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Spreading the control complexity in decentralized control of interconnected systems

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The problem is considered for large scale interconnected systems of designing decentralized dynamic controllers so that the sum of the orders of the local controllers is approximately the system order, while the separate controller orders are more evenly spread than permitted by the procedure of Corfmat and Morse, which generally leads to all memoriless controllers, save one of high complexity. The paper shows that under certain circumstances, it is possible to design dynamic feedback controllers for each channel which introduce stable uncontrollable or unobservable modes to the overall system, thus lowering the effective order of that system. This result is then used to spread the controller complexity.

Keywords: Decentralized control, Interconnected systems, Complexity, Linear systems, Local feedback.

1. Introduction

Among the significant results of decentralized control is a complete characterization of those linear time invariant systems which can be stabilized by decentralized (linear time invariant) dynamic output feedback (Wang and Davison [8]). This result, however, goes only part way towards answering problems of practical interest. For example, it is not known which (dynamic) orders can be chosen for the local controllers. This problem is difficult, since even in the centralized case, despite many efforts, the problem is far from having a full solution. Only estimates for the minimal order of a centralized stabilizing controller are known (see e.g. Brasch and Pearson [1], Byrnes and Stevens [2]). In the decentralized case, it is known (Corfmat and Morse [3]) that all but one local controller can be taken to be of dynamic order zero (i.e. memoriless feedback) and the remaining controller can be chosen to be of order $n-1$, if the system is strongly connected (which is no severe restriction). However, quite apart from lowering further the sum of the orders of the decentralized feedback controllers, it is of great practical importance to spread the control complexity more equally among the local controllers, i.e. to endow each local controller with some dynamics.

In this paper, a result in this direction will be presented for interconnected systems, where the subsystems are coupled only via their inputs and outputs. More specifically, it will be shown that for stabilization purposes, *the order of the i-th local controller can be taken to be at least the number of stable zeros of the i-th subsystem, while the total order of the decentralized controller (sum of the orders of the local controllers) does not exceed* $n-1$.

The general idea for the controller selection algorithm involves the following two steps. First, local dynamic feedbacks are applied in order to create uncontrollable and unobservable modes in the subsystems. These modes, then, result in a reduction of the dimensionality of the decentralized control problem, so that, in a second design step, the interconnected system can be stabilized by local controllers of reduced orders.

All proofs to be presented are constructive, so that (at least conceptual) algorithms for the design of these feedback laws may be given. Since, in the first step, calculations have to be done only on the subsystem level and, in the second step, a reduced problem has to be solved, a useful by-product of the proposed method is a simplified procedure for the design of decentralized feedback laws for interconnected systems.

The paper is organized as follows. In Sections 2 and 3, some preliminaries on fixed modes and interconnected systems are collected. In Section 4, the main result is presented and discussed and

Sections 5 and 6 are devoted to the proof of the main theorem. Finally, in Section 7, some directions for future research are pointed out.

2. Fixed modes

Consider the linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{2.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and B and C are partitioned as follows:

$$B = [B_1, \dots, B_l], \quad B_i \in \mathbb{R}^{n \times m_i},$$
$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_1 \end{bmatrix}, \quad C_i \in \mathbb{R}^{p_i \times n}. \tag{2.2}$$

First, we recall the properties of using memoriless, decentralized feedback of the form

$$u = Fy + v \tag{2.3}$$

where

$$F = \operatorname{diag}(F_1, \dots, F_l), \quad F_i \in \mathbb{R}^{m_i \times p_i}. \tag{2.4}$$

The *decentralized fixed polynomial* of (2.1) is defined by Wang and Davison (1973) as follows:

$$\psi_{\mathrm{D}}(s; A, B, C) = \gcd\{\det(sI - A - BFC) \mid F \in \mathscr{F}\}, \tag{2.5}$$

where $\mathscr{F}$ denotes the set of all decentralized feedback matrices satisfying (2.4). Correspondingly,

$$\psi_{\mathrm{C}}(s; A, B, C) = \gcd\{\det(sI - A - BFC) \mid F \in \mathbb{R}^{m \times p}\} \tag{2.6}$$

is called the *centralized fixed polynomial* of (2.1). Thus, $\psi_{\mathrm{D}}(s; A, B, C)$ (resp. $\psi_{\mathrm{C}}(s; A, B, C)$) represents the part of the characteristic polynomial of the system, which cannot be changed by any static decentralized (resp. centralized) feedback law.

The zeros of the centralized fixed polynomial $\psi_{\mathrm{C}}(s; A, B, C)$ are the uncontrollable or unobservable eigenvalues of A and ψ_{C} obviously divides ψ_{D}. The zeros of $\psi_{\mathrm{D}}(s; A, B, C)$ (resp. $\psi_{\mathrm{C}}(s; A, B, C)$) are called *decentralized* (resp. *centralized*) *fixed modes* of the system (2.1).

A system with no decentralized fixed modes (or with only stable decentralized fixed modes in general) may not be able to be stabilized by a feedback law of the form (2.3), (2.4). However, stabilization can be achieved with dynamic feedback. Accordingly, consider decentralized dynamic feedback laws of the form

$$\begin{aligned} u &= Ez + Hy, \\ \dot{z} &= Dz + Gy, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} E &= \operatorname{diag}(E_1, \dots, E_l), & E_i &\in \mathbb{R}^{m_i \times r_i}, \\ H &= \operatorname{diag}(H_1, \dots, H_l), & H_i &\in \mathbb{R}^{m_i \times p_i}, \\ D &= \operatorname{diag}(D_1, \dots, D_l), & D_i &\in \mathbb{R}^{r_i \times r_i}, \\ G &= \operatorname{diag}(G_1, \dots, G_l), & G_i &\in \mathbb{R}^{r_i \times p_i}, \end{aligned} \tag{2.8}$$

and $r_1, \dots, r_l$ are nonnegative integers. The relation between static control laws (2.3), (2.4) and dynamic control laws (2.7), (2.8) is clarified by the following theorem of Wang and Davison [8], as far as stabilization is concerned.

Theorem 1. (a) *For any decentralized feedback law of the form* (2.7), (2.8), *the decentralized fixed polynomial* $\psi_{\mathrm{D}}(s; A, B, C)$ *is a factor of the characteristic polynomial of the closed loop system defined from* (2.1) *and* (2.7).

(b) *For every open subset S of the complex plane, symmetric about the real axis, there exists a decentralized dynamic feedback law of the form* (2.7), (2.8), *such that the characteristic polynomial of the closed loop system* (2.1), (2.7) *has the form* $\psi_{\mathrm{D}}(s; A, B, C) \cdot \psi(s)$, *where the zeros of* $\psi(s)$ *are contained in S.*

Roughly speaking, Theorem 1 says that the fixed modes cannot be moved and the non-fixed modes can be shifted into arbitrary subsets by decentralized dynamic output feedback.

From now on, it will be assumed that there is given an open subset S of the complex plane, which is symmetric about the real axis. A complex number is then called *stable* if it is contained in S.

Theorem 1 is a pure existence statement. Although the proof of Wang and Davison [8] is constructive, their construction suffers from the following three main disadvantages:

(i) the computations to construct the controller are very involved,

(ii) the orders r_i of the local controllers may be quite high,

(iii) the orders r_i of the local controllers may differ very much in magnitude.

In elaboration of these statements, suppose by way of example that the system has order n, that there are two scalar channels and no fixed modes, and that all modes are unstable. Assume also that $n-1$ modes are controllable and observable through the first channel. Then a controller of order $n-2$ is required for the first channel to place $n-1$ poles. The resulting system has order $2n-2$. Now a controller of order $2n-3$ is required for the second channel in order to stabilize the system.

The second disadvantage was removed by Corfmat and Morse [3] for strongly connected systems (strong connectedness is no severe restriction). They showed that almost all control laws of the form (2.4), (2.5) result in a closed loop system which is stabilizable through every channel. Hence, in order to stabilize a strongly connected system with no unstable fixed modes, $l-1$ local memoriless controllers and one local dynamic controller suffice, where the order of the dynamic controller is at most $n-1$. This construction also partly removes the first disadvantage, but the third disadvantage remains.

3. Interconnected systems

The results of this paper are valid for so-called interconnected systems, where the subsystems are interconnected only through their inputs and outputs. These systems have been studied in the decentralized control literature before (Sezer and Hüseyin [6,7], Saeks [5]). Some preliminaries concerning this class of systems are collected in this section.

Consider the subsystems

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i v_i, \\ y_i &= C_i x_i, \quad i = 1,\ldots,l, \end{aligned} \tag{3.1}$$

which are interconnected according to the rule

$$v_i = \sum_{\substack{j=1 \\ j \neq i}}^{l} L_{ij} y_j - u_i, \quad i = 1,\ldots,l. \tag{3.2}$$

Here,

$$A_i \in \mathbb{R}^{n_i \times n_i}, \quad B_i \in \mathbb{R}^{n_i \times m_i},$$

$$C_i \in \mathbb{R}^{p_i \times n_i}, \quad L_{ij} \in \mathbb{R}^{m_i \times p_j}.$$

These systems may be written in the form (2.1), where

$$A = \begin{bmatrix} A_1 & B_1 L_{12} C_2 & \cdots & B_1 L_{1l} C_l \\ B_2 L_{21} C_1 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{l-1} L_{l-1,l} C_l \\ B_l L_{l1} C_1 & \cdots & B_l L_{l,l-1} C_{l-1} & A_l \end{bmatrix},$$

$$B = \operatorname{diag}(B_1,\ldots,B_l), \tag{3.3}$$

$$C = \operatorname{diag}(C_1,\ldots,C_l).$$

Now we can state the following result, which is a slight generalization of the main result of [5].

Theorem 2. *For interconnected systems of the form (3.1), (3.2),*

$$\psi_{\mathrm{D}}(s; A, B, C) = \psi_{\mathrm{C}}(s; A, B, C) = \prod_{i=1}^{l} \psi_{\mathrm{C}}(s; A_i, B_i, C_i). \tag{3.4}$$

We remark that Saeks [5] establishes that the sets of zeros of the three polynomials in the theorem statement are the same.

Proof of Theorem 2. Define

$$\bar{A} = \operatorname{diag}(A_1, A_2,\ldots,A_l),$$

$$L = \begin{bmatrix} 0 & L_{12} & \cdots & L_{1l} \\ L_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & L_{l-1,l} \\ L_{l1} & \cdots & L_{l,l-1} & 0 \end{bmatrix},$$

then $A = \bar{A} + BLC$, which implies

$$\psi_{\mathrm{C}}(s; A, B, C) = \psi_{\mathrm{C}}(s; \bar{A} + BLC, B, C) = \psi_{\mathrm{C}}(s; \bar{A}, B, C).$$

Moreover, as can be checked with the Kalman structure theorem,

$$\psi_{\mathrm{C}}(s; \bar{A}, B, C) = \prod_{i=1}^{l} \psi_{\mathrm{C}}(s; A_i, B_i, C_i).$$

This proves the second equality of the theorem and shows that, without loss of generality, we can assume (A_i, B_i, C_i) to be controllable and observable. The theorem then follows from the result in [5]. □

Theorem 2 and its derivation show that we can assume without loss of generality that the systems in (3.1) are controllable and observable: If they are not controllable and observable, we can form the Kalman decomposition and restrict ourselves to the controllable and observable part of the subsystems.

4. The main result

The method to spread the control complexity more equally among the controllers and to simplify the design will now be described.

Let the decentralized controller (2.7), (2.8) be connected to the system (3.1), (3.2). Then the closed loop system is given by the matrices

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_{12} & \cdots & \tilde{A}_{1l} \\ \tilde{A}_{21} & \tilde{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{A}_{l-1,l} \\ \tilde{A}_{l1} & \cdots & \tilde{A}_{l,l-1} & \tilde{A}_l \end{bmatrix},$$

$$\tilde{B} = \operatorname{diag}(\tilde{B}_1, \ldots, \tilde{B}_l),$$

$$\tilde{C} = \operatorname{diag}(\tilde{C}_1, \ldots, \tilde{C}_l),$$

where

$$\tilde{A}_i = \begin{bmatrix} A_i + B_i H_i C_i & B_i E_i \\ G_i C_i & D_i \end{bmatrix},$$

$$\tilde{A}_{ij} = \begin{bmatrix} B_i L_{ij} C_j & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \qquad \tilde{C}_i = [C_i \quad 0],$$

for $i, j = 1, \ldots, l$, $i \neq j$. Hence the closed loop system is again of the form (3.1), (3.2) and, by Theorem 2,

$$\psi_D(s; \tilde{A}, \tilde{B}, \tilde{C}) = \prod_{i=1}^{l} \psi_C(s; \tilde{A}_i, \tilde{B}_i, \tilde{C}_i). \tag{4.1}$$

The strategy will now be to design for each $i = 1, \ldots, l$ the local controller matrices H_i, E_i, G_i and D_i such that $\psi_C(s; \tilde{A}_i, \tilde{B}_i, \tilde{C}_i)$ has degree $2r_i$ and only stable zeros (recall that r_i denotes the order of the i-th compensator). Then, by (4.1), the $2r_i$ stable zeros of $\psi_C(s; \tilde{A}_i, \tilde{B}_i, \tilde{C}_i)$ are also eigenvalues of the compensated interconnected system. Moreover, these eigenvalues cannot be changed by any further decentralized feedback which is applied to the system. Also, in accordance with the observation following the proof of Theorem 2, the dimensionality of the control problem can be reduced by forming the Kalman decompositions of the subsystems and working with the controllable, observable parts alone of these subsystems. Note that the i-th local controller depends only on the i-th subsystem (A_i, B_i, C_i). Therefore, all calculations can be done on the subsystem level in a completely decentralized and parallel manner.

It would be ideal if one could choose $r_i = n_i$. Then there would remain no non-fixed modes and the interconnected system is stabilized. But this is, of course, not possible. The analysis of the following sections will show that r_i can be taken to be the number of stable zeros (cf. Rosenbrock [4], and Sections 5, 6 of this paper) of the i-th subsystem, but no larger. More exactly, the following theorem will be constructively established in the next two sections.

Theorem 3. *Let $s_i^1, \ldots, s_i^{r_i}$ be the stable zeros of the subsystem (A_i, B_i, C_i), each zero occurring according to its multiplicity. Then a dynamic controller, described by H_i, E_i, G_i, D_i, and of order r_i can be designed, such that*

$$\psi_C(s; \tilde{A}_i, \tilde{B}_i, \tilde{C}_i) = \prod_{j=1}^{r_i} (s - s_i^j)^2.$$

Since the i-th controller adds an order r_i to the i-th subsystem but introduces $2r_i$ fixed modes, the effective dimensionality of the decentralized control problem is reduced by $r = \sum_{i=1}^{l} r_i$. Moreover, if $r_1, \ldots, r_l$ are approximately of the same size, this is done by equally distributing the control complexity.

The reduction that is achieved by this method can obviously be very sizeable. For instance, if a subsystem is minimum phase with relative order 1, then $r_i = n_i - 1$ and in the reduced decentralized control problem this subsystem is of order one.

The reduced decentralized control problem can now be solved by the method of Corfmat and Morse [3]. Then, if the system is strongly connected, all but one local controller have dynamic order r_i and the remaining controller (the j-th, say) has dynamic order

$$n - \sum_{\substack{i=1 \\ i \neq j}}^{l} r_i - 1.$$

The obvious advantage of using the method of Corfmat and Morse for the reduced problem is that a possible uniform distribution of the complexity is destroyed. However, compared to the original method of Corfmat and Morse, the proposed method has the following two advantages:

(a) the control complexity is spread amongst the controllers (but not necessarily distributed equally);

(b) the dimensionality of the control problem is reduced by only doing subsystem calculations.

5. The single input, single output case

Sections 5 and 6 are devoted to the proof of Theorem 3. Since the theorem is only concerned with subsystems, without reference to their connection to the interconnected system, one may restrict attention to the study of a single, given system (bearing in mind that this system is a subsystem of the interconnected system).

In this section, single input, single output systems will be analyzed in some detail although the results given here are only special cases of the multivariable results of the next section. This part is included for the convenience of the reader: The single variable proof is not burdened with too many technical details (especially the definition of a zero of a system is straightforward), and the general ideas become much clearer than in the multivariable case.

As it turns out, it is advantageous to work in the frequency domain; and as we are now interested in a single subsystem, we shall omit subscripts i entering the subsystem. Consider now a system defined by the transfer function

$$G(s)=N(s)/D(s),$$

where $N(s)$ and $D(s)$ are coprime polynomials with $\deg N < \deg D =: n$. Write

$$N(s)=N_-(s)\cdot N_-(s),$$

where N_- has only stable and N_+ has only unstable zeros.

First it will be shown that the claim of Theorem 3 is the best one can expect to prove. For this, suppose the compensator is given by

$$K(s)=P(s)/Q(s),$$

where $P(s)$ and $Q(s)$ are coprime polynomials with $\deg P(s) \le \deg Q(s)$. The degree of the closed loop transfer function

$$N(s)Q(s)/(D(s)Q(s)-N(s)P(s))$$

is given by $\deg DQ - \deg \psi$, where ψ is the greatest common factor of NQ and $DQ-NP$. Since ψ should have all zeros stable, we have

$$\deg \psi \le \deg N_- + \deg Q,$$

i.e.

$$\deg DQ - \deg \psi \ge n - \deg N_-.$$

This means that there is no controller such that the effective degree of the closed loop system is smaller than $n - \deg N_-$ and the fixed modes are all stable.

In the present context, Theorem 3 is equivalent to saying that one can choose P and Q (coprime; $\deg P_- \le \deg Q = \deg N_- =: r$) such that $\psi(s) = [N_-(s)]^2$. This will now be shown using the following lemma on solvability of polynomial equations.

Lemma 1. *Let $a(s)$, $b(s)$ and $c(s)$ be polynomials with a and b coprime. Then there exist polynomials $x(s)$ and $y(s)$, such that*

$$a(s)x(s)+b(s)y(s)=c(s),$$
$$\deg y(s) < \deg a(s).$$

If a, b and c are coprime, then x and b as well as a and y are coprime.

Proof. Since a and b are coprime, there exist polynomials $\bar{x}$ and $\bar{y}$ with

$$a\bar{x}+b\bar{y}=1.$$

Also, by dividing $\bar{y}c$ by a, one obtains polynomials $\hat{y}$ and y with

$$\bar{y}c=a\hat{y}+y, \quad \deg y < \deg a.$$

Hence

$$c=a\bar{x}c+b\bar{y}c=a(\bar{x}c+b\hat{y})+by,$$

giving the desired solution. Finally, $x(s_0)=b(s_0)=0$ as well as $y(s_0)=a(s_0)=0$ imply $c(s_0)=0$, which proves the supplement of the lemma. □

By the above lemma, there exist polynomials $L(s)$ and $P(s)$ satisfying

$$N_-L+N_+P=D$$

with $\deg P < \deg N_- = r$, P and N_- coprime and L and N_+ coprime. Now, if $Q := N_-$, then P and Q are coprime and have correct degrees. Moreover,

$$NQ = (N_-)^2 N_+$$

and

$$\begin{aligned} DQ - NP &= DN_- - N_- N_+ P \\ &= N_-(D - N_+ P) = N_-(N_- L). \end{aligned}$$

Hence, $(N_-)^2$ is a common factor of NQ and $DQ - NP$. It is actually the greatest common factor, since N_+ and L are coprime.

Thus, Theorem 3 is proved for the special case of single input, single output subsystems. Moreover, we have shown that the controller may be chosen to be strictly proper and stable. Note though that there is no assertion that the closed loop characteristic polynomial has all stable zeros.

6. The multivariable case

The result of the previous section will now be generalized to the multivariable case. This is done by considering the McMillan form of the transfer function matrix and thus reducing the multivariable problem to a scalar one.

As in Section 5, we start our analysis with a transfer function $p \times m$ matrix $G(s)$, assumed to be strictly proper and rational. For notational convenience we assume $p \le m$. The reverse case can be analyzed in a strictly analogous manner.

Consider the McMillan form [4] of G,

$$G(s) = U(s) F(s)^{-1} E(s) V(s),$$

where $U(s)$ is a $p \times p$ polynomial unimodular matrix, $V(s)$ is an $m \times m$ polynomial unimodular matrix and $F(s)$ and $E(s)$ are polynomial matrices with the structure

$$F(s) = \operatorname{diag}(\psi^1(s), \ldots, \psi^p(s)),$$

$$E(s) = \begin{bmatrix} \varepsilon^1(s) & 0 & \cdots & & \cdots & 0 \\ 0 & \varepsilon^2(s) & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & \varepsilon^p(s) & \underbrace{0 \ \cdots}_{m-p} & 0 \end{bmatrix}$$

with polynomials $\psi^1, \ldots, \psi^p$, $\varepsilon^1, \ldots, \varepsilon^p$, such that ψ^i and ε^i are coprime for $i = 1, \ldots, p$ (one may have $\psi^i \equiv 0$ and $\varepsilon^i \equiv 0$ for some i), and having divisibility properties as described in [4]. The divisibility properties will not be exploited here, and are therefore not explicitly stated. Note that

$$G = D^{-1} N$$

with

$$D = FU^{-1}, \quad N = EV,$$

is a left coprime factorization of G. Define the *zero polynomial* by

$$\varepsilon(s) := \varepsilon^1(s) \cdot \varepsilon^2(s) \cdot \cdots \cdot \varepsilon^p(s)$$

and the *pole polynomial* by

$$\psi(s) := \psi^1(s) \cdot \psi^2(s) \cdot \cdots \cdot \psi^p(s).$$

The zeros of ε are called the *zeros* of G and the zeros of ψ are called the *poles* of G. As in the previous section, we split the polynomial matrix E into factors,

$$E = E_- \cdot E_+,$$

$$E_- = \operatorname{diag}(\varepsilon_-^1, \ldots, \varepsilon_-^p),$$

$$E_+ = \begin{bmatrix} \varepsilon_+^1 & 0 & \cdots & & & \cdots & 0 \\ 0 & \varepsilon_+^2 & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & 0 & \varepsilon_+^p & 0 & \cdots & 0 \end{bmatrix}$$

where $\varepsilon_- := \varepsilon_-^1 \cdot \varepsilon_-^2 \cdot \cdots \cdot \varepsilon_-^p \neq 0$ has only stable zeros and $\varepsilon_+ := \varepsilon_+^1 \cdot \cdots \cdot \varepsilon_+^p$ has only unstable zeros or is identically zero. Define $r := \deg \varepsilon_-$.

We now have to show that there exists a proper feedback compensator of order r, such that the fixed polynomial of the closed loop system is $(\varepsilon_-(s))^2$.

For this, assume that the compensator has the form

$$K(s) = P(s) Q^{-1}(s),$$

where P and Q are relatively right prime polynomial matrices of sizes $m \times p$ and $p \times p$, respectively, and $K(s)$ is strictly proper. A short calculation yields the closed loop transfer function

$$Q(s)(D(s)Q(s) - N(s)P(s))^{-1} N(s).$$

For a definition of input, output and input–output decoupling zeros of such a representation, see [4].

The following equivalent version of Theorem 3 can then be stated.

Theorem 4. *Under the above assumptions, there exist polynomial matrices P and Q with* $\deg\det Q = r$ *and PQ^{-1} strictly proper, such that the closed loop system described by*

$$Q(DQ-NP)^{-1}N$$

has exactly r input decoupling zeros, exactly r output decoupling zeros and no input–output decoupling zeros. The input decoupling zeros as well as the output decoupling zeros are the zeros of $\varepsilon_-(s)$, and are thus stable.

Proof. The derivation of the single input, single output case will be imitated. However, some technical complications arise, since polynomial matrices in general do not commute.

Define

$$Q = UE_-.$$

Then

$$\deg\det Q = \deg \varepsilon_- = r$$

and

$$DQ - NP = FU^{-1}UE_- - E_-E_+VP$$
$$= E_-(F - E_+VP).$$

Hence, since $N = E_-E_+V$, $DQ - NP$ and N have the common left factor E_-. We now want to choose P, such that $F - E_+V_P$ and Q have the common right factor E_-. For this, P has to satisfy

$$F - E_+VP = LE_- \tag{6.1}$$

for some polynomial matrix L. If (6.1) holds, then the zeros of ε_- are input decoupling as well as output decoupling zeros of the closed loop system, but none of these zeros is an input–output decoupling zero. In order for the system to have no further input decoupling or output decoupling zeros, the representation

$$Q(DQ-NP)^{-1}N = UE_-(E_-LE_-)^{-1}E_-E_+V$$
$$= UL^{-1}E_+V$$

has to be of least order, i.e. L and E_+ have to be left coprime. Summing up, the proof will be completed once we have shown that there exists an $m \times p$ polynomial matrix P and a $p \times p$ polynomial matrix L such that

$$E_+VP + LE_- = F, \tag{6.2}$$

$$PQ^{-1} = P(UE_-)^{-1} \text{ is strictly proper,} \tag{6.3}$$

$$P \text{ and } Q = UE_- \text{ are right coprime,} \tag{6.4}$$

$$L \text{ and } E_+V \text{ are left coprime.} \tag{6.5}$$

Equivalently, the more symmetric version of (6.2),

$$(E_+V)P + L(UE_-) = F, \tag{6.6}$$

with the same constraints (6.3)–(6.5) can be analysed.

For the special case $V = \mathrm{Id}_m$ and $U = \mathrm{Id}_p$, (6.6) may be viewed as a collection of p scalar, decoupled equations. Hence Lemma 1 ensures a solution $\tilde{P}$, $\tilde{L}$, where $\tilde{P}$ and $\tilde{L}$ are diagonal. But then

$$\bar{P} := V^{-1}\tilde{P} \quad \text{and} \quad \bar{L} := \tilde{L}U^{-1}$$

solve (6.6) for general unimodular U and V. Note that $\bar{P}$ in general will not satisfy (6.3). In order to take into account this requirement, divide $\bar{P}$ from the right by UE_- to obtain

$$\bar{P} = \hat{P}(UE_-) + P,$$

where $\hat{P}$ and P are $m \times p$ polynomial matrices and $P(UE_-)^{-1}$ is strictly proper. Thus,

$$F = (E_+V)\tilde{P} + \bar{L}(UE_-)$$
$$= (E_+V)(\hat{P}(UE_-) + P) + \bar{L}(UE_-)$$
$$= (E_+V)P + (\bar{L} + E_+V\hat{P})(UE_-),$$

which gives the desired solution of (6.6) satisfying (6.3).

Finally, right coprimeness of E_- and F and so UE_- and F implies (6.4), and left coprimeness of E_+ and F and so E_+V and F implies (6.5). □

The statement of Theorem 4 only offers an existence statement. The proof, however, is constructive, so that an algorithm to calculate P and Q may be obtained. Unfortunately, the algorithm requires very tedious calculations with polynomial matrices.

It should be noted that solvability of (6.6) (without constraints (6.3) through (6.5)) also follows from the results in [9] on solvability of general matrix equations. However, since the matrices in (6.6) bear some very special structure, the direct proof chosen above appears to be more attractive.

7. Conclusions

A partial solution has been given to the problem of stabilizing a linear system using a decentralized control structure with controller complexity spread as evenly as possible through the different channels. The key restrictions which prevent the paper from offering a complete solution are, first, that the large scale system must be of the so + called interconnected type, and, second, that complexity can only be spread when the individual subsystems have stable zeros, in which case the spread of complexity is limited by the number of such stable zeros. The overall stabilization process results in uncontrollable, unobservable modes at the stable subsystem zeros.

Some of the ideas of this paper will be developed in a forthcoming work considering a different class of problems. A two channel set-up is postulated that is not of the 'interconnected system' structure. For purposes of either decentralized control, or for considering disturbance rejection problems, we study the procedure for choosing a controller to place round one channel such that the system, viewed now as having only one input and output channel, has as many uncontrollable and unobservable stable modes as possible.

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