SIAM J. APPL. MATH.
Vol. 44, No. 4, August 1984

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THE UNIT CIRCLE CAUCHY INDEX: DEFINITION, CHARACTERIZATION AND POLYNOMIAL ZERO DISTRIBUTION*

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Abstract. The Cauchy index of a rational transfer function evaluated over an interval of the real line has proved a useful tool in various linear systems applications.

In this paper, we define the unit circle Cauchy index and give some computational methods of it. This provides a systematic view of many problems and results on unit circle positivity, polynomial root distribution, and stability. Results on positivity of polynomials in z and z^{-1} which are real on $|z|=1$ are relevant in checking discrete positive realness and the stability of two-dimensional digital filters, while results on the zero distribution of a polynomial relative to the boundary of the unit circle are of relevance in studying the stability of discrete-time systems.

Key words. Cauchy index, stability, discrete-time systems, Toeplitz matrix

1. Introduction. The Cauchy index of a rational function evaluated over an interval of the real line has proved a useful tool in various applications, allowing as it does the formulation of tests for checking the positivity of a given polynomial in that interval, or checking that the roots of a polynomial all have negative real parts [7].

In this paper, we set out some parallel theory with the unit circle playing the role of the real axis or of the imaginary axis. Results on positivity of polynomials in z and z^{-1} which are real on $|z|=1$ are relevant in checking discrete positive realness, a property arising in various applications including adaptive control [12] and the stability of two-dimensional digital filters [1], while results on the zero distribution of a polynomial relative to the boundary of the unit circle are of relevance in studying the stability of sampled-data control systems [10].

There is of course much literature on the latter topic, see e.g. [11] and the references therein, as well as some literature on the positivity problem. Our aim here is to give a unifying picture for these problems in the same manner that the conventional Cauchy index can unify the picture for real axis positivity, half-plane root counting, etc.

We note that the concept of a unit circle Cauchy index is not new, see [14], [2]; these references do not contain the systematic development of this paper.

2. Definition of the unit circle Cauchy index. First, we shall define the unit circle Cauchy index as follows.

DEFINITION. Let $F(z)$ be a function which is real on $|z|=1$; then define the unit circle Cauchy index,

$$\underset{\text{U.C.}}{I}\ F(z)$$

as the number of jumps of $F(z)$ from $-\infty$ to $+\infty$ less the number of jumps from $+\infty$ to $-\infty$ in moving once around the unit circle in a counterclockwise direction.

Before considering the case where $F(z)$ is a rational function, we shall give an example of the use of the Cauchy index in calculating the number of unit circle zeros of a polynomial in z and z^{-1} which is real on $|z|=1$. (This gives of course a check for positivity also.)

* Received by the editors January 27, 1983.

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Consider the polynomial in z and z^{-1}

$$P(z, z^{-1}) = \sum_{i=0}^{m} \frac{a_i}{2}(z^i + z^{-i}) \tag{2.1}$$

which, with the a_i real, is real on $|z|=1$. (Any polynomial in z and z^{-1} with real coefficients must be of this form if it is to be real on $|z|=1$.)

Let us consider how we might count the number of distinct zeros of $P(z, z^{-1})$. Write

$$P(z, z^{-1})|_{z=e^{j\theta}} = a_0 + a_1 \cos \theta + \cdots + a_m \cos m\theta. \tag{2.2}$$

Then

$$\frac{d}{d\theta}[P(z, z^{-1})|_{z=e^{j\theta}}] = -[a_1 \sin \theta + 2a_2 \sin 2\theta + \cdots + ma_m \sin m\theta]$$

$$= -\left[a_1\left(\frac{z-z^{-1}}{2j}\right) + 2a_2\left(\frac{z^2-z^{-2}}{2j}\right) + \cdots + ma_m\left(\frac{z^m-z^{-m}}{2j}\right)\right]_{z=e^{j\theta}} \tag{2.3}$$

$$= jz\frac{dP(z, z^{-1})}{dz}\bigg|_{z=e^{j\theta}}.$$

With x a real variable, $f(x)$ a real (usually rational) function of x, taking possibly infinite values, it is known [7] that the number of distinct zeros of $f(x)$ in (a, b) is given by $I_a^b[f'(x)/f(x)]$. Consequently, if $P(z, z^{-1})$ is nonzero at $z=-1$, the number of distinct zeros of $P(z, z^{-1})$ on $|z|=1$ is

$$I_{-\pi}^{\pi} - \frac{a_1 \sin \theta + 2a_2 \sin 2\theta + \cdots + ma_m \sin m\theta}{a_0 + a_1 \cos \theta + \cdots + a_m \cos m\theta} = I_{-\pi}^{\pi}\, j\frac{z\frac{d}{dz}P(z, z^{-1})}{P(z, z^{-1})}\bigg|_{z=e^{j\theta}}. \tag{2.4}$$

Notice that

$$p(z) = z^m P(z, z^{-1}), \qquad q(z) = z^m z\frac{d}{dz}P(z, z^{-1}) \tag{2.5}$$

are both polynomials with real coefficients and $jq(z)/p(z)$ is real on $|z|=1$ with $jq(-1)/p(-1)$ finite.

This suggests an approach to defining a unit circle Cauchy index for a rational function as

$$I_{\text{U.C.}}\, jw(z) \tag{2.6}$$

where $w(z)$ is

$$w(z) = \frac{g(z)}{f(z)} = \frac{g_0 + g_1 z + \cdots + g_n z^n}{f_0 + f_1 z + \cdots + f_n z^n} \tag{2.7}$$

with the f_i, g_i in general complex and $jw(z)$ is real on $|z|=1$. (As f_n or g_n is permitted to be zero, there is no loss of generality in assuming equal degrees for numerator and denominator.)

As with the conventional Cauchy index, we can virtually display the unit circle Cauchy index with an appropriate partial fraction expansion. Let $\bar{w}(z)$ denote $w(z)$ with each f_i, g_i replaced by its complex conjugate $\bar{f}_i$, $\bar{g}_i$; now $jw(z)$ is real on $|z|=1$ if and only if $w(z) = -\bar{w}(z^{-1})$. Accordingly if λ is a pole of $w(z)$ of multiplicity m, λ^{-1} is a pole of $\bar{w}$ of the same multiplicity, and so $\bar{\lambda}^{-1}$ is a pole of $w(z)$ of the same multiplicity. Let ε_i, $i=1,2,\cdots,p$ be the distinct poles of $w(z)$ on $|z|=1$. We can write a Caratheodory partial fraction expansion [15]

$$w(z)=d_0+d_1z+\cdots+d_mz^m+\sum_{i=1}^{p}\sum_{k=1}^{m_i}\alpha_{i,k}\left(\frac{\varepsilon_i+z}{\varepsilon_i-z}\right)^k+\frac{d_{-1}}{z}+\cdots+\frac{d_{-m}}{z^m}+\eta(z) \tag{2.8}$$

where $\eta(z)$ is a function of the form

$$\eta(z)=\sum_{i=1}^{q}\sum_{k=1}^{e_i}\left[\gamma_{i,k}\left(\frac{\lambda_i+z}{\lambda_i-z}\right)^k+\gamma_{-i,k}\left(\frac{\bar{\lambda}_i^{-1}+z}{\bar{\lambda}_i^{-1}-z}\right)^k\right] \tag{2.9}$$

with $|\lambda_i|\neq 0, 1$ and $\eta(z)=-\bar{\eta}(z^{-1})$. The requirement $w(z)=-\bar{w}(z^{-1})$ yields

$$\begin{aligned} &d_i=-\bar{d}_{-i}, \qquad i=0,1,\cdots,m,\\ &\bar{\alpha}_{i,k}=(-1)^{k+1}\alpha_{i,k}, \qquad \gamma_{i,k}=(-1)^{k+1}\bar{\gamma}_{-i,k}. \end{aligned} \tag{2.10}$$

Notice from (2.10) that for odd k, $\alpha_{i,k}$ is real.

We now assert:

THEOREM 2.1. *Let $w(z)$ be rational with $w(z)=-\bar{w}(z^{-1})$. Let $w(z)$ have a partial fraction expansion of the form of (2.8), where ε_i, $i=1,\cdots,p$ are the distinct poles on $|z|=1$ of $w(z)$. Then*

$$\underset{\text{U.C.}}{\mathrm{I}}\, jw(z)=\sum_{i=1}^{p}(-1)^{(m_i+1)/2}\operatorname{sign}(\alpha_{i,m_i}) \quad \textit{for odd } m_i.$$

Proof. A direct calculation shows easily that

$$\underset{\text{U.C.}}{\mathrm{I}}\, j\alpha_{i,m_i}\left(\frac{\varepsilon_i+z}{\varepsilon_i-z}\right)^{m_i}=\begin{cases}(-1)^{(m_i+1)/2}\operatorname{sign}(\alpha_{i,m_i}), & m_i \text{ odd},\\ 0, & m_i \text{ even}.\end{cases}$$

Then, since

$$\sum_{k=1}^{m_i}\alpha_{i,k}\left(\frac{\varepsilon_i+z}{\varepsilon_i-z}\right)^k=\alpha_{i,m_i}\left(\frac{\varepsilon_i+z}{\varepsilon_i-z}\right)^{m_i}[1+\delta_i(z)]$$

where $\delta_i(z)=0$ at $z=\varepsilon_i$, the result follows easily.

A further result we shall have occasion to use can be obtained just as for the Cauchy index on the real line [7].

THEOREM 2.2. *Let $w(z)$ be rational with $w(z)=-\bar{w}(z^{-1})$. Then*

$$\underset{\text{U.C.}}{\mathrm{I}}\, jw(z)+\underset{\text{U.C.}}{\mathrm{I}}\,\frac{1}{jw(z)}=0.$$

3. Computation of the unit circle Cauchy index by Toeplitz and Bezout forms.

DEFINITION OF TOEPLITZ AND BEZOUT MATRICES. Let $w(z)$ be any function of z with $w(0)<\infty$ and possessing a power series expansion

$$w(z)=w_0+w_1z+w_2z^2+\cdots. \tag{3.1}$$

Define the Nth order Toeplitz matrix associated with $w(z)$ by

$$A_N[w]=\begin{bmatrix} w_0 & w_1 & \cdots & w_{N-1}\\ 0 & w_0 & \cdots & w_{N-2}\\ 0 & 0 & \cdots & w_{N-3}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & & w_0 \end{bmatrix}. \tag{3.2}$$

The following properties are very easily shown, where α, β are complex numbers and $v(z)$ has the same properties as $w(z)$:

$$A_N[\alpha w+\beta v]=\alpha A_N[w]+\beta A_N[v], \tag{3.3}$$

$$A_N[wv]=A_N[w]A_N[v]=A_N[v]A_N[w]. \tag{3.4}$$

If $w(0)=w_0\neq 0$,

$$A_N^{-1}[w]=A_N[w^{-1}]. \tag{3.5}$$

Suppose now w is rational, with $w=g/f$, g and f being coprime polynomials. Let $f(z)=f_0+f_1z+\cdots+f_nz^n$, so that $w(0)<\infty$ implies $f_0\neq 0$ and $A_N^{-1}[f]$ exists. It is easily verified that

$$A_N[w]=A_N[g]A_N^{-1}[f]=A_N^{-1}[f]A_N[g]. \tag{3.6}$$

With a superscript prime denoting *matrix transposition*, define the $N\times N$ hermitian Toeplitz matrix $T_N[w]$ by

$$T_N[w]=A_N[w]+\bar{A}'_N[w] \tag{3.7}$$

and the $N\times N$ hermitian matrix $B_N[w]$, which we shall term a Bezout matrix for reasons to be clarified later, by

$$B_N[w]=\bar{A}'_N[f]A_N[g]+\bar{A}'_N[g]A_N[f]. \tag{3.8}$$

Notice that the coprimeness of f and g means that f and g are individually defined to within a complex nonzero constant α, and $B_N[w]$ accordingly is only defined to within a positive constant $|\alpha|^2$. This will not concern us. Notice also that $B_N[w]$ is congruent to $T_N[w]$:

$$\begin{aligned} T_N[w]&=\bar{A}_N^{-T}[f]\{\bar{A}'_N[f]A_N[g]+\bar{A}'_N[g]A_N[f]\}A_N^{-1}[f]\\ &=\bar{A}_N^{-T}[f]B_N[w]A_N^{-1}[f]. \end{aligned} \tag{3.9}$$

Finally, notice that both the $T_N[w]$ and the $B_N[w]$ for variable N are nested; thus for $M>N$, $B_N[w]$ is a top left submatrix of $B_M[w]$.

Relation of Toeplitz and Bezout matrices to Bezout forms. We shall now demand that $w(z)$ be rational with $w(z)=-\bar{w}(z^{-1})$. To compute $I_{\text{U.C.}}jw(z)$, it is enough to evaluate $-I_{\text{U.C.}}[1/jw(z)]$, see Theorem 2.2. Accordingly, *without loss of generality*, we can assume $w(0)<\infty$ for the purposes of computing the index. Note that since $w(z)=-\bar{w}(z^{-1})$, this means that also $w(\infty)<\infty$. These restrictions on $w(z)$ now imply that for all N, we can form $T_N[w]$ and $B_N[w]$. We shall relate these matrices to quantities formed from certain Bezout forms.

Define scalars t_{ij} by

$$\frac{1}{1-xy}[\bar{w}(x)+w(y)]=\sum_{i,j=1}^{\infty} t_{ij}x^{i-1}y^{j-1}. \tag{3.10}$$

Now, it can easily be derived that

$$t_{ij}=\begin{cases} w_{j-i}, & j>i,\\ w_0+\bar{w}_0, & j=i,\\ \bar{w}_{i-j}, & i>j. \end{cases} \tag{3.11}$$

Hence

$$T_N[w]=[t_{ij}]_{i,j=1}^{N}. \tag{3.12}$$

Next, define scalars b_{ij} by

$$\frac{\bar{f}(x)g(y)+\bar{g}(x)f(y)}{1-xy}=\sum_{i,j=1}^{n} b_{ij}x^{i-1}y^{j-1} \tag{3.13}$$

which is a Bezout form for f and g. It is easy to verify the validity of the summation limits on the right side of (3.13). In fact, the following formula can easily be obtained:

$$b_{ij}=\bar{b}_{ji}=\sum_{k=1}^{i}(i-k, j-k), \qquad i\leqq j \tag{3.14}$$

where

$$(i, j)=\bar{g}_i f_j+g_j\bar{f}_i.$$

We shall now relate b_{ij} to the matrix $B_N[w]$. Observe that

$$\sum_{i,j=1}^{n} b_{ij}x^{i-1}y^{j-1}=\frac{\bar{f}(x)f(y)}{1-xy}[\bar{w}(x)+w(y)]=\bar{f}(x)f(y)\sum_{i,j=1}^{\infty} t_{ij}x^{i-1}y^{j-1}. \tag{3.15}$$

Now for $i\leqq n$

$$x^{i-1}f(x)=f_0x^{i-1}+f_1x^i+\cdots+f_{n-i}x^{n-i}+\text{higher order}$$

and since the left side of (3.15) is known to have no terms of higher degree than $(n-1)$ in either x or y, we must have

$$\sum_{i,j=1}^{n} b_{ij}x^{i-1}y^{j-1}=\pi_n'(x)\bar{A}_n'[f]T_n[w]A_n[f]\pi_n(y)=\pi_n'(x)B_n[w]\pi_n(y) \tag{3.16}$$

where $\pi_n(x)=[1xx^2\cdots x^{n-1}]'$. This calculation shows that b_{ij} is the i-j entry of $B_n[w]$.

A simple modification shows that for any $N\geqq n$,

$$\sum_{i,j=1}^{n} b_{ij}x^{i-1}y^{j-1}=\pi_N'(x)\bar{A}_N'(f)T_N[w]A_N[f]\pi_N(y) \tag{3.17}$$

which shows that $B_n[w]$ is the top left $n\times n$ block of $B_N[w]$, and that the remainder of $B_N[w]$ is zero; $T_N[w]$ is congruent to a direct sum of $B_n[w]$ and a zero matrix of appropriate size.

In defining $T_n[w]$, it turns out to be crucial to have $w(0)<\infty$; which enables $w(\infty)<\infty$ and $\deg g\leqq\deg f$. However, the definition of $B_n[w]$ via (3.13) places no such restriction, so long as

$$n=\max[\deg f, \deg g].$$

The rank of Toeplitz and Bezout matrices. For future reference, we note the following easy result.

LEMMA 3.1. *Let $w(z)=g(z)/f(z)$ for polynomial f, g have $w(z)=-\bar{w}(z^{-1})$ with $w(0)<\infty$. Let $T_n[w]$ be formed as described above. Then* rank $T_N[w]=n$ *for all $N\geqq n$ if and only if g and f are coprime, with $n=\deg f$.*

Proof. Without loss of generality, assume f and g are coprime. Then rank $T_N[w]=$ rank $B_N[w]=$ rank $B_n[w]$. So we must show rank $B_n[w]=n$. Now, because $w(z)=-\bar{w}(z^{-1})$, we have

$$g(y)\bar{f}(y^{-1})+\bar{g}(y^{-1})f(y)=0$$

and then

$$g(y)[y^n\bar{f}(y^{-1})]+[y^n\bar{g}(y^{-1})]f(y)=0.$$

Because f and g are coprime, $f(y)=\alpha y^n \bar{f}(y^{-1})$ for some nonzero constant α. Setting $y=1$ identifies α as $f(1)/\bar{f}(1)$. Now

$$\sum_{i,j=1}^{n} b_{ij}x^{i-1}y^{j-1}=\frac{\bar{f}(x)g(y)+\bar{g}(x)f(y)}{1-xy}=\frac{\bar{\alpha}x^n f(x^{-1})g(y)-\bar{\alpha}x^n g(x^{-1})f(y)}{1-xy}$$

$$=\bar{\alpha}x^{n-1}\frac{f(x^{-1})g(y)-g(x^{-1})f(y)}{x^{-1}-y}.$$

Hence, with $u=x^{-1}$,

$$\sum_{i,j=1}^{n} b_{ij}u^{n-i}y^{j-1}=\bar{\alpha}\frac{f(u)g(y)-g(u)f(y)}{u-y}.$$

The basic result on Bezoutians [9] shows that the matrix with i, j element $b_{n-i+1,j}$ is nonsingular since f and g are coprime, whence the nonsingularity of $B_n[w]$ is immediate. □

Notice that if a Bezout matrix is formed from a non-coprime f and g with $\max(\deg f, \deg g)=m>n$, then this Bezout matrix though of size $m\times m$, would still have rank n, being congruent to $T_n[w]$.

These results are of course consistent with results applicable to the real line, as opposed to the unit circle, with the Toeplitz matrix here playing the role of the Hankel matrix in the real line case. The same is true of the next result.

Signature of Toeplitz and Bezout matrices.

THEOREM 3.1. *Let $w(z)$ be rational with $w(z)=-\bar{w}(z^{-1})$. Let $w(z)=g/f$, with f, g coprime and* $\max(\deg f, \deg g)=n$. *Then if $w(0)<\infty$*

$$I_{\text{U.C.}}\, jw(z)=-\sigma(B_N[w])=-\sigma(T_N[w]) \tag{3.18}$$

for all $N\geqq n$ where $\sigma(\cdot)$ denotes the signature of a hermitian matrix. If $w(0)$ is infinite, only the first equality holds.

For a proof, see the appendix.

Determinantal forms and the unit circle Cauchy index. Let B_i be the ith principal minor of the matrix $B_n[w]$, defined previously. Then the signature of $B_n[w]$ is given by

$$\sigma\{B_n[w]\}=n-2V(1, B_1, B_2, \cdots, B_n) \tag{3.19}$$

where $V(\cdots)$ denotes the number of variations in sign of the sequence in $(\cdots)$. In case entries in the sequence are zero, special rules apply, see [8, p. 304]. Let us now observe how the B_i can be obtained as the values of successive Schur–Cohn type determinants.

Schur's result [15, p. 216] and the commutativity of $A_i[f]$ and $A_i[g]$, (see (3.4)), imply that

$$\det\begin{bmatrix}\bar{A}_i'(f) & A_i(f)\\ -\bar{A}_i'(g) & A_i(g)\end{bmatrix}=\det[\bar{A}_i'(f)A_i(g)+\bar{A}_i'(g)A_i(f)]=B_i \tag{3.20}$$

on using (3.8) and the observation immediately following. In more extended fashion,

$$\det\left[\begin{array}{cccc|cccc}
\bar f_0 & & & & f_0 & f_1 & \cdots & f_{i-1}\\
\bar f_1 & \bar f_0 & & & 0 & f_0 & \cdots & f_{i-2}\\
\vdots & \vdots & \ddots & & \vdots & \vdots & & \\
\bar f_{i-1} & \bar f_{i-2} & \cdots & \bar f_0 & 0 & 0 & \cdots & f_0\\
\hline
-\bar g_0 & & & & g_0 & g_1 & \cdots & g_{i-1}\\
-\bar g_1 & -\bar g_0 & & & 0 & g_0 & \cdots & g_{i-2}\\
\vdots & & \ddots & & \vdots & \vdots & & \vdots\\
-\bar g_{i-1} & & & -\bar g_0 & 0 & 0 & \cdots & g_0
\end{array}\right] = B_i.$$

Also, B_i is the ith inner [11] of the following matrix:

$$\left[\begin{array}{cccc|cccc}
\bar f_0 & \bar f_1 & \cdots & \bar f_{n-1} & 0 & \cdots & 0 & f_0\\
0 & \bar f_0 & \cdots & \bar f_{n-2} & 0 & & f_0 & f_1\\
\vdots & & \ddots & & & \cdot^{\cdot^{\cdot}} & & \vdots\\
0 & 0 & \cdots & \bar f_0 & f_0 & \cdots & f_{n-2} & f_{n-1}\\
\hline
0 & 0 & \cdots & -\bar g_0 & g_0 & \cdots & g_{n-2} & g_{n-1}\\
\vdots & & \cdot^{\cdot^{\cdot}} & & \vdots & \ddots & & \\
0 & -\bar g_0 & \cdots & -\bar g_{n-2} & 0 & & g_0 & g_1\\
-\bar g_0 & -\bar g_1 & \cdots & -\bar g_{n-1} & 0 & & & g_1
\end{array}\right].$$

4. Other approaches to unit circle Cauchy index evaluation.

Method based on state-space realization. This technique is suggested by the availability of a similar technique for the conventional Cauchy index [3]. We begin with the following preliminary result.

LEMMA 4.1. *Let $w(z) = -\bar w(z^{-1})$ be rational, with $w(\infty) < \infty$. Suppose that*

$$w(z) = d + c'(zI - A)^{-1}b \tag{4.1}$$

with $[A, b]$ and $[A, c]$ controllable and observable. Then

$$d + \bar d = c'A^{-1}b \tag{4.2}$$

and there exists a unique hermitian T such that

$$\bar A'TA = T, \qquad \bar A'Tb = \bar c. \tag{4.3}$$

Outline of proof. Since $w(z) = -\bar w(z^{-1})$, simple manipulations show that

$$w(z) = [-\bar d + \bar b'(\bar A')^{-1}\bar c] + \bar b'(\bar A')^{-1}[zI - (\bar A')^{-1}]^{-1}(\bar A')^{-1}\bar c. \tag{4.4}$$

Minimality in (4.1) then implies (4.2) and (4.3) for a unique T. Since, as is easily checked $\bar T'$ also satisfies (4.3), T is hermitian.

The following result mirrors one for the conventional Cauchy index.

THEOREM 4.1. *Let $w(z) = -\bar w(z^{-1})$ be rational with $w(\infty) < \infty$. Let A, b, c, d and T be as defined in Lemma 4.1. Then*

$$\underset{\text{U.C.}}{I}\, jw(z) = \sigma\{T\}. \tag{4.5}$$

Proof. Equations (4.3) imply

$$TW = V \tag{4.6}$$

where $W=[b, Ab, \cdots, A^{n-1}b]$, $V=[(\bar{A}')^{-1}\bar{c}, (\bar{A}')^{-2}\bar{c}, \cdots, (\bar{A}')^{-n}\bar{c}]$ with obvious definition of the nonsingular matrices W and V. Now consider $\bar{W}'V=\bar{W}'TW$. The $i-j$ entry is

$$\bar{b}'(\bar{A}')^{i-1}(\bar{A}')^{-j}\bar{c}=\bar{b}'(\bar{A}')^{i-j-1}\bar{c}=\bar{c}'(\bar{A})^{i-j-1}\bar{b}. \tag{4.7}$$

Now

$$-\bar{w}(z^{-1})=w(z)=w_0+w_1z+w_2z^2+\cdots. \tag{4.8}$$

Hence

$$w(z)=-\bar{w}_0-\bar{w}_1z^{-1}-\bar{w}_2z^{-2}+\cdots \tag{4.9}$$

(where the expansions in (4.8) and (4.9) are valid in different regions). Now (4.9) means that

$$d=-\bar{w}_0, \qquad c'A^{i-j-1}b=-\bar{w}_{i-j}, \qquad i\geqq j+1. \tag{4.10}$$

It is then easy to derive

$$T_n[w]=-\bar{W}'TW \tag{4.11}$$

and accordingly the theorem is proved by Theorem 3.1. □

Connection to the Bezout matrix. By appropriate selection of the state-space coordinate basis, we can virtually identify $-T$ with the Bezout matrix $B_n[w]$. Let a minimal realization of

$$w(z)=\frac{g(z)}{f(z)}=\frac{g_0+g_1z+\cdots+g_nz^n}{f_0+f_1z+\cdots+f_nz^n}$$

be

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \\ \dfrac{-f_0}{f_n} & \dfrac{-f_1}{f_n} & \dfrac{-f_2}{f_n} & \cdots & \dfrac{-f_{n-1}}{f_n} \end{bmatrix},$$

$$b=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad c=\begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_{n-1} \\ \tilde{g}_n \end{bmatrix}, \qquad d=\frac{g_n}{f_n}. \tag{4.12}$$

where the $\tilde{g}_i$ are defined by

$$\tilde{g}_nz^{n-1}+\cdots+\tilde{g}_1=\frac{1}{f_n}g(z)-\frac{g_n}{f_n^2}f(z). \tag{4.13}$$

Then, the solution T of (4.3) is

$$T = -\frac{1}{|f_n|^2} B_n[w].$$

This can be derived by methods similar to those in [3].

Sturm sequence approach. There are various approaches to the use of a Sturm sequence to evaluate the Cauchy index, which we shall now outline. Suppose we are evaluating, for some constants a_i, b_i, c_i, d_i

$$I_{-\pi}^{\pi} \frac{\sum (a_i \cos i\theta + b_i \sin i\theta)}{\sum (c_i \cos i\theta + d_i \sin i\theta)}$$

where it is assumed that at $\theta = \pi$, the denominator is nonzero, so that a unit circle Cauchy index is involved. This may be rewritten for another set of constants α_i, β_i, γ_i, δ_i as

$$I_{-\pi}^{\pi} \frac{\sum [\alpha_i(\cos\theta)^i + \beta_i \sin\theta(\cos\theta)^i]}{\sum [\gamma_i(\cos\theta)^i + \delta_i \sin\theta(\cos\theta)^i]}.$$

Now set $t = \tan \theta/2$, and notice that $\cos\theta = (1-t^2)(1+t^2)^{-1}$ and $\sin\theta = 2t(1-t^2)^{-1}$. Moreover, as θ ranges from $-\pi$ to π, t ranges from $-\infty$ to $+\infty$. Accordingly, we may replace the unit circle Cauchy index evaluation by the evaluation of an index

$$I_{-\infty}^{+\infty} \quad (\text{rational in } t)$$

and this can be achieved using Sturm sequences, see e.g. [7].

In case the Cauchy index is defined for a ratio of two polynomials in z, or in z and z^{-1}, the introduction of the $\cos\theta$, $\sin\theta$ terms can be bypassed by simply setting

$$z = \frac{1+jt}{1-jt}. \tag{4.14}$$

It is sometimes the case that the index to be evaluated has the special form

$$I_{-\pi}^{\pi} \frac{\sum b_i \sin i\theta}{\sum a_i \cos i\theta}.$$

(Consider for example the positivity question discussed in § 2). Then, as pointed out in [14], one has

$$I_{-\pi}^{\pi} \frac{\sum b_i \sin i\theta}{\sum a_i \cos i\theta} = 2\, I_{0}^{\pi} \frac{\sum b_i \sin i\theta/\sin\theta}{\sum a_i \cos i\theta} = 2\, I_{-1}^{1} \frac{P_1(x)}{P_0(x)} \tag{4.15}$$

where $P_0(x) = \sum a_i T_i(x)$, $P_1(x) = \sum b_i T_i'(x)$, and $T_i(x)$, $T_i'(x)$ are Chebyshev polynomials of the first and second types.

Again, a Sturm sequence approach can be used. Actually, by using known relations among the Chebyshev polynomials, one does not have to evaluate explicitly the polynomials $P_0(x)$ and $P_1(x)$. Instead, one can by simple rules obtain the coefficients $c_i, d_i, \cdots$ where $P_2(x) = \sum c_i T_i'(x)$, $P_3(x) = \sum d_i T_i'(x), \cdots$ are successively lower degree polynomials of a Sturm chain with first two members $P_0(x)$ and $P_1(x)$. See [14] for details.

5. Polynomial zero distribution and the unit circle. Let

$$d(z) = d_n z^n + d_{n-1} z^{n-1} + \cdots + d_0 \tag{5.1}$$

(where the d_i are not necessarily real); we wish to use the Cauchy index to count the number of zeros of $d(z)$ within the unit circle.

Suppose temporarily there are no zeros on the unit circle while there are k zeros inside. Then

$$2\pi k = \Delta[\arg d(e^{j\theta})]_{-\pi}^{\pi}.$$

Suppose that there is no zero of $\operatorname{Re}[d(z)]$ at $z=-1$. Then we have furthermore,

$$2\pi k = \Delta\left[\arctan\frac{\operatorname{Im} d(e^{j\theta})}{\operatorname{Re} d(e^{j\theta})}\right]_{-\pi}^{\pi} = -\pi \, I_{-\pi}^{\pi} \frac{\operatorname{Im} d(e^{j\theta})}{\operatorname{Re} d(e^{j\theta})}.$$

(If there is a zero of $\operatorname{Re}[d(z)]$ at $z=-1$, one adopts a different start/end point on the unit circle for the Cauchy index computation.) A similar argument also yields

$$2\pi\left(k-\frac{n}{2}\right) = -\pi \, I_{-\pi}^{\pi} \frac{\operatorname{Im}[e^{-jn\theta/2}d(e^{j\theta})]}{\operatorname{Re}[e^{-jn\theta/2}d(e^{j\theta})]}$$

provided that the denominator is nonzero at $\theta=-\pi$. Now define

$$\hat{d}(z) = \bar{d}_n + \bar{d}_{n-1}z + \cdots + \bar{d}_0 z^n = z^n \bar{d}(z^{-1}).$$

Then if $z=e^{j\theta}$,

$$I_{-\pi}^{\pi} \frac{\operatorname{Im}[e^{-jn\theta/2}d(e^{j\theta})]}{\operatorname{Re}[e^{-jn\theta/2}d(e^{j\theta})]} = I_{\text{U.C.}} \frac{\dfrac{z^{-n/2}d(z) - z^{n/2}\bar{d}(z^{-1})}{2j}}{\dfrac{z^{-n/2}d(z) + z^{n/2}\bar{d}(z^{-1})}{2}}$$

or

$$2k - n = I_{\text{U.C.}}\, j\frac{d(z)-\hat{d}(z)}{d(z)+\hat{d}(z)}. \tag{5.2}$$

The restriction on the behaviour of the denominator at -1 is inconsequential, and not needed in (5.2); however, (5.2) does require adjustment in case there are unit circle zeros of $d(z)$. We shall show that if $d(z)$ in (5.1) has k zeros inside the unit circle, and $\tilde{m}$ zeros (counting multiplicities) on the unit circle, then

$$2k + \tilde{m} - n = I_{\text{U.C.}}\, j\frac{d(z)-\hat{d}(z)}{d(z)+\hat{d}(z)}. \tag{5.3}$$

To establish (5.3), let $d(z) = e(z)f(z)$ where $e(z)$ has all zeros on $|z|=1$ and $f(z)$ has no zero on $|z|=1$. Then it is easily seen that

$$\hat{d}(z) = \hat{e}(z)\hat{f}(z), \tag{5.4a}$$

$$\hat{e}(z) = (\bar{e}_{\tilde{m}}/e_0)e(z) \tag{5.4b}$$

with $|\bar{e}_{\tilde{m}}/e_0| = 1$. Let $e^{2j\phi} = \bar{e}_{\tilde{m}}/e_0$. Then

$$I_{\text{U.C.}}\, j\frac{d(z)-\hat{d}(z)}{d(z)+\hat{d}(z)} = I_{\text{U.C.}}\, j\frac{f(z)-e^{2j\phi}\hat{f}(z)}{f(z)+e^{2j\phi}\hat{f}(z)} = I_{\text{U.C.}}\, j\frac{[e^{-j\phi}f(z)] - \widehat{[e^{-j\phi}f(z)]}}{[e^{-j\phi}f(z)] + \widehat{[e^{-j\phi}f(z)]}}. \tag{5.5}$$

The right-hand side of (5.5) can be evaluated, on recognizing that $f(z)$ has degree $n-\tilde{m}$, as $2k-n+\tilde{m}$, which yields (5.3).

For real $f(z)$, the formula (5.3) was presented in [2]. An important special case is the following condition for $d(z)$ to have all its zeros inside the unit circle.

(5.6) $$n = \underset{\text{U.C.}}{I}\, j\frac{d(z)-\hat{d}(z)}{d(z)+\hat{d}(z)}$$

which can in effect be found in [14].

Fujiwara [6] established the following result, which we can easily prove by using (5.3).

THEOREM 5.1. *Let $d(z)$ be as in (5.1) and suppose that no pair of zeros z_i, z_j satisfy $z_i\bar{z}_j = 1$ save those for which $|z_i| = |z_j| = 1$. Define the hermitian matrix F by*

(5.7) $$F = \bar{A}'_n[\hat{d}]A_n[\hat{d}] - \bar{A}'_n[d]A_n[d].$$

Then the number of zeros of $d(z)$ in $|z|<1$, $|z|>1$ and $|z|=1$ is $n_+[F]$, $n_-[F]$, $n_0[F]$, respectively, where n_+, n_-, n_0 denote the numbers of positive, negative and zero eigenvalues of F respectively.

Proof. Let k, $\tilde{m}$ denote the number of zeros inside and on the unit circle. By (5.3) and Theorem 3.1, with $w = (d-\hat{d})(d+\hat{d})^{-1}$

$$2k + \tilde{m} - n = -\sigma[B_n[w]] = -\sigma\{\bar{A}'_n[d+\hat{d}]A_n[d-\hat{d}] + \bar{A}'_n[d-\hat{d}]A_n[d+\hat{d}]\}$$

using (3.8). The additivity property of A_n, see (3.3), then yields

$$2k + \tilde{m} - n = \sigma\{2(\bar{A}'_n[\hat{d}]A_n[\hat{d}] - \bar{A}'_n[d]A_n[d])\}.$$

Hence

(5.8) $$2k + \tilde{m} - n = n_+[F] - n_-[F].$$

Also, the restrictions on the zero positions of $d(z)$ are easily seen to imply that $d(z)-\hat{d}(z)$ and $d(z)+\hat{d}(z)$ have $\tilde{m}$ common factors. So by Lemma 3.1,

(5.9) $$n_+(F) + n_-(F) = n - \tilde{m},$$

(5.10) $$n_0(F) = \tilde{m}.$$

Equations (5.8) through (5.10) yield the result. □

In case there are zero pairs z_i, z_j not on the unit circle satisfying $z_i\bar{z}_j = 1$, the theorem needs modification. Let the number of such zeros be $2p$. Then (5.8) is valid, while (5.9) and (5.10) are replaced, since $d(z)-\hat{d}(z)$ and $d(z)+\hat{d}(z)$ have a greatest common divisor of degree $\tilde{m}+2p$, by

(5.11) $$n_+(F) + n_-(F) = n - \tilde{m} - 2p,$$

(5.12) $$n_0(F) = \tilde{m} + 2p$$

and then $n_+(F) = k - p$, $n_-(F) = n - k - \tilde{m} - p$. Thus via F we can only classify the zeros into three sets: {inside unit circle and not satisfying $z_i\bar{z}_j = 1$}, {on unit circle and/or satisfying $z_i\bar{z}_j = 1$}, {outside unit circle and not satisfying $z_i\bar{z}_j = 1$}.

A Toeplitz type result is also possible. Let

(5.13) $$w(z) = \frac{d(z)-\hat{d}(z)}{d(z)+\hat{d}(z)} = m_0 + m_1 z^{-1} + m_2 z^{-2} + \cdots.$$

(Because $w(z)=-\bar{w}(z^{-1})$, this means that $m_i=-\bar{w}_i$, where $w(z)=\sum_{i=0}^{\infty} w_i z^i$.) Let

(5.14)
$$S=\begin{bmatrix} m_0+\bar{m}_0 & \bar{m}_1 & \cdots & \bar{m}_{n-1} \\ m_1 & m_0+\bar{m}_0 & \cdots & \bar{m}_{n-2} \\ m_2 & m_1 & \cdots & \bar{m}_{n-3} \\ \vdots & & & \\ m_{n-1} & m_{n-2} & \cdots & m_0+\bar{m}_0 \end{bmatrix}.$$

With the same hypothesis as Theorem 5.1, we have $n_+(S)$, $n_-(S)$ and $n_0(S)$ the same as $n_+(F)$, $n_-(F)$ and $n_0(F)$. This theorem was obtained by Schur for $d(z)$ with all zeros in $|z|<1$, [8], and follows for us by noting that $S=-T_n[w]$ and using the congruence of $T_n[w]$ and $B_n[w]$.

The expression (5.7) allows us to give a nice determinantal expression for $i\times i$ principal minors of F, call it F_i; arguing as in § 3, we have

(5.15)
$$F_i=\det\left[\begin{array}{cccc|cccc} \bar{d}_n & 0 & \cdots & 0 & d_0 & d_1 & \cdots & d_{n-1} \\ \bar{d}_{n-1} & \bar{d}_n & \cdots & 0 & 0 & d_0 & \cdots & d_{n-2} \\ \vdots & & & & \vdots & \vdots & & \\ \bar{d}_{n-i+1} & \bar{d}_{n-i+2} & \cdots & \bar{d}_n & 0 & 0 & \cdots & d_0 \\ \hline d_0 & 0 & \cdots & 0 & \bar{d}_n & \bar{d}_{n-1} & \cdots & \bar{d}_{n-i+1} \\ d_1 & d_0 & \cdots & 0 & 0 & \bar{d}_n & \cdots & \bar{d}_{n-i+2} \\ \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ d_{i-1} & d_{i-2} & \cdots & d_0 & 0 & 0 & & \bar{d}_n \end{array}\right]$$

which is known as Schur–Cohn's determinant [13, p. 152] and is also equal to the ith inner of Jury's formulation in [11, p. 29].

If the F_i are all nonzero, i.e. $d(z)$ has no unit circle zeros and/or zeros satisfying $z_i\bar{z}_j=1$, then

$$\sigma(F)=n-2V[1,F_1,\cdots,F_n]=n_+[F]-n_-[F]$$

while $n=n_+[F]+n_-[F]$. Hence the number of zeros of $d(z)$ in $|z|>1$ is

(5.16)
$$n_-[F]=V[1,F_1,\cdots,F_n]$$

and so

(5.17)
$$n_+[F]=V[1,-F_1,\cdots,(-1)^nF_n].$$

This result is known from [5].

6. Matrix Cauchy index. There is discussed in [4] the definition, properties and some applications of a Cauchy index of a square rational matrix $W(s)$ with $W(s)$ hermitian for all real s. The possibility arises of proceeding analogously in the unit circle case. We sketch here the main ideas.

Let $W(z)=-\bar{W}'(z^{-1})$ be $n\times n$ and rational in z. Thus $jW(z)$ will be hermitian on $|z|=1$. Matrices W_i are defined as in (3.1) and block Toeplitz matrices $A_N[W]$ and $T_N[W]$ are defined as in (3.2) and (3.7). If $W(z)=G(z)F^{-1}(z)$ is a right matrix factor description [16] of $W(z)$, define

(6.1)
$$B_N[F,G]=\bar{A}_N'[F]A_N[G]+\bar{A}_N'[G]A_N[F]$$

which means that

$$T_N[W]=\{\bar{A}'_N[F]\}^{-1}B_N[F,G]A_N^{-1}[F]. \tag{6.2}$$

If T_{ij} are defined by

$$\frac{\bar{W}'(x)+W(y)}{1-xy}=\sum_{i,j=1}^{\infty} T_{ij}x^{i-1}y^{j-1}, \tag{6.3}$$

then the T_{ij} are the block entries of $T_N[W]$. Also, the block entries B_{ij} of $B_N[F,G]$ are defined by

$$\frac{\bar{F}'(x)G(y)+\bar{G}'(x)F(y)}{1-xy}=\sum_{i,j=1}^{N} B_{ij}x^{i-1}y^{j-1} \tag{6.4}$$

provided that N is not less than the degree of $F(\cdot)$ or $G(\cdot)$.

In case $W(0)$ is infinite, it is not possible to construct $T_N[W]$; however, $B_N[F,G]$ may always be constructed.

One defines the Cauchy index of $W(z)$ as the number of jumps of any eigenvalue of $jW(z)$ from $-\infty$ to ∞ less the number of jumps of eigenvalues from $+\infty$ to $-\infty$ as z moves once round the unit circle in a counterclockwise direction, commencing at any point where $W(z)$ is finite. Then the main result is

$$\mathop{I}_{\text{U.C.}} jW(z)=-\sigma(B_N[W])=-\sigma(T_N[W]) \tag{6.5}$$

where $N\geqq\max(\deg F, \deg G)$. Also, in analogy to Lemma 3.1, we have: with $n=$ McMillan degree of $W(z)$,

$$\operatorname{rank} T_N[W]=\operatorname{rank} B_N[W]=n$$

(again with a lower bound on N, the precise nature of which involves the detailed structure of $W(z)$; in any case, this identity holds for $N\geqq n$).

7. Concluding remarks. In this paper, we have attempted to present parallels to known results on the conventional Cauchy index for the unit circle Cauchy index. This provides a systematic view of many problems and results on unit circle positivity, polynomial root distribution, and stability. Many of the unit circle results closely parallel results for the real line; it is however slightly surprising to see that the Hankel matrix of the real line is replaced by a Toeplitz matrix for the corresponding unit circle result.

Appendix: Proof of Theorem 3.1.

LEMMA A.1. *In case*

$$w(z)=\alpha_1\left(\frac{\lambda+z}{\lambda-z}\right)+\cdots+\alpha_l\left(\frac{\lambda+z}{\lambda-z}\right)^l \tag{A.1}$$

with $|\lambda|=1$, *the theorem is true.*

Proof. Vary α_i, $i=1,\cdots,l-1$ smoothly to zero, preserving $w(z)=-\bar{w}(z^{-1})$. By Lemma 3.1, $B_n[w]$ has constant rank, and therefore constant signature. Next, vary λ smoothly to 1. Again, $B_n[w]$ has constant rank l and constant signature.

Case 1. l is odd. (Then α_l is real.) Observe that $(1-x)(1+y)+(1+x)(1-y)=2(1-xy)$. Hence, using (3.13) with $f(x)=(1-x)^l$ and $g(x)=(1+x)^l$, we have

$$\sum_{i,j=1}^{l} b_{ij}x^{i-1}y^{j-1}=\alpha_l\frac{(1-x)^l(1+y)^l+(1+x)^l(1-y)^l}{1-xy}$$

$$=2\alpha_l\left[\frac{(1-x)^l(1+y)^l+(1+x)^l(1-y)^l}{(1-x)(1+y)+(1+x)(1-y)}\right]$$

$$=2\alpha_l[(1-x)^{l-1}(1+y)^{l-1}-(1-x)^{l-2}(1+y)^{l-2}(1+x)(1-y)\cdots+(1+x)^{l-1}(1-y)^{l-1}]$$

$$=2\alpha_l[(1-x)^{l-1}(1-x)^{l-2}(1+x)\cdots(1+x)^{l-1}]\times\operatorname{diag}[1,-1,1,-1,\cdots,1]\times\begin{bmatrix}(1+y)^{l-1}\\(1+y)^{l-2}(1-y)\\ \vdots\\(1-y)^{l-1}\end{bmatrix}$$

$$=[(1-x)^{l-1}(1-x)^{l-2}(1+x)\cdots(1+x)^{l-1}]\times\begin{bmatrix} & & & & +2\alpha_l\\ & & & \cdot^{\cdot^{\cdot}} & \\ & & +2\alpha_l & & \\ & -2\alpha_l & & & \\ +2\alpha_l & & & & \end{bmatrix}\begin{bmatrix}(1-y)^{l-1}\\(1-y)^{l-2}(1+y)\\ \vdots\\(1+y)^{l-1}\end{bmatrix}.$$

Since rank $B_l=l$, it is clear that $\sigma(B_l)$ is the signature of the constant $l\times l$ matrix in the last equality above. Now it is not hard to verify that the $l\times l$ matrix

$$\begin{bmatrix} & & & 1\\ & & -1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ -1 & & & \\ 1 & & & \end{bmatrix}$$

(where l is odd) has all eigenvalues at $+1$ and -1, and has trace equal to $(-1)^{(l-1)/2}$. Accordingly,

$$\sigma(B)=(-1)^{(l-1)/2}\operatorname{sgn}(\alpha_l).$$

By Theorem 2.1, the lemma is verified for Case 1. The Case 2 verification (l is even) proceeds similarly.

LEMMA A.2. *In case*

$$w(z)=\sum_{i=1}^{l}\left[\gamma_i\left(\frac{\lambda+z}{\lambda-z}\right)^i+\gamma_{-i}\left(\frac{\bar{\lambda}^{-1}+z}{\bar{\lambda}^{-1}-z}\right)^i\right]$$

with $|\lambda|\neq 0, 1$, *the theorem is true.*

Proof. As in Lemma A.1, we can assume $\gamma_1=\cdots=\gamma_{l-1}=0$, and also $\bar{\gamma}_{-l}=(-1)^{l+1}\gamma_l$, see (2.10). Then

$$\frac{\bar{w}(x)+w(y)}{1-xy}=\frac{\gamma_l}{1-xy}\left[\left(\frac{\lambda+y}{\lambda-y}\right)^l+(-1)^{l-1}\left(\frac{\lambda^{-1}+x}{\lambda^{-1}-x}\right)^l\right]$$
$$+\frac{(-1)^{l+1}\bar{\gamma}_l}{1-xy}\left[\left(\frac{\bar{\lambda}^{-1}+y}{\bar{\lambda}^{-1}-y}\right)^l+(-1)^{l+1}\left(\frac{\bar{\lambda}+x}{\bar{\lambda}-x}\right)^l\right].$$

Now

$$\frac{1}{1-xy}=\frac{2}{(\lambda^{-1}-x)(\lambda-y)}\left\{\frac{1}{(\lambda^{-1}+x)/(\lambda^{-1}-x)+(\lambda+y)/(\lambda-y)}\right\}$$
$$=\frac{2}{(\bar{\lambda}-x)(\bar{\lambda}^{-1}-y)}\left\{\frac{1}{(\bar{\lambda}+x)/(\bar{\lambda}-x)+(\bar{\lambda}^{-1}+y)/(\bar{\lambda}^{-1}-y)}\right\}$$

so that

$$\frac{\bar{w}(x)+w(y)}{1-xy}=\frac{2\gamma_l}{(\lambda^{-1}-x)(\lambda-y)}\left[\left(\frac{\lambda+y}{\lambda-y}\right)^{l-1}-\left(\frac{\lambda+y}{\lambda-y}\right)^{l-2}\left(\frac{\lambda^{-1}+x}{\lambda^{-1}-x}\right)\right.$$
$$\left.+\cdots+(-1)^{l-1}\left(\frac{\lambda^{-1}+x}{\lambda^{-1}-x}\right)^{l-1}\right]$$
$$+\frac{2(-1)^{l+1}\bar{\gamma}_l}{(\bar{\lambda}-x)(\bar{\lambda}^{-1}-y)}\left[\left(\frac{\bar{\lambda}^{-1}+y}{\bar{\lambda}^{-1}-y}\right)^{l-1}-\left(\frac{\bar{\lambda}^{-1}+y}{\bar{\lambda}^{-1}-y}\right)^{l-2}\left(\frac{\bar{\lambda}+x}{\bar{\lambda}-x}\right)\right.$$
$$\left.+\cdots+(-1)^{l-1}\left(\frac{\bar{\lambda}+x}{\bar{\lambda}-x}\right)^{l-1}\right].$$

When $w(y)$ is expressed as a rational function, one may adopt as a denominator $f(y)=(\lambda-y)^l(\bar{\lambda}^{-1}-y)^l$, which is certainly coprime with the numerator. Define

$$\xi_1(y)=(\bar{\lambda}^{-1}-y)^l(\lambda+y)^{l-1},\qquad \xi_{l+1}(y)=(\lambda-y)^l(\bar{\lambda}^{-1}+y)^{l-1},$$
$$\xi_2(y)=(\bar{\lambda}^{-1}-y)^l(\lambda+y)^{l-2}(\lambda-y),\qquad \xi_{l+2}(x)=(\lambda-y)^l(\bar{\lambda}^{-1}+y)^{l-2}(\bar{\lambda}^{-1}-y),$$
$$\vdots\qquad\qquad\vdots$$
$$\xi_l(y)=(\bar{\lambda}^{-1}-y)^l(\lambda-y)^{l-1},\qquad \xi_{2l}(y)=(\lambda-y)^l(\bar{\lambda}^{-1}-y)^{l-1}.$$

There results

$$\frac{\bar{f}(x)f(y)[\bar{w}(x)+w(y)]}{1-xy}$$
$$=2\gamma_l[\xi_1(y)\bar{\xi}_{2l}(x)-\xi_2(y)\bar{\xi}_{2l-1}(x)+\cdots+(-1)^{l-1}\xi_l(y)\bar{\xi}_{l+1}(x)]$$
$$+2(-1)^{l+1}\bar{\gamma}_l[\xi_{l+1}(y)\bar{\xi}_l(x)-\xi_{l+2}(y)\bar{\xi}_{l-1}(x)+\cdots+(-1)^{l-1}\xi_{2l}(y)\bar{\xi}_1(x)]$$
$$=[\xi_1(y)\xi_{2l}(y)\xi_2(y)\xi_{2l-1}(y)\cdots]\operatorname{diag}\left\{\begin{bmatrix}0 & 2\gamma_l\\ 2\bar{\gamma}_l & 0\end{bmatrix},\begin{bmatrix}0 & -2\gamma_l\\ -2\bar{\gamma}_l & 0\end{bmatrix},\begin{bmatrix}0 & 2\gamma_l\\ 2\bar{\gamma}_l & 0\end{bmatrix},\cdots\right\}$$
$$\times\begin{bmatrix}\bar{\xi}_1(x)\\ \bar{\xi}_{2l}(x)\\ \bar{\xi}_2(x)\\ \bar{\xi}_{2l-1}(x)\\ \vdots\end{bmatrix}.$$

By Lemma 3.1, we know that this quadratic form has rank $2l$. The block diagonal matrix also has $2l$ entries. Hence its signature, viz zero, is the signature of the quadratic form. Then Lemma A.2 follows on using Theorem 2.1.

Proof of Theorem 3.1. Without loss of generality, let $w(\infty)<\infty$. Write $w(z)=\sum w_i(z)$ where each $w_i(z)$ is of the form considered in Lemma A.1 or Lemma A.2, and $i\neq j$ implies $w_i(z)$ and $w_j(z)$ have distinct poles. Let n be the degree of the denominator of $w(z)$. Now $T_n[w]=\sum T_n[w_i]$, and $\sum \text{rank } T_n[w_i]=n=\text{rank } T_n[w]$. Hence $\sum \sigma\{T_n[w_i]\}=\sigma\{T_n[w]\}$. The result then is a consequence of Theorem 2.1 and Lemmas A.1 and A.2.

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