\ &= H - \sigma_{k+1}(H\tilde{G})\phi\tilde{G}^{-1}, \end{aligned} \tag{2.11}$$

we can take

$$\bar{W} = [W]_+ = H - \sigma_{k+1}(H\tilde{G})\left[\phi\tilde{G}^{-1}\right]_+ \tag{2.12}$$

as a stable k-th order approximation to $H(z)$.

The importance of having $\tilde{G}$ with unstable poles and zeros should be clear from analysis of the algorithm presented above. Let us however give further intuition. Let us define L_2^- by $\eta(z) \in L_2^-$ if

$$\eta(z) = \sum_{i=-\infty}^{0} \eta_i z^{-i}, \qquad \sum_{i=-\infty}^{0} \eta_i^2 < \infty. \tag{2.13}$$

Then we have [6]

$$\|F - \chi\|_H = \sup_{\substack{\hat{\eta}(z):\, \|\hat{\eta}\|_2 = 1 \\ \hat{\eta}(z) \in L_2^-}} \left\|\left([F(z) - \chi(z)]_+ \eta(z)\right)_+\right\|_2 \tag{2.14}$$

so that $\|\cdot\|_H$ is an operator norm associated with mappings from L_2^- to L_2^+ (where L_2^+ is obviously defined). To preserve the idea of having a mapping L_2^- to L_2^+, $\tilde{G}$, which is an invertible mapping from L_2^- to L_2^-, is introduced:

$$\begin{aligned} &\|(F - \chi)\tilde{G}\|_H \\ &= \sup_{\substack{\eta(z):\, \|\eta(z)\|_2 = 1 \\ \eta(z) \in L_2^-}} \left\|\left\{[F\tilde{G} - \chi\tilde{G}]_+ \eta(z)\right\}_+\right\|_2 \\ &= \sup_{\substack{\eta(z):\, \|\eta(z)\|_2 = 1 \\ \eta(z) \in L_2^-}} \left\|\left\{[F - \chi]_+ \tilde{G}\eta(z)\right\}_+\right\|_2 \\ &= \sup_{\substack{\lambda(z):\, \|\tilde{G}^{-1}(z)\lambda(z)\|_2 = 1 \\ \lambda(z) \in L_2^-}} \left\|\left\{[F - \chi]_+ \lambda(z)\right\}_+\right\|_2. \end{aligned} \tag{2.15}$$

3. L^∞-errors of the approximation

With the approximation (2.11), which retains an unstable part, the error in the weighted gain HG is

$$HG - WG = \sigma_{k+1}(H\tilde{G})\phi\tilde{G}^{-1}G. \tag{3.1}$$

Because both ϕ and $\tilde{G}^{-1}G$ are all-pass functions, the magnitude of the difference in (3.1) is constant over all frequencies, that is

$$\left|HG(e^{i\omega}) - WG(e^{i\omega})\right| = \sigma_{k+1}(H\tilde{G}) \tag{3.2}$$

for all real ω. The role played by G as the frequency weighting is clearly displayed in this formula. Using the stable k-th order approximation of (2.12), we obtain the lower bound

$$\|HG - \bar{W}G\|_\infty \geqslant \sigma_{k+1}(H\tilde{G}). \tag{3.3}$$

Note that the actual weighting in (3.2) and (3.3) is determined by $|G|$, rather than $|G|$ and arg G. The latter is only important in that it leads to the minimum-phase, strictly stable property of G on which the algorithm rests. One could thus conceive of an arbitrary weighting function free of poles and zeros on $|z|=1$, and replace it by a strictly minimum-phase strictly stable equivalent with the same amplitude on $|z|=1$.

Although an explicit upper bound has not yet been obtained, it seems fortuitously characteristic of most examples that the maximum error is near the lower bound. The error in H using the approximation (2.11) is

$$H-W=\sigma_{k+1}(H\tilde{G})\phi\tilde{G}^{-1}$$

giving a frequency-weighted error of

$$\begin{aligned}|H(\mathrm{e}^{\mathrm{i}\omega})-W(\mathrm{e}^{\mathrm{i}\omega})|&=\sigma_{k+1}(H\tilde{G})|\tilde{G}^{-1}(\mathrm{e}^{\mathrm{i}\omega})|\\&=\sigma_{k+1}(H\tilde{G})|G^{-1}(\mathrm{e}^{\mathrm{i}\omega})|.\end{aligned} \quad (3.4)$$

A small error given by (3.4) at a particular frequency does not however guarantee a small error in $|H-\overline{W}|$. This error is bounded below by

$$\|H-\overline{W}\|_\infty \geqslant \sigma_{k+1}(H\tilde{G})\|G\|_\infty^{-1}. \quad (3.5)$$

Again, an explicit upper bound for this error has not been derived.

We note finally that if P and C are plant and nominal series compensator transfer functions respectively, the use of $G=P(1+CP)^{-1}$ seems appropriate in considering the problem of controller approximation. For if $\Delta=C-\overline{C}$ represents the error between C and an approximation $\overline{C}$, C is stabilizing and C and $\overline{C}$ have the same number of unstable poles, then $\overline{C}$ will be stabilizing when

$$\|\Delta G\|_\infty \leqslant 1. \quad (3.6)$$

4. Continuous-time case

Frequency weighting in continuous time can be performed by using the method of Section 3 after transforming to discrete time via the bilinear transformation $s=(z+1)/(z-1)$.

This is detailed in detail by Lin and Kung [2]. Note however that if $G(s)$ is proper, strictly stable and minimum phase and $\lim_{s\to\infty}G(s)$ is finite and nonzero, then the obvious analogue to Definition 1 is

$$\tilde{G}(s)=G(-s). \quad (4.1)$$

It is not immediately clear however, how to define $\tilde{G}$ for strictly proper systems since after bilinear transform, a zero at $z=1$ results. Now an optimal Hankel-norm reduction can be performed on the system

$$F(s)=[H(s)G(-s)]_+,$$

where $H(s)$ is strictly stable and proper of order n. Again $[\cdot]_+$ denotes the operation of taking the strictly stable part. The resulting k-th order approximation has the same form as (2.7). We again take

$$\tilde{W}=[\chi\tilde{G}^{-1}]_+ \quad (4.2)$$

as the stable k-th order frequency-weighted approximation to $H(s)$. Similarly error bounds to those of Section 3 apply with L^∞-norms taken on the $\mathrm{i}\omega$ axis. In particular, the corresponding result to (3.4) is

$$|H(\mathrm{i}\omega)-W(\mathrm{i}\omega)|=\sigma_{k+1}(H\tilde{G})|G(\mathrm{i}\omega)|^{-1}. \quad (4.3)$$

5. An example

As an example of the method developed in Sections 3 and 4, we apply the algorithms of Glover [3] to a continuous-time example. It should be noted however that instead of the approximation (4.2), these algorithms take

$$W_1=[\chi\tilde{G}^{-1}]_+ + D \quad (5.1)$$

as the stable k-th order approximation to $H(s)$. The choice of the constant D is specific and is explained in [3]. It is chosen to reduce the final L^∞-error.

We take for $H(s)$ a sixth-order Butherworth filter with 3dB point at $\omega=1.0$. The transfer function is $H(s)=Q^{-1}(s)$ where

$$\begin{aligned}Q(s)=s^6+3.8637s^5+7.4641s^4+9.1416s^3\\+7.4641s^2+3.8637s+1.\end{aligned}$$

The weighting is given by the second-order system

$$G_\alpha(s)=\frac{(s+1)^2}{s^2+2\alpha s+1} \quad (5.2)$$

where $\alpha\leqslant 1$ is chosen to vary $\|G_\alpha\|_\infty$.

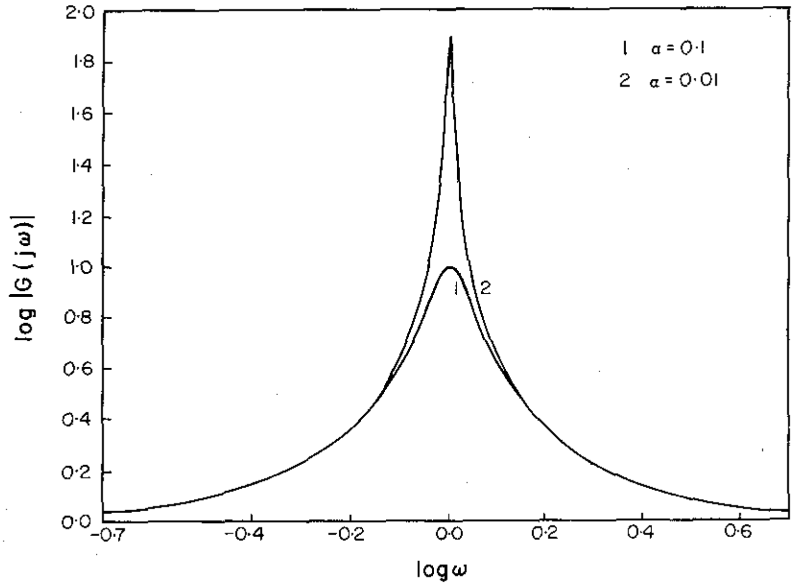

Fig. 1. The magnitude of the frequency weighting.

Fig. 1 shows the magnitudes $|G_\alpha|$ plotted against angular frequency for two particular choices of α used in this example. Curve 1 has $\alpha = 0.1$ and attains an L^∞-norm of 10, and curve 2 has $\alpha = 0.01$ with a corresponding L^∞-norm of 100. The L^∞-norms are attained at $\omega = 1.0$. For convenience of notation we denote by $E_\alpha(s)$ the error system given by

$$E_\alpha(s) = H(s) - W_1(s)$$

where W_1 is given by (5.1) and the subscript α indicates that G_α is the frequency weighting used in (5.1). Note that with this notation, $E_{1.0}(s)$ is the error obtained by performing a Hankel-norm approximation of $H(s)$ with no frequency weighting.

The singular values of the system H are

$$\bar{\sigma}_1 = 0.94707, \quad \bar{\sigma}_2 = 0.70013, \quad \bar{\sigma}_3 = 0.32544,$$
$$\bar{\sigma}_4 = 0.08278, \quad \bar{\sigma}_5 = 0.00113, \quad \bar{\sigma}_6 = 0.00630.$$

This set of singular values indicates $\bar{\sigma}_4$ as a natural cutoff ($\bar{\sigma}_4/\bar{\sigma}_5 \simeq 8$) and so we perform a fourth-order frequency-weighted optimal Hankel-norm approximation of $H(s)$.

Example 1. $\alpha = 0.1$. In this case the singular values of the system

$$F(s) = [H(s) G_{0.1}(-s)]_+$$

are

$$\sigma_1 = 2.6790, \quad \sigma_2 = 2.1589, \quad \sigma_3 = 0.84239,$$
$$\sigma_4 = 0.19287, \quad \sigma_5 = 0.021903, \quad \sigma_6 = 0.0011311.$$

Although each σ_i is larger than the corresponding $\bar{\sigma}_i$, the new set of singular values still indicates a fourth-order approximation is appropriate ($\sigma_4/\sigma_5 \simeq 9$). Fig. 2 shows the magnitude of the resulting error systems $E_{1.0}$(s) and $E_{0.1}$(s). We see that $|E_{0.1}(\mathrm{i}\omega)|$ is significantly smaller than $|E_{1.0}(\mathrm{i}\omega)|$ in a frequency band about $\omega = 1.0$ and is frequency shaped according to $|G_{0.1}^{-1}(\mathrm{i}\omega)|$, the oscillation near $\omega = 1.0$ being a typical consequence of neglecting the unstable part of the approximation in (2.11). Away from $\omega = 1.0$ the magnitude of $E_{0.1}(\mathrm{i}\omega)$ exceeds significantly that of $E_{1.0}(\mathrm{i}\omega)$; however this is a region where we can tolerate a less accurate approximation. The method of frequency weighting has thus performed better than the direct Hankel-norm approximation in the pass band of $G_{0.1}(s)$ and worse in the stop band of $G_{0.1}(s)$.

The magnitudes of the errors in the weighted gains, namely

$$|E_{1.0}(\mathrm{i}\omega) G_{0.1}(\mathrm{i}\omega)| \quad \text{and} \quad |E_{0.1}(\mathrm{i}\omega) G_{0.1}(\mathrm{i}\omega)|$$

appear in Fig. 3. We see that $\|E_{0.1} G_{0.1}\|_\infty \simeq 0.031$ which as expected is close to the lower bound in (3.3), namely $\sigma_5 = 0.0219$. In fact the magnitude of

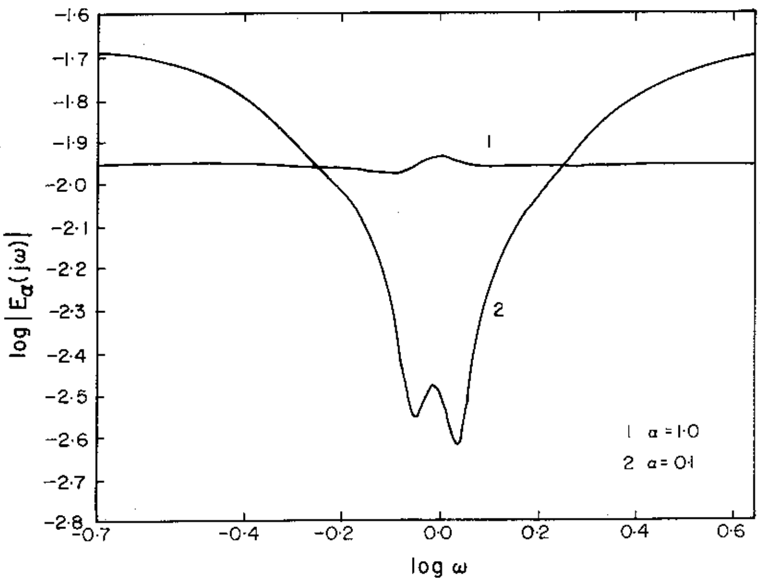

Fig. 2. A comparison of the magnitudes of the errors for the unweighted and frequency-weighted Hankel-norm approximations.

the errors over all frequencies is close to this value and again in the pass band of $G_{0.1}(s)$ it is much less than the error obtained when $E_{1.0}(s)$ is taken in series with the same frequency weighting.

Example 2. $\alpha = 0.01$. Here the singular values of $F(s) = [H(s)G_{0.01}(-s)]_+$ are

$$\sigma_1 = 3.6669, \quad \sigma_2 = 2.7631, \quad \sigma_3 = 0.94358,$$
$$\sigma_4 = 0.22032, \quad \sigma_5 = 0.024776, \quad \sigma_6 = 0.001228.$$

Fig. 4 shows the magnitudes of $E_{1.0}(s)$ and $E_{0.01}(s)$. The shape of the curve 2 is similar to Fig. 2. Although the minimum error is significantly less

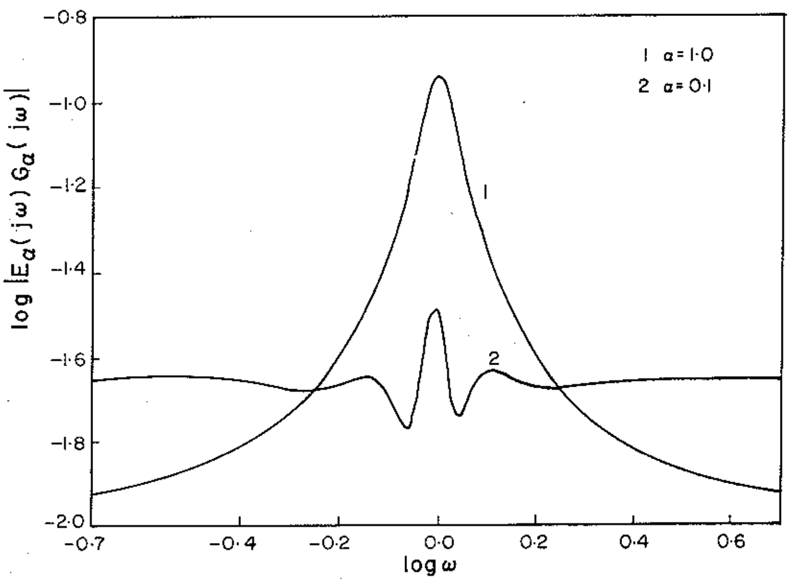

Fig. 3. A comparison of the magnitudes of the weighted error for the approximations shown in Fig 2.

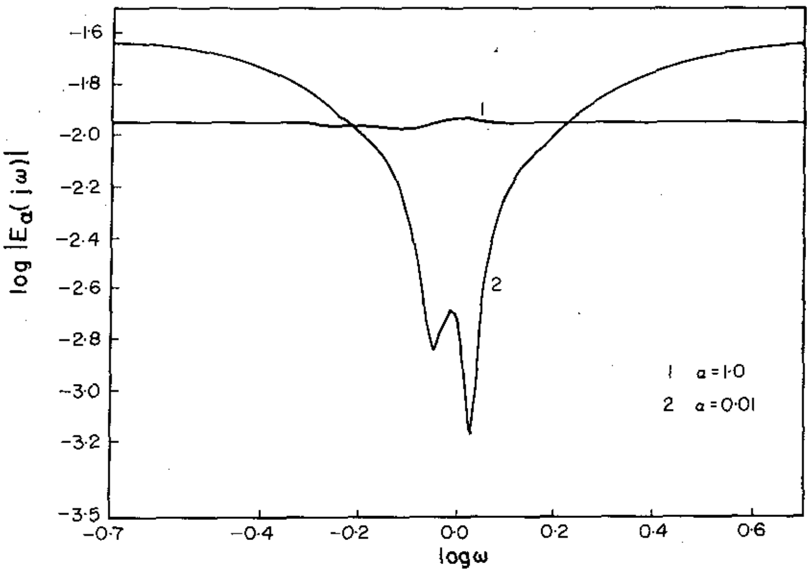

Fig. 4. The magnitudes of the errors for a frequency weighting with L^{∞}-norm 10 times that in Fig. 2.

than that for $E_{0.1}(s)$, the oscillation near $\omega = 1.0$ causes an improvement by only a factor of about 2 in the pass band of $G_{0.01}(s)$. The frequency band in which $|E_{0.01}(i\omega)| < |E_{1.0}(i\omega)|$ is less than the corresponding band in Fig 2. The L^{∞}-norm

$$\|E_{0.01}(i\omega)G_{0.01}(i\omega)\|_{\infty}$$

can be seen from Fig. 5 to be approximately 0.15 which is much greater than the lower bound in (3.3), here $\sigma_5 = 0.024226$. In the pass band of $G_{0.01}(s)$, the magnitude of $E_{0.01}(i\omega)G_{0.01}(i\omega)$ is still much less than that for the direct Hankel-norm approximation used in series with $G_{0.01}(s)$ (curve 1 of Fig. 5). The increase in the L^{∞}-norm of $E_{\alpha}(i\omega)G_{\alpha}(i\omega)$ for small α suggests that the L^{∞}-norm of the frequency weighting $G_{\alpha}(s)$ needs to be

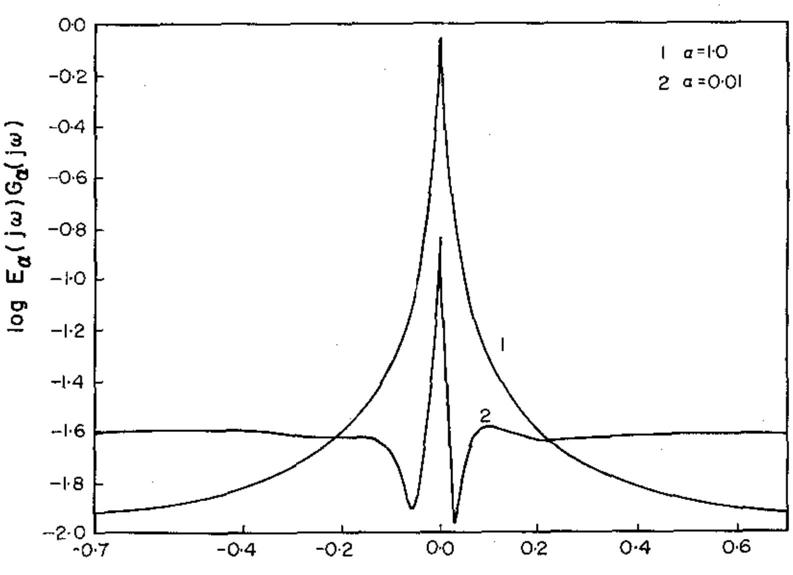

Fig. 5. The magnitudes of the weighted errors corresponding to the approximations in Fig 4.

carefully chosen to obtain a good compromise in the size of the loop gain error and the error in the approximation to $H(s)$.

6. Conclusion

We have presented a method for arbitrarily shaping the error obtained when performing a reduced-order optimal Hankel-norm approximation to a stable linear system. It has been seen that like Hankel-norm approximation without frequency weighting, the method performs well despite neglecting the unstable part of the approximation (2.11).

The most significant unanswered question about the new method is the determination of an explicit upper bound (in terms of the singular values) on the L^∞-error of the resulting stable approximation, and this is the subject of ongoing investigation. Also to be investigated is the method's performance in the context of controller reduction in a closed-loop control system. Extension to multivariable systems appears routine.

Acknowledgement

Helpful discussions with K. Glover and J. Moore are gratefully acknowledged.

References

[1] V.M. Adamjan, D.Z. Arov and M.G. Krein, Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur–Takagi problem, *Math. USSR Sbornik* **15** (1971) 31–73.

[2] D.W. Lin and S.-Y. Kung, Optimal Hankel-norm approximation of continuous-time linear systems, *Circuits Systems and Signal Processing* **1** (1982) 407–431.

[3] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds, *Internat. J. Control*, to appear.

[4] F.R. Gantmacher, *Theory of Matrices*, 2nd Ed. (Chelsea, New York, 1959).

[5] H. Widom, Toeplitz matrices, in: I.I. Hirschman, Jr., Ed., *Studies in Real and Complex Analysis*, Vol. 3 (Mathematical Association of America, 1965).

[6] S.-Y. Kung and D.W. Lin, Optimal Hankel-norm model reductions: Multivariable systems, *IEEE Trans. Automat. Control* A **26** (1981) 832–852.