Global Adaptive Pole Positioning

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Abstract—**Adaptive pole positioning for linear time-invariant discrete-time systems is considered, under the constraint that the controller consists of an identifier, the gains in which do not go to zero, an observer, and a state feedback law (the latter two elements being viewed in transfer function form). A globally convergent algorithm is presented for achieving a prescribed set of closed-loop poles. Persistency of excitation of an external input is required, together with an underbound on the magnitude of the Sylvester resultant of the plant numerator and denominator polynomials, this in effect being an underbound on the product of a measure of pole-zero separation and a generalized gain.**

I. Introduction

The aim of this paper is to give a self-contained formal solution to a global adaptive pole placement problem in discrete time. There is a time-invariant linear system with strictly proper transfer function $n(z)/d(z)$, with the polynomials $n(\cdot)$, $d(\cdot)$ having unknown coefficients. However, one knows: 1) the degrees s, t, of $d(z)$, $n(z)$ and that these are coprime; 2) that $d(z)$ is monic, together with other data noted below. Adaptive pole placement is also considered in, e.g., [1]–[15] but as far as we know no global robust solution is available for general discrete-time plants other than that of this paper. (This paper in fact provides a generalization of a global solution procedure for the first-order problem described in [15]. For those familiar with [15], the results should not be a great surprise.)

The aim is to adaptively determine a controller defined by

$$u_k = u_k^* + \frac{\bar{k}(z)}{z^{s-1}}\, u_k + \frac{\bar{h}(z)}{z^{s-1}}\, y_k \tag{1.1}$$

where $\deg \bar{k}(z) < s-1$, $\deg \bar{h}(z) \leq s-1$, and $\bar{h}(z)$, $\bar{k}(z)$ satisfy

$$d(z)\bar{k}(z) + n(z)\bar{h}(z) = z^{s-1}[d(z) - f(z)]. \tag{1.2}$$

Here f is a degree s monic polynomial with zeros in $|z| < 1$. As is easily verified, this controller produces a closed-loop transfer function $n(z)/f(z)$ [16].

The plants we shall consider are in the first instance noiseless and time-invariant, and there is an external input u_k^* which is persistently exciting. (A precise definition is given subsequently; roughly u_k^* must be frequency rich.) As many individuals have pointed out to the authors, a formal solution to the adaptive pole positioning problem then follows immediately: one runs the plant with no feedback control for a finite interval long enough to guarantee exact identification (this is possible because the input is persistently exciting). Then one implements the correct exact control thereafter. Although formally correct, we wish to reject such a solution here since it disallows the taking into account of several practical considerations—for example, if the real plant is actually time-varying, there is a need to update the knowledge of the plant, rather than to rely on a single identification to begin with. What we need is a solution to the noiseless, time-invariant problem which will still maintain some measure of applicability in the face of noise or time-variation, i.e., a robust solution.

We shall seek such a solution by modifying a known nonglobal result. By assuming that the external control signal u_k^* is persistently exciting (as defined subsequently) and the plant poles and zeros are separated by a certain amount (the measure being defined subsequently) we find that a simple modification to the known nonglobal algorithm suffices to get over "difficult" patches in the algorithm and turn nonglobal algorithms into a global one. The modification involves connecting one or more fixed controllers for a limited period.

In Section II the nonglobal result is reviewed and marginal extensions stated in Section III. In Section IV we list together all assumptions required and state the adaptive control algorithm. We also state the theorem which confirms the validity of the proposed algorithm. Sections V and VI provide the proof of the main theorem, and Section VII contains concluding remarks.

One might well ask the question: why do adaptive pole positioning? The answer to that question can well be pursued by asking the question: why do pole positioning? A long answer is out of place. Let it simply be noted that, especially for plants with no finite zeros, transient response of a desired character can often be tied to the achieving of particular closed-loop poles. The adaption becomes necessary when the plant is unknown or, more likely, reasonably well known but possibly slowly varying (perhaps on account of shifts in an operating point of a large system where linearized equations define the plant).

Another work dealing with global adaptive pole placement is [26], and it is pertinent to draw some comparisons. The parameterizations of the controller here and in [26] are different. Also, [26] requires piecewise time-invariant controllers to be used for all time, whereas in this work, they are only required when the usual algorithm introduces a difficulty. This means that in the scheme of this paper, there is less delay in adapting the controller.

II. Review of Nonglobal Result

Indirect adaptive control proceeds by 1) running an identifier and 2) implementing a controller as if the parameter estimate obtained from the identifier were correct. We shall now explain aspects of each of these steps in turn, emphasizing those points which are relevant to our global adaptive pole placement scheme.

Theorem 2.1: (Equation error identification; see, e.g., [20].) Let $d(z) = z^s + d_1 z^{s-1} + \cdots + d_s$ and $n(z) = n_0 z^t + \cdots + n_t$. Let $\theta_0 = [n_t\ n_{t-1} \cdots n_0\ \ -d_s\ \ -d_{s-1} \cdots -d_1]$ and let $\hat{\theta}_k$ denote an estimate of θ_0 available after taking measurements at time k. Define $\tilde{\theta}_k$, $\hat{n}_{i,k}$, $\hat{d}_{i,k}$, $\hat{n}_k(z)$, and $\hat{d}_k(z)$ by

$$\tilde{\theta}_k = \theta_0 - \hat{\theta}_k \tag{2.1}$$

$$\hat{\theta}_k = [\hat{n}_{t,k} \cdots \hat{n}_{0,k}\ \hat{d}_{s,k} \cdots -\hat{d}_{1,k}] \tag{2.2}$$

$$\hat{n}_k(z) = \sum_{i=0}^{t} \hat{n}_{i,k} z^{t-i}\quad \hat{d}_k(z) = z^s + \sum_{i=1}^{s} \hat{d}_{i,k} z^{s-i}. \tag{2.3}$$

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With u_k, y_k denoting the input and output sequences of the plant of transfer function $n(z)/d(z)$, define

$$\phi_{k-1}=[u_{k-s}\ u_{k-s+1}\ \cdots\ u_{k-s+t}\ y_{k-s}\ y_{k-s+1}\ \cdots\ y_{k-1}]^T \tag{2.4}$$

and the "equation error" by

$$e_k=\tilde{\theta}_{k-1}^T\phi_{k-1}. \tag{2.5}$$

Also, with μ a positive constant (normally small, but certainly less than 2), define the parameter estimate update law by

$$\hat{\theta}_k=\hat{\theta}_{k-1}+\frac{\mu\phi_{k-1}}{(1+\phi_{k-1}^T\phi_{k-1})}\,(y_k-\hat{\theta}_{k-1}\phi_{k-1}) \tag{2.6a}$$

which is analytically equivalent to

$$\hat{\theta}_k=\hat{\theta}_{k-1}+\frac{\mu\phi_{k-1}}{(1+\phi_{k-1}^T\phi_{k-1})}\,e_k. \tag{2.6b}$$

Then with z denoting a unit advance (i.e., $zw_k = w_{k+1}$)

$$e_k=[n(z)-\hat{n}_{k-1}(z)]u_{k-s}+[\hat{d}_{k-1}(z)-d(z)]y_{k-s} \tag{2.7}$$

$$\|\tilde{\theta}_k\|\leqslant\|\tilde{\theta}_{k-1}\| \tag{2.8}$$

$$\|\hat{\theta}_k-\hat{\theta}_{k-1}\|=\|\tilde{\theta}_k-\tilde{\theta}_{k-1}\|\to 0 \quad \text{as } k\to\infty \tag{2.9}$$

$$\frac{|\phi_{k-1}^T\tilde{\theta}_{k-1}|}{(1+\phi_{k-1}^T\phi_{k-1})^{1/2}}\to 0 \quad \text{as } k\to\infty. \tag{2.10}$$

Additionally, if there exist positive constants K_1, K_2 such that

$$|y_k|,\ |u_k|<K_1\sup_{0\leqslant l\leqslant k}|e_{l+1}|+K_2\forall k \tag{2.11}$$

then

$$e_k\to 0 \quad \text{as } k\to\infty. \tag{2.12}$$

Remark 2.1: The content of the theorem is a statement of an algorithm for generating estimates $\hat{n}_k(z)$, $\hat{d}_k(z)$ of the numerator and denominator polynomials of a transfer function of known numerator and denominator degree, together with certain properties of the estimates which are valid irrespective of whether the estimates converge to their correct value.

This identification procedure can be used in an adaptive pole placement algorithm as follows.

Theorem 2.2: (After [5].) Let $f(z)$ be a degree s polynomial, the zeros of which are the desired closed-loop poles, and are all inside the unit circle. Let $\hat{n}_k(z)$, $\hat{d}_k(z)$ be estimates of the plant numerator and denominator polynomials $n(z)$, $d(z)$ obtained as described in Theorem 2.1. Assume that the following equation can be solved for all k to find polynomials $\hat{k}_k(z)$ and $\hat{h}_k(z)$ with coefficients bounded as a function of k and of degrees, respectively, less than and less than or equal to $s-1$:

$$\hat{d}_k(z)\hat{k}_k(z)+\hat{n}_k(z)\hat{h}_k(z)=z^{s-1}[\hat{d}_k(z)-f(z)]. \tag{2.13}$$

Then the controller

$$u_k=u_k^*+\frac{\hat{k}_k(z)}{z^{s-1}}\,u_k+\frac{\hat{h}_k(z)}{z^{s-1}}\,y_k \tag{2.14}$$

with u_k^* bounded when used on the actual plant of transfer function $n(z)/d(z)$ ensures if $\hat{n}_k(z)\to n(z)$, $\hat{d}_k(z)\to d(z)$ that the closed-loop transfer function is $n(z)/f(z)$, and the closed-loop characteristic polynomial is $z^{s-1}f(z)$. In case $\hat{n}_k(z)$ and $\hat{d}_k(z)$ do not converge to $n(z)$, $d(z)$, the controller produces behavior like that associated with closed-loop poles at the zeros of f, in a sense made precise in [5]. In all cases, y_k and u_k are bounded sequences.

Remark 2.2: If $\hat{d}_k(z)$, $\hat{n}_k(z)$ have a common zero, or equivalently if the Sylvester resultant determinant of $\hat{d}_k(z)$ and $\hat{n}_k(z)$ is zero, (2.13) is not solvable for generic f.

Remark 2.3: If the initial estimates $\hat{n}_0(z)$, $\hat{d}_0(z)$ of $n(z)$, $d(z)$ define a transfer function $\hat{n}_0(z)/\hat{d}_0(z)$ with a different Cauchy index to the true transfer function $n(z)/d(z)$, it is impossible to define a continuous deformation of the initial parameter estimates to the true parameters which does not produce a pole-zero cancellation at some point in the deformation [19]. This means that the Sylvester determinant in a discrete-time stepping algorithm has to change sign. While it may be possible to give a with probability one argument that it will never be zero at any one occasion, it seems impossible to rule out the possibility that zero could be a limit point. Hence, there is a clear possibility that the above scheme might not work as a global algorithm.

Remark 2.4: Suppose that $n(z)$, $d(z)$ have no common zero. Note that

$$\theta_0-\hat{\theta}_k=[n_t-\hat{n}_{t,k}\ \cdots\ n_0-\hat{n}_{0,k}\ -d_s+\hat{d}_{s,k}\ \cdots\ -d_1+\hat{d}_{1,k}] \tag{2.15}$$

has a norm which is a measure of the error between $\hat{n}_k(z)$, $n(z)$ and $\hat{d}_k(z)$, $d(z)$ and by Theorem 2.1 is thus decreasing. It is clear that there is a ball around $n(z)$, $d(z)$ definable using this norm in which the Sylvester determinant will be everywhere nonzero, since it is a continuous function of its coefficients. It follows from the monotonicity of $\|\tilde{\theta}_k\|$ that if $\hat{n}_0(z)$, $\hat{d}_0(z)$ are chosen in this ball, the Sylvester determinant of $\hat{n}_k(z)$, $\hat{d}_k(z)$ will be nonzero for all k. In this way, a local result is obtained.

Remark 2.5: Note that the sequence of calculations is $y_k\to\hat{\theta}_k\to\hat{n}_k(z),\ \hat{d}_k(z)\to\hat{k}_k(z),\ \hat{h}_k(z)\to u_k$. If processing cannot be done fast enough one may have to compute u_k using $\hat{k}_{k-1}(z)$, $\hat{h}_{k-1}(z)$ instead of $\hat{k}_k(z)$, $\hat{h}_k(z)$. Such a change introduces only inessential adjustments to the basic ideas.

Remark 2.6: The last sentence in the statement of the theorem hides within it the following sort of problem. Suppose u_k^* is identically 1. What the identification algorithm almost certainly will obtain is values of $\hat{n}_k(z)$, $\hat{d}_k(z)$ which are unconstrained except that $\hat{n}_k(1)/\hat{d}_k(1) = n(1)/d(1)$, i.e., the correct dc gain of the plant is learned but nothing else. As a consequence, the controller designed using (2.13) may not even stabilize the real plant. This possibility is not pointed out in [5], part of the argument which should therefore really be regarded as purely formal. The only way this situation can be avoided is by having a frequency rich, or persistently exciting u_k^* sequence. In the next section we make this adjustment to the result of [5] in Theorem 3.1.

III. Modest Extension of the Nonglobal Result

In order to obtain a global adaptive pole-positioning algorithm, we need to note first some minimum extensions of the conclusions of Section II. The first such extension involves a modified "linear-boundedness" lemma.

Lemma 3.1: Assume the same hypotheses as Theorem 2.1, save that (2.11) is replaced by: for some positive K_1, K_2 and all k in some infinite subset of the integers K [compare to (2.11)]

$$\|\phi_{k-1}\|<K_1|e_k|+K_2. \tag{3.1}$$

Then

$$e_k\to 0 \quad \text{for } k\in K,\ k\to\infty. \tag{3.2}$$

Proof: Recall from (2.5) that $e_k = \tilde{\theta}_{k-1}^T\phi_{k-1}$ and from (2.10) that

$$\frac{|\tilde{\theta}_{k-1}^T\phi_{k-1}|}{(1+\phi_{k-1}^T\phi_{k-1})^{1/2}}\to 0.$$

Using (3.1), this implies

$$\frac{|e_k|}{[1+(K_1|e_k|+K_2)^2]^{1/2}}\to 0. \tag{3.3}$$

If lim sup $|e_k|$ evaluated for $k \in K$, $k \to \infty$ were nonzero, (3.3) would be immediately contradicted. ∇∇∇

Next, we recall the notion of persistency of excitation [17].

Definition 3.1: The sequence u_k^* is persistently exciting, with richness r and persistency interval L if there exist positive α_1, α_2 such that for all j

$$\alpha_2 I > \sum_{k=j}^{j+L-1} \begin{bmatrix} u_k^* \\ u_{k-1}^* \\ \vdots \\ u_{k-r+1}^* \end{bmatrix} [u_k^* \; u_{k-1}^* \cdots u_{k-r+1}]^T > \alpha_1 I. \tag{3.4}$$

Remark 3.1: It is not hard to secure satisfaction of this condition: if $u_k^* = \Sigma_{l=1}^{\sigma} \alpha_l \exp(j\beta_l k)$ with complex $\alpha_l \neq 0$ for all l, real $\beta_{l1} \neq \beta_{l2}$ for $l_1 \neq l_2$, and u_k^* real, then (3.4) holds if and only if $\sigma \geqslant r$.

The above definition allows us to state an improvement on Theorem 2.2. This improvement depends on two lemmas which relate persistency of excitation of an external control to persistency of excitation of a regression vector given, first, a time-invariant controller and second, a slowly varying controller.

Lemma 3.2: Consider the plant $d(z)y_k = n(z)u_k$ with $n(z)$ and $d(z)$ coprime, and as in Theorem 2.1 suppose that a controller

$$u_k = u_k^* + \frac{\bar{k}(z)}{z^{s-1}}\, u_k + \frac{\bar{h}(z)}{z^{s-1}}\, y_k \tag{3.5}$$

is used and that u_k^*, u_k, y_k are all bounded. Suppose further that

$$\alpha_2 I > \sum_{k=j+(s+t)-1}^{j+L-1+(s+t)-1} \begin{bmatrix} u_k^* \\ \vdots \\ u_{k-(s+t)+1}^* \end{bmatrix} \times [u_k^* \cdots u_{k-(s+t)+1}^*] > \alpha_1 I \tag{3.6}$$

for some positive α_1, α_2, some integer L, and some fixed j. Then with ϕ_k as in (2.4), repeated as

$$\phi_k = [u_{k-s+1} \cdots u_{k-s+t+1}\, y_{k-s+1} \cdots y_k]^T \tag{3.7}$$

there holds

$$\beta_2 I > \sum_{j}^{j+L-1+2s-1} \phi_k \phi_k^T > \beta_1 I \tag{3.8}$$

for some positive β_1, β_2.

Proof: The upper bound is trivial. Suppose the lower bound fails. Then there exists a vector

$$[\gamma^T\, \delta^T] = [\gamma_t\, \gamma_{t-1} \cdots \gamma_0\, \delta_s\, \delta_{s-1} \cdots \delta_1]$$

of unit length such that

$$[\gamma^T\, \delta^T]\phi_k = 0 \tag{3.9}$$

for $k = j$ to $j + L - 1 + (2s - 1)$. Set $\gamma(z) = \Sigma_{i=0}^{t} \gamma_i z^{t-i}$, $\delta(z) = \Sigma_{i=1}^{s} \delta_i z^{s-i}$. Then (3.9) is equivalent to

$$\gamma(z)u_{k-s+1} + \delta(z)y_{k-s+1} = 0 \tag{3.10}$$

for $k = j$ to $j + L - 1 + (2s - 1)$. Now the plant and controller equations together imply that

$$u_k = \frac{d(z)z^{s-1}}{d(z)z^{s-1} - \bar{k}(z)d(z) - \bar{h}(z)n(z)}\, u_k^* \tag{3.11a}$$

$$y_k = \frac{n(z)z^{s-1}}{d(z)z^{s-1} - \bar{k}(z)d(z) - \bar{h}(z)n(z)}\, u_k^*. \tag{3.11b}$$

Denote the denominator by $d_w(z)$. It has degree $2s - 1$. Then (3.10) implies

$$d_w(z)\gamma(z)u_{k-s+1} + d_w(z)\delta(z)y_{k-s+1} = 0$$

for $k = j$ to $j + L - 1$, and then, using (3.11),

$$\gamma(z)d(z)z^{s-1}u_{k-s+1}^* + \delta(z)n(z)z^{s-1}u_{k-s+1}^* = 0$$

or

$$[\gamma(z)d(z) + \delta(z)n(z)]u_k^* = 0 \tag{3.12}$$

for $k = j$ to $j + L - 1$.

Now observe that $\epsilon(z) \triangleq \gamma(z)d(z) + \delta(z)n(z)$ must be nonzero. Otherwise, $n(z)/d(z) = -\gamma(z)/\delta(z)$ with $\delta(z)$ of lower degree than $d(z)$, contradicting the coprimeness of $d(z)$ and $u(z)$. Let $\|\epsilon\|^2$ denote the sum of the squares of the coefficients of $\epsilon(z)$.

Note that $\inf_{\|\gamma\|^2+\|\delta\|^2=1} \|\epsilon\|^2$ is nonzero; otherwise we could construct a bounded sequence of $\gamma_i(z)$, $\delta_i(z)$ with $\|\gamma_i\|^2 + \|\delta_i\|^2 = 1$ such that $\lim_{i\to\infty} \|\epsilon_i\|^2 = 0$. There would be a convergent subsequence of the $\gamma_i(z)$, $\delta_i(z)$, converging to say $\bar{\gamma}(z)$, $\bar{\delta}(z)$, with $\bar{\gamma}(z)d(z) + \bar{\delta}(z)n(z) = \bar{\epsilon}(z) = 0$, a contradiction.

Noting that $\epsilon(z)$ has degree $s + t$, we see that (3.12) implies a contradiction to (3.6) ∇∇∇

Corollary 3.2: Assume the same hypotheses as Lemma 3.2, save that (3.6) holds for all $j \geqslant J$. Then (3.8) holds for all $j \geqslant J$.

Proof: The upper bound is trivial. If the lower bound fails, for arbitrary η and some j, one can find $[\gamma^T\, \delta^T]$ of unit length with $|\gamma(z)u_{k-s+1} + \delta(z)y_{k-s+1}| < \eta$ for $k = j$ to $j + L - 1 + (2s - 1)$. The same arguments as in the lemma proof yield $|\epsilon(z)u_k^*| < 0(\eta)$ for $k = j$ to $j + L - 1$, contradicting (3.6) if η is small enough.

Remark 3.2: If (3.6) holds for all $j \geqslant J$, it becomes a persistency of excitation condition, as does (3.8). Condition (3.8) is precisely then the condition required to secure exponential convergence of the error estimate $\|\tilde{\theta}_k\|$ to zero in the gradient algorithm noted in Theorem 2.1; see [25].

Remark 3.3: The above argument is very similar to ones we have used elsewhere; see, e.g., [17], [18]. Let us note, because the point is not always appreciated, that we have proved that the lower bound condition in (3.6) implies the lower bound condition in (3.8) without, in either Lemma 3.2 or Corollary 3.2, making use of the upper bound condition. The corresponding main result in [17] summarizes within the one theorem statement this result and a second result, the proofs of which are independent. On the other hand, if the controller were time-varying, as in part of [18] and as in Lemma 3.3 below, it does become necessary to use the upper bound condition within the argument which proves one lower bound condition from another. As we noted in the Introduction, the scheme of [26] uses piecewise constant controllers (thus allowing an external signal lower bound persistency of excitation condition to become a regression vector lower bound persistency of excitation signal), and by using recursive least squares (*without* a forgetting factor) gives exact identification without needing an upper bound.

Remark 3.4: There exists an obvious modification of Corollary 3.2 to cope with the case where (3.6) holds for a subset of $\{j : j \geqslant J\}$.

Let us now consider the effective independence of the constant β_1 for the controller parameters.

Corollary 3.3: Assume the same hypotheses as Lemma 3.2 or Corollary 3.2. Then the constant β_1 can be assumed to depend only on $n(z)$, $d(z)$, α_1, α_2 and a real δ such that $\|\bar{h}\|^2 + \|\bar{k}\|^2 \leqslant \delta$, where $\|\bar{h}\|^2$ denotes the sum of the squares of the coefficients of $\bar{h}(z)$.

Proof: For any fixed $\bar{h}$, $\bar{k}$, we know that $\beta_1 \neq 0$ and is dependent on $n(z)$, $d(z)$, α_1, α_2, $\bar{h}(z)$, and $\bar{k}(z)$. The set of $\bar{h}$, $\bar{k}$ with $\|\bar{h}\|^2 + \|\bar{k}\|^2 \leqslant \delta$ is a bounded closed set. Assume there exists a sequence of $\{\bar{h}_i, \bar{k}_i\}$ $(i = 1, 2 \cdots)$ in this set such that the associated sequence of β_1, call it $\{\beta_{1i}\}$, approaches zero. Thus, there is a limit point of the $\{\bar{h}_i, \bar{k}_i\}$ for which $\beta_1 = 0$. This is a contradiction. Hence, there exists a least β_1 which is nonzero. ▽▽▽

We shall now strengthen Lemma 3.2 and Corollary 3.2 to cope with a time-varying controller (which varies more and more slowly).

Lemma 3.3: Assume the same condition as Theorem 2.2, and assume that u_k^* is persistently exciting with richness $s + t$ and persistency interval L. Then there exists a k_1 such that ϕ_k is persistently exciting with richness 1 and persistency interval $L + 2s - 1$ for all $k \geqslant k_1$.

Proof: If the time-varying controller of (2.14) were time-invariant and such as to provide a stable closed-loop system the result would be immediate by Corollary 3.2. Recall from Theorem 2.1 [see (2.9)] that the rate of variation with k of the coefficients of $\hat{n}_k(z)$ and $\hat{d}_k(z)$ tends to zero. Since $\bar{h}_k(z)$, $\bar{k}_k(z)$ are obtained by solving a (nonsingular) linear equation with bounded solution involving $\hat{n}_k(z)$, $\hat{d}_k(z)$ [see (2.13)] $\bar{h}_k(z)$, $\bar{k}_k(z)$ will also exhibit increasingly slow coefficient variations. Consider now the mapping of u_k^* into ϕ_k over an interval $[k', k' + L + 2s - 2]$ for arbitrary k' where u_k^* is bounded but is otherwise unspecified. Let $\bar{\phi}_k$ denote the ϕ_k sequence which would result if $\bar{h}_k(z)$, $\bar{k}_k(z)$ were not varied throughout the interval but frozen at the values taken at the start of the interval. Noting that the assumption of the theorem ensures that ϕ_k is bounded, it is evident that given $\epsilon > 0$, there exists a k, such that if $k' > k$,

$$\|\phi_j - \bar{\phi}_j\| < \epsilon \tag{3.13a}$$

$$\|\phi_j\phi_j^T - \bar{\phi}_j\bar{\phi}_j^T\| < 0(\epsilon) \tag{3.13b}$$

for all $j \in [k', k' + L + 2s - 2]$. (The scaling constant on the right side of (3.13) involves the bound on the ϕ_k sequence.)

Now recall that $\bar{h}_k(z)$, $\bar{k}_k(z)$ have bounded coefficients by the theorem assumption. Applying Corollary 3.2 as strengthened by Corollary 3.3, we see that with $\epsilon = o(\beta)$, ϕ_k will have the desired persistency of excitation property. ▽▽▽

Remark 3.5: Lemma 3.3 puts together several key ideas. First, for a linear time-invariant system, persistency of excitation of an external driving signal implies persistency of excitation of an internal signal (with different richness and persistency interval). Second, this idea carries through to a linear time-varying system, provided the rate of time variation is suitably small. Third (and this is a development of the first idea, not made explicit in [15], [17], [18] which cover the first two ideas) the richness of the external persistently exciting signal is the same as that required for a signal applied directly at the plant input to identify it with exponentially decaying parameter error. This richness is the number of unknown plant parameters. One might have conjectured that the presence of an external compensator could introduce a requirement for a greater degree of richness of the external signal—but this is not apparently the case.

Now we can state the promised modest extension of Theorem 2.2.

Theorem 3.1: Assume the same conditions as Theorem 2.2 and assume further that u_k^* is persistently exciting with richness $s + t$ and persistency interval L. Then $\hat{n}_k(z) \to n(z)$, $\hat{d}_k(z) \to d(z)$ and $e_k \to 0$, with each convergence occurring exponentially fast.

Proof: By assumption, $\{\phi_k\}$ is a bounded sequence, and by Lemma 3.2 it is a persistently exciting sequence for $k \geqslant k_1$. It follows from the main result in [17] that $\|\tilde{\theta}_k\| \to \infty$ exponentially fast. This is equivalent to $\hat{n}_k(z) \to n(z)$, $n(z)$, $\hat{d}_k(z) \to d(z)$ exponentially fast. Also, since $e_k = \tilde{\theta}_{k-1}^T\phi_{k-1}$ with ϕ_{k-1} bounded, $e_k \to 0$ exponentially fast. ▽▽▽

Remark 3.6: Theorem 3.1 is of course still a local result but has the advantage over Theorem 2.2 of offering a robustness result, on account of the exponential stability.

Remark 3.7: Use of a persistently exciting input raises the following possibility for a global (but not robust) algorithm. Run the system with no feedback and with a persistently exciting u_k^*. Then

$$\sum_{j=0}^{s+t-1} \phi_j\phi_j^T > 0. \tag{3.14}$$

Now $y_k = \phi_{k-1}^T\theta_0$, so

$$\sum_0^{s+t-1} \phi_j\phi_j^T\theta_0 = \sum_0^{s+t-1} \phi_j y_{j+1}$$

hence

$$\theta_0 = \left(\sum_0^{s+t-1} \phi_j\phi_j^T\right)^{-1} \sum_0^{2+t-1} \phi_j y_{j+1}. \tag{3.15}$$

Thus, θ_0 is obtainable in a finite number of time instants. Once the plant is identified exactly, the nonglobal algorithm is run. Such a procedure will give some robustness in the presence of plant parameter variation (provided of course the persistency of input excitation is maintained). But should the current estimates of the plant parameters be perturbed by noise or plant parameter variation to the point where $\hat{n}_k(z)$ and $\hat{d}_k(z)$ have a common zero, the algorithm will still break down. Put another way, the capacity to recover should an error arise in the parameter estimates will be limited.

IV. Assumptions, Algorithm for, and Statement of Global Result

In this section, we list all the assumptions required for a global result, together with the algorithm and statement of the global result.

A1) The plant has transfer function $n(z)/d(z)$ with $n(\cdot)$, $d(\cdot)$ coprime, of degrees t and s, and $d(\cdot)$ monic.

A2) Let S be the Sylvester resultant determinant of $n(z)$ and $d(z)$. Then a number σ is known for which

$$|S| \geqslant \sigma.$$

A3) A controller of the form

$$u_k = u_k^* + \frac{\bar{k}_k(z)}{z^{s-1}} u_k + \frac{\bar{h}_k(z)}{z^{s-1}} y_k \tag{4.1}$$

is implemented, where u_k^* is an external input and u_k, y_k are the plant input and output: $\bar{k}_k$, $\bar{h}_k$ have degrees at most $s - 2$, $s - 1$, respectively.

A4) u_k^* is persistently exciting with richness $s + t$ and persistency interval L.

A5) The desired closed-loop characteristic polynomial $f(z)$ has all zeros inside $|z| < 1$.

Remark 4.1: The only new assumptions, relative to those for the nonglobal Theorem 2.2, are A2) and A4). Note that A2) implies the coprimeness part of A1). Assumption A4) appeared in the robust version of Theorem 2.2, viz. Theorem 3.1.

Remark 4.2: The quantity σ in effect is an underbound on the product of the pole-zero separation and the high frequency gain, since $S = \Pi_{i,j} n_0^s(z_{id} - z_{jn})$, where z_{id} is a zero of $d(z)$, z_{jn} a zero of $n(z)$ and n_0 the coefficient of the highest power of z in $n(z)$ [21]. A corresponding condition appears also in [15].

The algorithm makes use of a number of auxiliary polynomials, defined as follows.

A6) There are available pairs of polynomials $\bar{h}_i(z)$, $\bar{k}_i(z)$ for $i = 1, \cdots, 2(s+t)+1$ with $\bar{h}_i(z)$, $\bar{k}_i(z)$ possessing degrees at most $(s-1)$, $(s-2)$, respectively, with $z^{s-1} - \bar{k}_i$ and $\bar{h}_i$ coprime and with no three of the transfer functions $\bar{h}_j(z^{s-1} - \bar{k}_j)^{-1}$[for $j = 1, \cdots, 2(s+t)+1$] taking the same value (including ∞) at any one point in the complex plane.

Remark 4.3: It is clear that randomly selected coefficients for $\bar{h}_i(z)$, $\bar{k}_i(z)$ will almost surely meet the requirements of the assumption.

Algorithm Statement: While $|S[\hat{n}_k(z), \hat{d}_k(z)]| \geq \sigma/2$, use the scheme of Theorem 2.2 for determining $\hat{h}_k(z)$ and $\hat{k}_k(z)$. When $|S[\hat{n}_{k_j}(z), \hat{d}_{k_j}(z)]| < \sigma/2$, while $|S[\hat{n}_{k_j-1}(z), \hat{d}_{k_j-1}(z)]| \geq \sigma/2$, employ the following *special strategy*. Use $\bar{h}_1(z)$, $\bar{k}_1(z)$ for $L + 2s - 1$ time instants commencing at time k_j, then $\bar{h}_2(z)$, $\bar{k}_2(z)$ for the next $L + 2s - 1$, and so on through to use of $\bar{h}_{2(s+t)+1}(z)$, $\bar{k}_{2(s+t)+1}(z)$ for the last $L + 2s - 1$ instants of special strategy use. At the next time instant check $|S[\hat{n}_k(z), \hat{d}_k(z)]|$ and proceed as before.

Remark 4.4: It is precisely when $|S[\hat{n}_k(z), \hat{d}_k(z)]|$ becomes small that the algorithm of Theorem 2.2 runs into difficulty. Our algorithm provides a temporary replacement for the algorithm of Theorem 2.2, based on using a sequence of controllers which are fixed over intervals of prescribed length. As will be seen, this promotes the identification process.

Remark 4.5: The idea of using a sequence of fixed controllers is not new in identification; see [22].

Remark 4.6: This style of special control strategy was used in [15] for the first-order problem. There, with $s = 1$, all feedback controllers were memoryless.

Remark 4.7: It will be seen later that the special strategy needs only to be used a finite number of times.

Remark 4.8: Various simplifying possibilities exist, based on using memoryless controllers and using a lesser number than $2(s+t)+1$ distinct controllers. On occasions, *but not always*, no feedback, i.e., open-loop operation could be used. These points will be discussed subsequently. They are of course of great practical significance.

Remark 4.9: To the extent that it might be possible, $\bar{h}_i$ and $\bar{k}_i$ should be chosen to stabilize the system—for otherwise the growth of variables in the closed loop over the interval of use of any one $\bar{h}_i$, $\bar{k}_i$ may be very substantial. In this connection, we note that if the estimates of $\hat{n}_k$, $\hat{d}_k$ have a common *stable* zero cancellation, it is easy to form a pair of $\bar{h}_i$, $\bar{k}_i$ which, were $\hat{n}_k$ and $\hat{d}_k$ the true system numerator and denominator polynomials, would stabilize the system—one simply requires that the common zero also be made a zero of the closed-loop polynomial which is $\hat{d}_k(z^{s-1} - \bar{k}_i) - \hat{n}_k\bar{h}_i$. If $\hat{d}_k$, $\hat{n}_k$ nearly have a stable pole-zero cancellation, a stabilizing $\bar{h}_i$, $\bar{k}_i$ pair can be found without coefficients which are large on account of the near cancellation. Now if the true n and d are close to $\hat{n}_k$, $\hat{d}_k$ (which might well be the case if the difference is due to noise or slow time-variation in an otherwise well-identified setup), then the true system will be stabilized by $\bar{k}_i$, $\bar{h}_i$. We stress however that a choice of stabilizing $\bar{h}_i$, $\bar{k}_i$, although desirable, is not essential. Also, the problem of what this adaptive scheme does during a learning phase began with poor plant parameter estimates is *no different in principle* to the same problem for some other scheme, e.g., reference trajectory following.

The main result is as follows.

Theorem 4.1: Given the assumptions A1)–A6) and algorithm statement above, $\hat{n}_k(z) \to n(z)$, $\hat{d}_k(z) \to d(z)$ and $e_k \to 0$, all exponentially fast. Also,

$$\hat{d}_k(z)\hat{k}_k(z) + \hat{n}_k(z)\hat{h}_k(z) \to z^{s-1}[d(z) - f(z)] \tag{4.2}$$

with exponentially fast convergence. The sequences u_k and y_k remain bounded.

The proof of this theorem is complicated, and we will proceed with a number of lemmas. Most of the work is associated with showing (by a contradiction argument) that the special strategy can only be used on a finite number of occasions.

Before proceeding with the proof of the theorem, we note the following simple corollary, which shows that the behavior of the plant with the adaptive controller approaches the behavior obtainable when the correct fixed controller is used.

Corollary 4.1: Assume the same hypotheses as for Theorem 4.1. Let $\bar{h}(z)$, $\bar{k}(z)$ solve

$$d(z)\bar{k}(z) + n(z)\bar{h}(z) = z^{s-1}[d(z) - f(z)] \tag{4.3}$$

and let $\bar{u}_k$, $\bar{y}_k$ denote the plant input and output when the controller is defined using $\bar{h}(z)$, $\bar{k}(z)$. Then $|u_k - \bar{u}_k|$, $|y_k - \bar{y}_k| \to 0$ exponentially fast.

Proof: Let x_k, $\bar{x}_k$ be the state vectors of systems implemented by using controllers around the plant defined by $\hat{h}_k(z)$, $\hat{k}_k(z)$ and $\bar{h}(z)$, $\bar{k}(z)$. Then the closed-loop systems are represented by

$$x_{k+1} = A_k x_k + B_k u_k^* \quad \begin{bmatrix} y_k \\ u_k \end{bmatrix} = C_k x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k^* \tag{4.4}$$

and

$$\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}u_k^* \quad \begin{bmatrix} \bar{y}_k \\ \bar{u}_k \end{bmatrix} = \bar{C}\bar{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k^* \tag{4.5}$$

with A_k, B_k, C_k, containing entries which are integral in the coefficients of $n(z)$, $d(z)$, $\hat{h}_k(z)$, $\hat{k}_k(z)$, and $\bar{A}$, $\bar{B}$, $\bar{C}$ similarly; further, by Theorem 4.1, $\|A_k - \bar{A}\|$, $\|B_k - \bar{B}\|$, and $\|C_k - \bar{C}\|$ approach zero exponentially fast, and x_k, $\bar{x}_k$ are bounded. Last, $\bar{A}$ has all eigenvalues inside the unit circle since the closed loop is stable.

Now observe that

$$x_{k+1} - \bar{x}_{k+1} = \bar{A}(x_k - \bar{x}_k) + (A_k - \bar{A})x_k + (B_k - \bar{B})u_k^*.$$

The second two summands on the right are exponentially decaying. Hence, $\|x_{k+1} - \bar{x}_{k+1}\|$ is exponentially decaying. The result follows from the second equation in each of (4.4) and (4.5).

V. Preliminary Lemmas

The four lemmas in this section all develop properties flowing from the choice of the $\bar{h}_i$, $\bar{k}_i$ pair mentioned in assumption A6).

Lemma 5.1: Let assumptions A1) and A6) hold. Let

$$r_i(z) = d(z)[z^{s-1} - \bar{k}_i(z)] - n(z)\bar{h}_i(z) \qquad i = 1, 2, \cdots, 2(s+t)+1 \tag{5.1}$$

$$2\epsilon = \min_{i,j}\{|z_{0i} - z_{0j}| : z_{0i} \neq z_{0j},\ r_i(z_{0i}) = 0,\ r_j(z_{0j}) = 0\}. \tag{5.2}$$

Let $q(z)$ be an arbitrary $(s+t)$th or lesser degree polynomial. Then for at least one i, every zero of $q(z)$ will be separated from every zero of $r_i(z)$ by at least ϵ.

Remark 5.1: The lemma is saying that any $(s+t)$th or lesser degree polynomial cannot have a nontrivial common divisor with each r_i, but must be "uniformly" coprime with at least one. The $r_i(z)$ are the closed-loop characteristic polynomials resulting from use of the ith controller of the special strategy sequence.

Proof of Lemma 5.1: Observe first that for any triplet i, j, k with no two of i, j, k the same, $r_i(z)$, $r_j(z)$, and $r_k(z)$ cannot all have a common zero. For if there holds $r_i(z_0) = r_j(z_0) = r_k(z_0 = 0$, then

$$0=\begin{bmatrix} z_0^{s-1}-\bar{k}_i(z_0) & -\bar{h}_i(z_0) \\ z_0^{s-1}-\bar{k}_j(z_0) & -\bar{h}_j(z_0) \\ z_0^{s-1}-\bar{k}_l(z_0) & -\bar{h}_l(z_0) \end{bmatrix} \begin{bmatrix} d(z_0) \\ n(z_0) \end{bmatrix}.$$

Now either $d(z_0)$ or $n(z_0)$ is nonzero since $d(z)$ and $n(z)$ are coprime by A1). Hence,

$$\frac{\bar{h}_i(z_0)}{z_0^{s-1}-\bar{k}_i(z_0)}=\frac{\bar{h}_j(z_0)}{z_0^{s-1}-\bar{k}_j(z_0)}=\frac{\bar{h}_l(z_0)}{z_0^{s-1}-\bar{k}_l(z_0)}$$

which is a contradiction of A6).

Now let $z_1, \cdots, z_{s+t}$, be the zeros of $q(z)$ and consider the following procedure. List the polynomials $r_1(z), r_2(z), \cdots, r_{2(s+t)+1}$ and examine the zero separation of z_1 with successive polynomials in the list. If there is a zero of $r_j(z)$ for any j within ϵ of z, cross r_j off the list. In making this check, observe that at most two of the r_j can be crossed off. For if three are crossed off, say r_1, r_2, r_3, then each has a zero within ϵ of z_1, and thus there is a pair of zeros of r_1, r_2 within 2ϵ of one another, and likewise of r_2, r_3 and r_1, r_3. But the definition of 2ϵ in (5.2) then ensures that these zeros must be identical, and this contradicts the fact that there cannot be a common zero for three or more of the r_i.

Now having compared z_1 to the zeros of all the $r_j(z)$, and deleted at most two $r_j(z)$ as a result, one repeats the procedure with z_2, then $z_3, z_4, \cdots, z_{s+t}$, working with the reduced list of the $r_j(z)$ remaining after earlier crossings off. In working with z_2, at most 2 of the $r_j(z)$ can be crossed off the list, and likewise with $z_3, z_4, \cdots$. Hence, at most $2(s+t)$ of the $r_j(z)$ in all can be crossed off. There are however $2(s+t)+1$ of the $r_j(z)$. So at least one $r_j(z)$ has all its zeros separated by at least ϵ from all zeros of $q(z)$.

Corollary 5.1: Suppose the special strategy of the main algorithm is used at time $k_1, k_2, k_3, \cdots$,. Then for each j, there exists an i, call it i_j, such that all zeros of $r_{i_j} = d(z)[z^{s-1} - \bar{k}_{i_j}] - n(z)\bar{h}_{i_j}$ are at a distance of at least ϵ from all zeros of $\hat{d}_{k_j}(z)n(z) - \hat{n}_{k_j}(z)d(z)$.

Proof: Identify $q(z)$ with $\hat{d}_{k_j}(z)n(z) - \hat{n}_{k_j}(z)d(z)$, which has degree at most $s+t$. ∇∇∇

Having established a zero-spacing property, we shall now turn it into a property of the Sylvester resultant.

Lemma 5.2: Let $r(z)$ be a fixed, degree p, monic polynomial, and let $q(z)$ be an arbitrary polynomial of degree at most p, save that $\eta_2 > \Sigma q_i^2 > \eta_1 > 0$ with η_i prescribed and the q_i being coefficients of $q(z)$. Suppose each zero of $q(z)$ is separated by at least ϵ from all zeros of $r(z)$. Then $|S[q(z), r(z)]| > \lambda > 0$ for *some* λ *independent of* $q(z)$ *but depending on* ϵ, η_1, η_2, and $r(z)$.

Proof: Suppose the claim were not true. Then there exists an infinite sequence $q_1(z), q_2(z), \cdots$, meeting the conditions on the $q(z)$ for which $|S[q_i(z), r(z)]| \to 0$. Since the coefficients of the $q_i(z)$ are bounded, there exists a limit point; call it $\bar{q}(z)$. Since S is a continuous function of the polynomial coefficients, $|S[\bar{q}(z), r(z)]| = 0$, so that $\bar{q}(z)$ and $r(z)$ have a common zero. Let $q_{i_1}(z), q_{i_2}(z), \cdots$, be the sequence approaching $\bar{q}(z)$. In case the highest order coefficients of the $q_{i_j}(z)$ do not approach zero, the zeros of $\bar{q}(z)$ are limits of the zeros of $q_{i_j}(z)$, and a contradiction is immediate. In case the highest order coefficients of the $q_{i_j}(z)$ approach zero, it is not hard to see that the zeros of $\bar{q}(z)$ are the limits of a subset of zeros of $q_{i_j}(z)$ (those remaining finite as $j \to \infty$). Again, a contradiction is immediate. ∇∇∇

Corollary 5.2: Let $r_i(z)$, $i = 1, 2, \cdots, 2(s+t)+1$ be a collection of fixed degree $2s-1$ polynomials, and let $q(z)$ be an arbitrary polynomial of degree at most $2s-1$, save that $\eta_2 > \Sigma q_i^2 > \eta_1 > 0$. *Suppose each zero of* $q(z)$ *is separated by at least* ϵ from all zeros of one of the $r_i(z)$, say $r_j(z)$. Then $|S(q(z), r_j(z)| > \lambda > 0$ for some λ independent of $q(z)$ but depending on ϵ, η_1, η_2 and the collection $r_i(z)$.

Proof: Apply Lemma 5.2 with $r(z)$ in Lemma 5.2 identified with $r_j(z)$, and with λ in Lemma 5.2 replaced by λ_j, depending on ϵ, η_1, η_2, and $r_j(z)$. Take λ in the corollary as $\min_j \lambda_j$. ∇∇∇

The following lemma will also be used in the next section.

Lemma 5.3: Suppose that $\alpha, \beta, \gamma, \delta$ are polynomials with deg $\alpha \leqslant$ deg $\beta = s$, deg $\gamma \leqslant$ deg $\delta = s$, with the coefficient vector of all the polynomials known to lie in a ball of fixed radius, and with β, δ monic. Suppose further that $|S[\alpha, \beta]| < \sigma/2$ and $|S[\gamma, \delta]| > \sigma$. Let $q = \beta\gamma - \alpha\delta$. Then the sum of the squares of the coefficients of q is bounded away from zero by a function of σ and the ball radius.

Proof: We proceed by contradiction. Suppose the result is not true. Then there exists a sequence $\{\alpha_i, \beta_i, \gamma_i, \delta_i\}$ for $i = 1, 2, \cdots$, such that $\lim_{i\to\infty} q_i(z) \to 0$ while $|S[\alpha_i, \beta_i]| < \sigma/2$, $|S[\gamma_i, \delta_i]| > \sigma$, and the degree and monic constraints are fulfilled. Because of the hypothesis restricting the magnitude of the coefficient vector, there exists a limit point of the sequence, call it $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$, and by continuity we have $\bar{\beta}\bar{\gamma} = \bar{\alpha}\bar{\delta}$, $|S[\bar{\alpha}, \bar{\beta}]| < \sigma/2$ and $|S[\bar{\gamma}, \bar{\delta}]| > \sigma$. Then $\bar{\gamma}, \bar{\delta}$ have no common zeros and $\bar{\gamma}/\bar{\delta} = \bar{\alpha}/\bar{\beta}$. Since $\bar{\beta}, \bar{\delta}$ are monic and have the same degree s, it follows that $\bar{\beta} = \bar{\delta}$, $\bar{\alpha} = \bar{\gamma}$ and the contradiction is immediate. ∇∇∇

Finally, we note a property of the quantities $\bar{h}_k$, $\bar{k}_k$.

Lemma 5.4: Under assumptions A1)–A6), the algorithm ensures that $\bar{h}_k(z)$ and $\bar{k}_k(z)$ have bounded coefficients.

Proof: By Theorem 2.1, $\hat{n}_k(z)$ and $\hat{d}_k(z)$ have bounded coefficients. When $|S[\hat{n}_k(z), \hat{d}_k(z)]| \geqslant \sigma/2$, $\bar{h}_k(z)$ and $\bar{k}_k(z)$ are defined by

$$\hat{d}_k(z)\bar{k}_k(z)+\hat{n}_k(z)\bar{h}_k(z)=z^{s-1}[\hat{d}(z)-f(z)].$$

The coefficients of the polynomial on the right are bounded. With the lower bound on the Sylvester resultant, the subset of $\bar{h}_k(z)$, $\bar{k}_k(z)$ determined using this equation has bounded coefficients. Otherwise, $\bar{h}_k(z)$, $\bar{k}_k(z)$ are set by the algorithm to equal predetermined values. Hence, the result. ∇∇∇

VI. Finitely Often Use of Special Strategy

Our broad aim in this section is to show that the special strategy aspect of the algorithm needs only to be used on a finite number of occasions. To do this, we shall show that on each occasion of special strategy use, there is a significant improvement in the quality of our estimates of $n(z)$ and $d(z)$, so significant in fact that after a finite number of uses of it, $\|\tilde{\theta}_k\|$, which is a measure of the error between $\hat{n}_k(z)$ and $n(z)$, $\hat{d}_k(z)$, and $d(z)$, is so small that it is impossible to have $|S[n(z), d(z)]| > \sigma$ and $|S[\hat{n}_k(z), d_k(z)]| \leqslant \sigma/2$ for suitably large k. This means that no further special strategy use will be required.

The detail of the argument will proceed by contradiction.

The first lemma simply sets out a state-variable statement of the effect of using one of the fixed controllers of the special strategy, with a choice of state vector intimately related to the regression vector of the parameter identification algorithm.

Lemma 6.1: Under assumptions A1)–A6), and use of the algorithm of Section IV, suppose that uses of the special strategy occur commencing at times $k_1, k_2, \cdots$. Suppose that during the jth occasion, commencing at time k_j, the controller parameters $\bar{k}_{i_j}(z)$, $\bar{h}_{i_j}(z)$ are used over the interval $[l_{i_j}, l_{i_j} + (L + 2s - 2)]$ so that

$$u_k = u_k^* + \frac{\bar{k}_{i_j}(z)}{z^{s-1}}\, u_k + \frac{\bar{h}_{i_j}(z)}{z^{s-1}}\, y_k. \tag{6.1}$$

Define

$$x_k = [u_{k-1}\ u_{k-2} \cdots u_{k-s+1}\ y_k\ y_{k-1} \cdots y_{k-s+1}] \tag{6.2}$$

$$\bar{k}_{i_j}(z) = \sum_{i=0}^{s-2} k^{(i)} z^i \quad \bar{h}_{i_j}(z) = \sum_{i=0}^{s-1} h^{(i)} z^i \tag{6.3}$$

and

$$d(z)=\sum_{i=0}^{s} d^{(i)}z^i \quad n(z)=\sum_{i=0}^{t} n^{(i)}z^i \tag{6.4}$$

(this being a slightly different notation to that of Section II.). Then for $k = [l_{ij}, l_{ij} + (L + 2s - 1)]$

$$x_{k+1}=Ax_k+bu_k^* \tag{6.5}$$

where, if $t \leqslant s - 2$,

$$A = \left[\begin{array}{cccc|cccc} k^{(s-2)} & \cdots & k^{(1)} & k^{(0)} & h^{(s-1)} & \cdots & h^{(1)} & h^{(0)} \\ 1 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & & 1 & 0 & & & 0 & 0 \\ \hline & \cdots & n^{(1)} & n^{(0)} & -d^{(s-1)} & \cdots & -d^{(1)} & -d^{(0)} \\ 0 & \cdots & 0 & 0 & & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right] \quad b = \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right] \tag{6.6a}$$

and if $t = s - 1$,

$$A = \left[\begin{array}{cccc|cccc} k^{(s-2)} & \cdots & k^{(1)} & k^{(0)} & h^{(s-1)} & \cdots & h^{(1)} & h^{(0)} \\ 1 & & 0 & 0 & 0 & & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline p^{(s-2)} & \cdots & p^{(1)} & p^{(0)} & -e^{(s-1)} & \cdots & -e^{(1)} & -e^{(0)} \\ 0 & \cdots & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right] \quad b = \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hline n^{(s-1)} \\ 0 \\ \vdots \\ 0 \end{array}\right] \tag{6.6b}$$

where

$$p(z)=\sum_{i=0}^{s-2} p^{(i)}z^i=[n(z)-n^{(s-1)}z^{s-1}]+n^{(s-1)}\bar{k}_{ij}(z) \tag{6.7a}$$

$$e(z)=z^s+\sum_{i=0}^{s-1} e^{(i)}z^i=d(z)-n^{(s-1)}\bar{h}_{ij}(z). \tag{6.7b}$$

Moreover, for $t \leqslant s - 2$

$$(zI-A)^{-1}b=\frac{1}{z^{s-1}d(z)-\bar{k}_{ij}(z)d(z)-\bar{h}_{ij}(z)n(z)} \cdot [z^{s-2}d(z) \cdots d(z)\ z^{s-1}n(z) \cdots n(z)]^T \tag{6.8}$$

and for $t = s - 1$, the same equation holds save that every occurrence of $n(z)$ in the numerator vector is replaced by $n(z) - n^{(s-1)}z^{s-1}$.

Proof: When $t \leqslant s - 2$, the first row of (6.5) is equivalent to (6.1), and this accounts for the first row of A and b in (6.6a). Also, $d(z)y_k = n(z)u_k$, and this equation when rearranged leads to the sth row of (6.5) and accounts for the sth row of A and b. The remaining rows of A and b are easily accounted for by the form of x in (6.2).

When $t = s - 1$, observe that

$$\begin{aligned} y_{k+1}&=n^{(s-1)}u_k+\cdots+n^{(0)}u_{k-s+1}-d^{(s-1)}y_k\cdots-d^{(0)}y_{k-s+1} \\ &=n^{(s-2)}u_{k-1}+\cdots+n^{(0)}u_{k-s+1}-d^{(s-1)}y_k\cdots-d^{(0)}y_{k-s+1} \\ &\quad+n^{(s-1)}\{[k^{(s-2)}\cdots k^{(1)}\ k^{(0)}\ h^{(s-1)}\cdots h^{(1)}\ h^{(0)}]x_k+u_k^*\} \end{aligned}$$

and this accounts for the $(s + 1)$th row in (6.6b).

Next, simple algebra will verify if $t \leqslant s - 2$ that

$$[z^{s-1}d(z)-\bar{k}_{ij}(z)d(z)-\bar{h}_{ij}(z)n(z)]b =(zI-A)[z^{s-2}d(z)\cdots d(z)\ z^{s-1}n(z)\cdots n(z)]^T.$$

This yields (6.8). The variation when $t = s - 1$ is similarly easily obtained. ▽▽▽

Below, we shall be relating $\bar{x}_k$ to the error signal e_k, which we recall from (2.7) is

$$e_{k+1}=[n(z)-\hat{n}_k(z)]u_{k-s+1}+[\hat{d}_k(z)-d(z)]y_{k-s+1}. \tag{6.9}$$

To assist in setting up the relation, we shall also define

$$\bar{e}_{k+1}=[n(z)-\hat{n}_{k_j}(z)]u_{k-s+1}+[\hat{d}_{k_j}(z)-d(z)]y_{k-s+1} \tag{6.10}$$

(with k variable and k_j fixed). Also, define

$$n-\hat{n}_{k_j}\triangleq\bar{n}(z)=\bar{n}^{(t)}z^t+\cdots+\bar{n}^{(0)}$$

$$d-\hat{d}_{k_j}\triangleq\bar{d}(z)=\bar{d}^{s-1}z^s+\cdots+\bar{d}^{(0)}. \tag{6.11b}$$

The next lemmas and corollaries are concerned with $\bar{e}_k$. The key ideas are to define $\bar{e}_{k+1}$ both using state variable ideas and transfer function ideas, and then to argue that x_k must be observable from $\bar{e}_{k+1}$. This observability must not be lost in the limit as $k \to \infty$, but must be in some way uniform in the quantities $\hat{d}_{k_j}(z)$, $\hat{n}_{k_j}(z)$. This will take us to Lemma 6.3.

Lemma 6.2: With the same hypothesis as Lemma 6.1 and the above definition (6.10) of $\bar{e}_k$, there holds in case $t < s - 1$,

$$\bar{e}_{k+1} = c^T x_k \quad c^T = [0 \cdots 0 \; \tilde{n}^{(t)} \cdots \tilde{n}^{(0)} \; -\tilde{d}^{(s-1)} \cdots -\tilde{d}^{(0)}] \tag{6.12a}$$

and if $t = s - 1$,

$$\bar{e}_{k+1} = c^T x_k + j u_k^* \tag{6.12b}$$

for a different c and a nonzero j. Moreover, the transfer function mapping from u_k^* to $\bar{e}_{(k+1)}$ is defined, for $t \leqslant s - 1$, by

$$\bar{e}_{k+1} = \frac{\hat{d}_{k_j}(z)n(z) - \hat{n}_{k_j}(z)d(z)}{z^{s-1}d(z) - \bar{k}_{i_j}(z)d(z) - \bar{h}_{i_j}(z)n(z)} u_k \tag{6.13}$$

and the mapping is valid over the interval $[l_{i_j}, l_{i_j} + (L + 2s - 2)]$.

Proof: Equation (6.12) is an immediate consequence of the definition (6.2) of x_k, and (6.10) and (6.11). Using (6.8), one obtains for $t \leqslant s - 2$

$$\begin{aligned}\bar{e}_{k+1} &= c^T(zI - A)^{-1} b u_k^* \\ &= \frac{\tilde{n}(z)d(z) - \tilde{d}(z)n(z)}{z^{s-1}d(z) - \bar{k}_{i_j}(z)d(z) - \bar{h}_{i_j}(z)n(z)} u_k^* \\ &= \frac{\hat{d}_{k_j}(z)n(z) - \hat{n}_{k_j}(z)d(z)}{z^{s-1}d(z) - \bar{k}_{i_j}(z)d(z) - \bar{h}_{i_j}(z)n(z)} u_k^*.\end{aligned}$$

Next, if $t = s - 1$, (6.10) yields

$$\begin{aligned}e_{k+1} &= [\tilde{n}^{(s-2)} \cdots \tilde{n}^{(0)} \; -\tilde{d}^{(s-1)} \cdots \tilde{d}^{(0)}] \\ &\quad \cdot x_k + \tilde{n}^{(s-1)}[1 \;\; 0 \;\; \cdots \;\; 0] x_{k+1} \\ &= [\tilde{n}^{(s-2)} \cdots \tilde{n}^{(0)} \; -\tilde{d}^{(s-1)} \cdots \tilde{d}^{(0)}] \\ &\quad \cdot x_k + \tilde{n}^{(s-1)}[1 \;\; 0 \;\; \cdots \;\; 0](A x_k + b u_k^*)\end{aligned}$$

[where A, b are given by (6.6b)]

$$\begin{aligned}&= [\tilde{n}^{(s-2)} \cdots \tilde{n}^{(0)} \; -\tilde{d}^{(s-1)} \cdots \tilde{d}^{(0)}] x_k \\ &\quad + \tilde{n}^{(s-1)}[k^{(s-2)} \cdots k^{(0)} \; h^{(s-1)} \cdots h^{(0)}] x_k \\ &\quad + \tilde{n}^{(s-1)} u_k^*.\end{aligned}$$

Now use (6.8) with numerator occurrences of $n(z)$ replaced by $n(z) - n^{(s-1)}z^{s-1}$, as required when $t = s - 1$. There results

$$\begin{aligned}\bar{e}_{k+1} &= \frac{[\tilde{n}(z) - \tilde{n}^{(s-1)}z^{s-1}]d(z) - \tilde{d}(z)[n(z) - n^{(s-1)}z^{s-1}]}{z^{s-1}d(z) - \bar{k}_{i_j}(z)d(z) - \bar{h}_{i_j}(z)n(z)} u_k^* \\ &\quad + \tilde{n}^{(s-1)} \frac{\bar{k}_{i_j}(z)d(z) + \bar{h}_{i_j}(z)[n(z) - n^{(s-1)}z^{s-1}]}{z^{s-1}d(z) - \bar{k}_{i_j}(z)d(z) - \bar{h}_{i_j}(z)n(z)} u_k^* \\ &\quad + \tilde{n}^{(s-1)} u_k^* \\ &= \frac{\tilde{n}(z)d(z) - \tilde{d}(z)n(z)}{z^{s-1}d(z) - \bar{k}_{i_j}(z)d(z) - \bar{h}_{i_j}(z)n(z)} u_k^* \\ &= \frac{\hat{d}_{k_j}(z)n(z) - \hat{n}_{k_j}(z)d(z)}{z^{s-1}d(z) - \bar{k}_{i_j}(z)d(z) - \bar{h}_{i_j}(z)n(z)} u_k^*\end{aligned}$$

as before. ▽▽▽

Corollary 6.1: With the same hypothesis as Lemma 6.2 and with the assumption, validated by Corollary 5.1, that all zeros of $d(z)[z^{s-1} - \bar{k}_{i_j}(z)] - n(z)\bar{h}_{i_j}(z)$ are at a distance of at least ϵ from all zeros of $\hat{d}_{k_j}(z)n(z) - \hat{n}_{k_j}(z)n(z)$, the state-variable system mapping u_k^* into $\bar{e}_{k+1}$ defined by (6.5) and (6.12) is controllable and observable.

Proof: By Corollary 5.1, the numerator and denominator of the transfer function defined in (6.13) are coprime. Since the denominator has degree equal to the dimension of x_k, controllability and observability are immediate.

Lemma 6.3: With the same hypotheses as Corollary 6.1, the determinant of the observability matrix of the pair (A, c) has magnitude

$$|S[d(z), n(z)]^{-1} S[\hat{d}_{k_j}(z)n(z) - \hat{n}_{k_j}(z)d(z), \\ \cdot z^{s-1}d(z) - \bar{k}_{i_j}(z)d(z) - \bar{h}_{i_j}(z)n(z)]|.$$

Proof: Define the matrix T by

$$T = \begin{bmatrix} 1 & & d^{(s-1)} & \cdots & d^{(0)} & 0 & & \cdots & 0 \\ 0 & & 1 & \cdots & d^{(1)} & d^{(0)} & & \cdots & 0 \\ \vdots & & & & & \vdots & & & \vdots \\ 0 & & & & & & & & d^{(0)} \\ 0 & \cdots & n^{(t)} & \cdots & n^{(0)} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & & & n^{(1)} & n^{(0)} & 0 & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \cdots & & & & & & & n^{(0)} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} s-1 \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} s \end{matrix} \tag{6.14}$$

(If $t = s - 1$, the first entry in row s is $n^{(s-1)}$, not 0.) Then easy direct calculation using the definition (6.6) shows that in both cases $t < s - 1$ and $t = s - 1$

$$(zI - A)T \begin{bmatrix} z^{2s-2} \\ z^{2s-3} \\ \vdots \\ 1 \end{bmatrix} = T \begin{bmatrix} z^{s-1}d(z) - \bar{k}_{i_j}(z)\bar{d}(z) - \bar{h}_{i_j}(z)n(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$T\begin{bmatrix}1\\0\\\cdot\\\cdot\\\cdot\\0\end{bmatrix}=b$$

hence

$$(zI-T^{-1}AT)\begin{bmatrix}z^{2s-2}\\z^{2s-3}\\\cdot\\\cdot\\\cdot\\1\end{bmatrix}=\begin{bmatrix}z^{s-1}d(z)-\bar{k}_{i_j}(z)d(z)-\bar{h}_{i_j}(z)n(z)\\0\\\cdot\\\cdot\\\cdot\\0\end{bmatrix}\quad T^{-1}b=\begin{bmatrix}1\\0\\\cdot\\\cdot\\\cdot\\0\end{bmatrix}.$$

Therefore, $[T^{-1}AT,\ T^{-1}b]$ has the completely controllable canonical form. The observability matrix associated with $[c^TT,\ T^{-1}AT,\ T^{-1}b]$ will be

$$\begin{bmatrix}c^T\\c^TA\\\cdot\\\cdot\\\cdot\\c^TA^{2s-2}\end{bmatrix}^T$$

and (see [21], [23]) it has determinant equal in magnitude to the resultant of $\hat{d}_{k_j}(z)n(z)-\hat{n}_{k_j}(z)d(z)$ and $z^{s-1}d(z)-\bar{k}_{i_j}(z)d(z)-\bar{h}_{i_j}(z)n(z)$, while the determinant of T is $S[d(z),\ n(z)]$. The result of the lemma is then immediate. ∇∇∇

Corollary 6.2: With the same hypothesis as Lemma 6.3, the observability matrix of the pair $(A,\ c)$ has determinant magnitude bounded away from zero by a function of ϵ, α and the given quantities $n(z)$, $d(z)$, and $\|\tilde{\theta}_0\|$.

Proof: Note that $|S[\hat{n}_{k_j}(z),\ \hat{d}_{k_j}(z)]|<\sigma/2$ (this being the condition which gives rise to use of the special strategy) while $|S[n(z),\ d(z)]|>\sigma$, by assumption A2). Also, by Theorem 2.1, $\|\tilde{\theta}_k\|$ is monotone decreasing, so the coefficients of $\hat{n}_{k_j}(z)$, $\hat{d}_{k_j}(z)$ lie in some ball dependent on $\|\tilde{\theta}_0\|$. The polynomials $d(z)$ and $\hat{d}_{k_j}(z)$ are monic. By Lemma 5.3, the sum of the squares of the coefficients of $\hat{d}_{k_j}(z)n(z)-\hat{n}_{k_j}(z)d(z)$ is bounded away from zero by a function of σ, and the radius of the ball in which the coefficients of $n(z)$, $d(z)$, $\hat{n}_{k_j}(z)$, and $\hat{d}_{k_j}(z)$ lie, or equivalently by a function of σ, $n(z)$, $d(z)$, and $\|\tilde{\theta}_0\|$. The sum of the squares of the coefficients of $\hat{d}_{k_j}(z)n(z)-\hat{n}_{k_j}(z)n(z)$ is also bounded above (again in terms of given quantities and $\|\tilde{\theta}_0\|$).

By Corollary 5.2, identifying $r_j(z)$ with $z^{s-1}d(z)-\bar{k}_{i_j}d(z)-\bar{h}_{i_j}(z)n(z)$

$$|S[\hat{d}_{k_j}(z)n(z)-\hat{n}_{k_j}(z)d(z),\ z^{s-1}d(z)-\bar{k}_{i_j}(z)d(z)-\bar{h}_{i_j}(z)n(z)]|$$

is bounded away from zero by a function of ϵ, σ and given quantities $n(z)$, $\|\tilde{\theta}_0\|$, and the fixed collection $\bar{h}_i(z)$, $\bar{k}_i(z)$. Then by Lemma 6.3, the observability matrix of the pair $(A,\ c)$ has determinant bounded away from zero by a function of ϵ, σ and the given quantities while the determinant is always bounded in terms of the same given quantities. ∇∇∇

The whole point of proving observability is to allow us to bound x_k in terms of $\bar{e}_j$ and u_j, where j belongs to a certain interval. This bound will later be converted into a linear boundedness condition relating ϕ_k and e_{k+1} (see Lemma 6.6 below). The value of linear boundedness conditions for establishing desirable properties of adaptive schemes was demonstrated clearly in [20]. What is different here is the route by which the linear boundedness condition is derived—in [20] by appeal to a minimum phase property—here by appeal to observability. The fact that there is an alternative approach to demonstrating a linear boundedness condition is itself important, as the technique is likely to have application outside this paper.

Lemma 6.4: Assume the same hypotheses as Lemma 6.1, the definition (6.10) of $\bar{e}_k$, and assume (in accordance with Corollary 5.1) that all zeros of $z^{s-1}d(z)-\bar{k}_{i_j}(z)d(z)-\bar{h}_{i_j}(z)n(z)$ are a distance of at least ϵ from all zeros of $\hat{d}_{k_j}(z)n(z)-\hat{n}_{k_j}(z)n(z)$. Let

$$|\bar{e}_{m_j+1}|=\max\ |\bar{e}_{k+1}|\ \text{ for } k\in[l_{i_j},\ l_{i_j}+(L+2s-2)]\,. \tag{6.15}$$

Then

$$\|x_{m_j}\|\leqslant K_1|\bar{e}_{m_j}+1|+K_2 \tag{6.16}$$

where K_1, K_2, are positive constants depending only on ϵ, σ, $n(z)$, $d(z)$, and $\|\tilde{\theta}_0\|$ and the bound on u_k^*.

Proof: The idea of observability is that one can discern the state at some time k by observing the input and output up to time k, or from time k on, for an interval of suitable length. In our case, we are guaranteed the existence of constants $\bar{K}_1$, $\bar{K}_2$ with, for $k\in[l_{i_j}+2s-1,\ l_{i_j}+L+2s-2]$

$$\|x_k\|\leqslant\bar{K}_1\max_{j\in[k-2s+2,\ k]}|e_{j+1}|+\bar{K}_2\max_{j\in[k-2s+2,\ k-1]}|u_k^*| \tag{6.17a}$$

and also for $k\leqslant[l_{i_j},\ l_{i_j}+(L+2s-2)-(2s-2)]$

$$\|x_k\|\leqslant\bar{K}_1\max_{j\in[k,\ k+2s-2]}|\bar{e}_{j+1}|+\bar{K}_2\max_{j\in[k,\ k+2s-3]}|u_k^*|\,. \tag{6.17b}$$

Note that the constants $\bar{K}_1$, etc., can be bounded in the same manner as K_1, K_2 because 1) the observability matrix has a determinant underbound as described in Corollary 6.2, and 2) the system equations [see (6.5) and (6.12)] contain parameters which are bounded in the same terms. Equation (6.16) now follows from (6.17a) and (6.17b).

Our real interest is in e_k rather than $\bar{e}_k$; these two quantities differ in a sense made precise by comparing (6.9) and (6.10). Whereas $\bar{e}_{k+1}$ is the output of a time-invariant linear system [over l_{i_j}, $l_{i_j}+(L+2s-2)$], e_{k+1} is the output of a time-varying linear system. Note that the time-variation is totally confined to the vector c and possible scalar j in (6.12), while A, b as defined in (6.6) stay the same. The following result explains how Lemma 6.4 carries over to e_k rather than $\bar{e}_k$.

Lemma 6.5: Assume the same hypotheses as Lemma 6.4. If the sequence k_1, k_2, $\cdots$, is infinite, then for all sufficiently large k_j, Lemma 6.4 is true with e_k replacing $\bar{e}_k$.

Proof: By Theorem 2.2, $\|\hat{\theta}_k-\hat{\theta}_{k-1}\|\to 0$. Hence, $\|\hat{\theta}_{k+l}-\hat{\theta}_k\|\to 0$ for any fixed l. Hence, given arbitrary $\delta>0$ and an infinite sequence k_1, k_2, $\cdots$, there exists a k_J such that for all $k_j\geqslant k_J$ and all $k\in[l_{i_j},\ l_{i_j}+(L+2s-2)]$,

$$\|\hat{\theta}_k-\hat{\theta}_{k_j}\|<\delta. \tag{6.17}$$

Hence, the observability Gramian for the time-invariant system associated with $\bar{e}_{k+1}$ over $[l_{i_j},\ l_{i_j}+(L+2s-2)]$ will differ in norm from that associated with e_{k+1} over the same interval by $0(\delta)\times$ positive constants depending only on $n(z)$, $d(z)$ and $\|\tilde{\theta}_0\|$. [Note that this Gramian depends continuously on the entries of vector c in (6.12), which depends continuously on the estimates $\hat{n}_k(z)$, $\hat{d}_k(z)$.] For the $\bar{e}_k$ system, the observability Gramian is

$$\sum_{i=0}^{2s-2} (A^T)^i cc^T A^i = [cA^Tc \cdots (A^T)^{2s-2}c] \begin{bmatrix} c^T \\ c^TA \\ \cdot \\ \cdot \\ \cdot \\ c^TA^{2s-1} \end{bmatrix} \quad (6.18)$$

and the determinant is the square of the determinant of the observability matrix referred to in Corollary 6.2. Hence, by choosing δ sufficiently small, we can ensure that the observability Gramian associated with the e_{k+1} system also has determinant bounded away from zero in terms of ϵ, σ, $n(z)$, $d(z)$, and $\|\tilde{\theta}_0\|$. The remainder of the argument of Lemma 6.4 immediately carries over.

Remark 6.1: As noted in the proof, there exists a J such that for all $k_j \geqslant k_J$, Lemma 6.4 holds with $\bar{e}_k$ replaced by e_k. Observe that while Lemma 6.4 makes no reference to the particular u_k^* sequence used (as opposed to the bound on it), the value of J will in general depend on the particular sequence.

Lemma 6.6: Assume the same hypotheses as Lemma 6.4; if the sequence $k_1, k_2, \cdots$, is infinite, there exist constants K_3, K_4 depending only on ϵ, σ, $n(z)$, $d(z)$ and $\|\tilde{\theta}_0\|$ and a k_J such that for all $k_j \geqslant k_J$

$$\|\phi_{m_j}\| \leqslant K_3 |e_{m_j+1}| + K_4 \quad (6.19)$$

where ϕ_k is as defined in (2.4) and is repeated as

$$\phi_k = [u_{k-s+1}\ u_{k-2+2} \cdots u_{k-s+t+1}\ y_{k-s+1} \cdots y_k. \quad (6.20)$$

Proof: If $t \leqslant s-2$, every entry of ϕ_k appears as an entry of x_k [see (6.2)]. If $t = s-1$, the entries of ϕ_k comprise the entries of x_k together with u_k. The defining equation for u_k, viz. (6.1), expresses u_k as a linear functional of x_k plus u_k^*. The result follows by Lemma 6.5.

Corollary 6.3: Assume the same hypotheses as Lemma 6.4; if the sequence $k_1, k_2 \cdots$ is infinite, then $e_k \to 0$ on the collection of intervals $[l_{ij}, l_{ij} + (L + 2s - 2)]$ as $j \to \infty$, and $\|\phi_k\|$ is bounded on the union of these intervals.

Proof: Use Lemmas 6.6 and 3.1 to conclude that $e_{m_j+1} \to 0$ as $j \to \infty$, and so, with

$$|e_{m_j+1}| = \max_{k \in [l_{ij},\, l_{ij}+(L+2s-2)]} |e_k|$$

$e_k \to 0$ on $[l_{ij}, l_{ij}, + (L + 2s - 2)]$ as $j \to \infty$. By (6.1b), $\|x_k\|$ is bounded on $[l_{ij}, l_{ij}, + (L + 2s - 2)]$. The relation between $\|\phi_k\|$ and $\|x_k\|$, as explicated in the proof of Lemma 6.6, ensures that $\|\phi_k\|$ is bounded on the collection of intervals $[l_{ij}, l_{ij} + (L + 2s - 2)]$, $k_j \geqslant k_J$; the bound depends only on ϵ, σ, $n(z)$, $d(z)$ and $\|\tilde{\theta}_0\|$.

Remark 6.2: We have almost completed an argument by contradiction which establishes that the sequence $k_1, k_2, \cdots$, cannot be infinite. One can see heuristically from Corollary 6.3 in fact that the sequence is probably finite. For the controllers used during the special strategy may destabilize the system during their use. It follows that their use on an infinite number of occasions is likely to destabilize the system. Of course, the argument is only suggestive: it is easy to give examples of systems with periodically switched controllers where there is "temporary" instability, and yet the closed-loop system is exponentially stable.

Remark 6.3: Since for $j < J$ the number of intervals $[l_{ij}, l_{ij} + (L + 2s - 2)]$ is finite, there exists a bound on $\|\phi_k\|$ applying on all intervals. The bound for those with $j < J$ will be dependent on the particular u_k^* sequence.

Remark 6.4: So far in this section, no use has been made of the persistency of excitation property of u_k^*, assumption A4). Now it will prove crucial.

Lemma 6.7: Assume the same hypotheses as Lemma 6.4; assume the sequence $k_1, k_2, \cdots$, is infinite. Let J be such that for $k_j \geqslant k_J$, $\|\phi_k\|$ is bounded in terms of ϵ, σ, $n(z)$, $d(z)$, and $\|\tilde{\theta}_0\|$. Then there exists $\alpha < 1$, depending only on ϵ, σ, $n(z)$, $d(z)$, $\|\tilde{\theta}_0\|$, $\bar{k}_{ij}(z)$, $\bar{h}_{ij}(z)$, and the constants in the persistency of excitation definition, such that for $k_j \geqslant k_J$

$$\|\tilde{\theta}_{l_{ij}+(L+2s-2)}\| \leqslant \alpha \|\tilde{\theta}_{l_{ij}}\| \qquad \alpha < 1. \quad (6.21)$$

Proof: By Lemma 3.2 as extended by Corollary 3.3 (see also Remark 3.4) there exists a positive β_1 such that

$$\sum_{l_{ij}}^{l_{ij}+(L+2s-2)} \phi_k \phi_k^T > \beta_1 I \quad (6.22a)$$

with β_1 depending only on $n(z)$, $d(z)$, the persistency bounds on u_k^*, and the magnitude of the coefficients in $\bar{h}_{ij}(z)$, $\bar{k}_{ij}(z)$. Also, there exists β_2 such that

$$\beta_2 I > \sum_{l_{ij}}^{l_{ij}+(L+2s-2)} \phi_k \phi_k^T \quad (6.22b)$$

with the dependence of β_2 as explained in the lemma statement. Now note from (2.5) that

$$\tilde{\theta}_{k+1} = \left[I - \frac{\mu \phi_k \phi_k^T}{1 + \phi_k^T \phi_k} \right] \tilde{\theta}_k. \quad (6.23)$$

It follows from (6.22) and (6.23) by analysis as set out in [17], [25] that (6.14) holds where α depends on β_1, β_2 and μ, and thus on the quantities listed in the lemma statement.

Remark 6.5: The value of α above depends on the particular controller polynomials $\bar{k}_{ij}(z)$, $\bar{h}_{ij}(z)$ used on the jth occasion the special strategy is employed. Let α_i denote the value of α obtained with control polynomials $\bar{k}_{ij}(z)$, $\bar{h}_{ij}(z)$ and set $\alpha = \max_i \alpha_i$. Then $\alpha < 1$, since the maximization is over a finite set.

Finally, we can rule out the existence of an infinite number of k_j.

Lemma 6.8: Assume the same hypotheses as Lemma 6.4. Then the sequence $k_1, k_2, \cdots$, is finite.

Proof: Suppose it is infinite. Recall from Theorem 2.1 that $\|\tilde{\theta}_k\|$ is monotone nonincreasing. Lemma 6.7 and Remark 6.5 show that on the intervals $[l_{ij}, l_{ij} + L + 2s - 2]$, $\|\tilde{\theta}_k\|$ decreases by a minimum fixed percentage. It follows that if there are an infinite number of such intervals, $\tilde{\theta}_k \to 0$. But $|S[n(z), d(z)]| \geqslant \sigma$, and so $\lim_{k\to\infty} |S[\hat{n}_k(z), \hat{d}_k(z)]| \geqslant \sigma$. Hence, there exists a k' such that for all $k \geqslant k'$, $S[\hat{n}_k(z), \hat{d}_k(z)] > \sigma/2$. The algorithm statement then guarantees that there can be no special strategy use for $k \geqslant k'$. Hence, the sequence $k_1, k_2, \cdots$, is finite.

Lemma 6.8 is the key to proving the theorem.

Proof of Theorem: By Lemma 6.8, the special strategy will not be used after some time k', so that for all $k \geqslant k'$, $|S[\hat{n}_k(z), \hat{d}_k(z)]| \geqslant \sigma$. Then for $k \geqslant k'$ the conditions of Theorem 3.1 are fulfilled, and this theorem establishes the result.

VII. Other Comments and Concluding Remarks

Robustness: Exponential convergence of the algorithm will guarantee (by a variant on arguments of [24]) that the algorithm will be robust in the presence of noise, nonlinearity, unmodeled modes, and time-variation of plant parameters, in the sense that trajectories in these nonideal cases will be close to trajectories in the ideal case, provided that the departure from ideality is not great.

Other Feedback Structures: It is clear that other feedback structures could be contemplated, e.g., $u_k = u_k^* + p^{-1}(z)q(z)y_k$, and that if an observer/state-feedback law structure is contemplated, observers need not be deadbeat.

Reducing the Number of Different Controllers When Using the Special Strategy: In the equation error algorithm the vector ϕ_{k-1} plays a crucial role. The special strategy is designed to ensure this is persistently exciting on at least one of the $2(s + t) + 1$ intervals making up one use of the special strategy. If the persistency of excitation of ϕ_k can be monitored, then any one use of the special strategy could be terminated after the first of the $2(s + t) + 1$ intervals in which ϕ_k is found to be persistently exciting. Moreover, if the $\bar{h}_i$, $\bar{k}_i$ are selected randomly, then generically, the persistency of excitation will be achieved in the first of the $2(s + t) + 1$ intervals. Thus, one does not always have to incur the apparent inefficiency involved in using $2(s + t) + 1$ different, rather arbitrary, controllers.

Linear-Quadratic Adaptive Control: The same technique of introducing the special strategy controls appears applicable for tackling adaptive linear-quadratic problems, where typically difficulties arise when the estimated plant state-variable equations lack controllability or observability, or the estimated plant transfer function has a pole-zero cancellation.

Relation with the First-Order Problem: The argument here only differs from that of the first-order case [15] in the way that the "linear boundedness" condition of (6.19) is achieved. The core of the idea is that for an nth order observable system with bounded input, the state x_k can be bounded by $\|x_k\| \leqslant K_1$ max $[|y_{k-n+1}|, \cdots, |y_k|] + K_2$ and by K_1' max $[|y_k|, \cdots, |y_{k+n-1}|] + K_2'$, with y_k the output. In our case, x_k and y_k are replaced by ϕ_k and e_{k+1}.

Impossibility of Special Strategy Based on No Feedback: "Awkward" stages of the nonglobal algorithm are dealt with using special controllers; improvement in plant identification takes place during their use, apparently independently of whether they are stabilizing. Why could one not just use feedback of zero during such occasions, even if the plant is unstable? The answer is that one can use a feedback of zero, provided it is one of $2(s + t) + 1$ feedback law selections, or one finds that on each case when zero feedback is used that there is persistency of excitation. However, if during the period the zero control is used, we have $\hat{d}_{k_j} = d$, the transfer function linking u_k^* to $\bar{e}_{k+1}$ becomes $[\hat{d}_{k_j}(z)n(z) - \hat{n}_{k_j}(z)d(z)]/z^{s-1}d(z) = [n(z) - \hat{n}_{k_j}(z)]d(z)/z^{s-1}d(z)$ and there is a pole-zero cancellation. The observability which was crucial to the argument of Section VI is lost. The same question can be asked of the algorithm in the first-order case (see [15]) where the answer to the question is much simpler to comprehend.

It is the observability of course which ultimately guarantees the boundedness of the various signals. One might then ask how signals could be guaranteed bounded as a result of use of an arbitrary controller, the long term use of which could lead to unbounded signals. This paradox is resolved by noting that, precisely because of the observability, the arbitrary controllers are only used over intervals of finite length, and in total for a finite number of time intervals. But in any case, even if such controllers were used on an infinite number of occasions, this would not necessarily imply the presence of unbounded signals, since a succession in time of stabilizing and nonstabilizing controllers repeated infinitely often can quite readily stabilize a system.

Use of Memoryless Special Strategy Controllers: Using memoryless controllers is equivalent to taking $\bar{k}_i = 0$ and $\bar{h}_i = h_i z^{s-1}$ with h_i constant; the resulting closed-loop characteristic polynomial is $r_i = z^{s-1}(d - nh_i)$. If $h_i \neq h_j$, the only zeros in common are at $z = 0$. It then follows easily that if 2ϵ is the minimum spacing of the zeros of $2s$ such polynomials (excluding zeros at the origin), an arbitrary $(2s - 1)$th degree polynomial will have its zeros separated by at least ϵ from at least one of the r_i. Then the earlier arguments will carry through. It would therefore seem that there is an advantage in using only $2s$ rather than $2(s + t) + 1$ different special controllers. This may well be the case on some occasions. However, a typical applications situation may involve a fairly well-known plant, which is more easily or attractively stabilized with the more complicated $\bar{h}_i(z)$, $\bar{k}_i(z)$ pairs used earlier than with a constant control law.

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