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Behaviour of the output error identification algorithm for small stepsize gains

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In the output error identification algorithm, it is possible to adjust a gain parameter to trade off, for example, sensitivity to noise and capacity to track plant parameter variations. The rate of convergence of the ideal algorithm (obtained in the noise free, time-invariant plant parameter case) is related to the magnitude of this gain parameter. The analysis applies to a broader class of adaptive problems than output error identification, and in particular, a positive real transfer function appearing in the problem formulation is allowed to be nonrational. An alternative to the usual Lyapunov-based analysis is then needed.

Keywords: Identification, Recursive identification, Output error.

1. Introduction

The following equation arises in the study of adaptive identification filtering and control problems, see e.g. [1]:

$$\phi(k+1)=\phi(k)-\frac{\varepsilon x(k)x'(k)}{1+\varepsilon x'(k)x(k)}\phi(k). \qquad (1.1)$$

Typically, $\phi(\cdot)$ is a vector measuring the error between an estimate of a parameter vector and the true value of that parameter, while $x(k)$ is a vector derived from certain measurements, often termed a regression vector. In [4], the transient behaviour of (1.1) is studied under the conditions ε small, and

$$\alpha_2 I > \sum_{j=k}^{k+S-1} x(j)x'(j) > \alpha_1 I \qquad (1.2)$$

for some positive α_1, α_2, integer S and all k. The key conclusion is that one can find positive constants a, b, and c independent of ε for small ε, such that

$$\|\phi(0)\|^2(\mathrm{e}^{-c\varepsilon})^k \leqslant \|\phi(k)\|^2 \leqslant a\|\phi(0)\|^2(\mathrm{e}^{-b\varepsilon})^k. \qquad (1.3)$$

This result is used in [4] to discuss the effects of noise and other implementation errors in the adaptive algorithm. Small ε for example give better protection against measurement noise but poorer ability to track plant parameter variations. References [5–7] consider variants and specializations of these ideas.

In this work, we shall consider a variant on (1.1) which is important in a number of adaptive systems problems, especially but not only in output error identification. Let $Z(z)$ be a strictly positive real discrete-time transfer function, i.e.

$$\operatorname{Re} Z(\mathrm{e}^{j\omega}) > \delta > 0 \quad \text{for all real } \omega, \qquad (1.4a)$$

$$\text{all singularities of } Z(z) \text{ lie in } |z| < \rho < 1. \qquad (1.4b)$$

The notation $Z\{u(\cdot)\}(k)$ for some time sequence $u(\cdot)$ with support on $k \geqslant 0$ will be taken to mean the value at time k of the time sequence resulting at the output of a linear system of transfer function matrix $Z(z)$, driven by $u(\cdot)$, and with a possibly nonzero initial condition at $k=0$. The variant of (1.1), see e.g. [8–10], can be written as

$$\phi(k+1)=\phi(k)-\varepsilon\Gamma x(k)Z\{x'(\cdot)\phi(\cdot+1)\}(k). \qquad (1.5)$$

It is proved in [8] that if $e(k)$ denotes the state of the system with a rational transfer function $Z(z)$, then $\|e'(k)\phi'(k)\|$ decays exponentially fast to zero under condition (1.2).

Our purpose here is to establish a similar dependence of the convergence rate on ε to that applying for (1.1), and to allow coverage of the nonrational $Z(z)$ case. We needed this result in studying identification and control problems where the plant order is mismodelled and there will doubtless be other situations in which the result will be of use, e.g. those where there is noise

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present, or time-variation of plant parameters occurs. It is of course generally important to establish semiquantitative results on the behaviour of various adaptive algorithms which will give guidance for their use in practice.

By way of preliminary simplification, let us note that we can, without loss of generality, take $\Gamma = I$, for we can replace ϕ by $\Gamma^{-1/2}\phi$ and x by $\Gamma^{1/2}x$ in (1.5) and reduce the problem to one where $\Gamma = I$.

2. Lyapunov-like functions for positive real transfer function matrices

In this section we shall first introduce a form of Lyapunov function for an arbitrary, i.e. not necessarily rational, discrete-time positive real transfer function matrix. In the next section, we shall use this to exhibit a form of Lyapunov function for (1.5).

Lemma 2.1. *Let $Z(z)$ be a positive real, discrete-time, transfer function matrix, not necessarily rational, and let S be a linear, time-invariant, discrete-time system with $Z(\cdot)$ as its transfer function matrix. Suppose S is in the zero state at time 0. Then for each k there exists a function $Q(k)$ of the input $u(l)$, $0 \leqslant l \leqslant k-1$, with the following properties*:

$$Q(k) \geqslant 0 \quad \forall k, \tag{2.1a}$$

$$Q(k) < \infty \quad \text{if } \|u(l)\| < \infty, l \in [0, k-1], \tag{2.1b}$$

$$Q(k+1) - Q(k) \leqslant u'(k) y(k). \tag{2.1c}$$

Proof. Define $Q(k)$ by

$$Q(k) \doteq \sup_{u(l),\, l \in [k, \infty]} \left[-\sum_{l=k}^{\infty} u'(l) y(l) \right]. \tag{2.2}$$

(Obviously $Q(k)$ depends on the state of the system at time k. Since the system begins at time 0 in the zero-state, we can identify the state at time k with $u(l)$, $l \in [0, k-1]$, to conclude that $Q(k)$ is a function of $u(l)$, $l \in [0, k-1]$.)

Notice from (2.2) that

$$Q(k) \geqslant \left[-\sum_{l=k}^{\infty} u'(l) y(l)|_{u(l)=0 \forall l \in [k,\infty]} \right]$$

or

$$Q(k) \geqslant 0.$$

Also, the initial conditions and the positive real property guarantee for all K and $u(l)$, $l \in [0, K]$,

$$\sum_{l=0}^{K} u'(l) y(l) \geqslant 0,$$

whence

$$-\sum_{l=k}^{K} u'(l) y(l) \leqslant \sum_{l=0}^{k-1} u'(l) y(l)$$

so that

$$Q(k) \leqslant \sum_{l=0}^{k-1} u'(l) y(l).$$

This proves (2.1b). Last, (2.2) implies for all $u(k)$,

$$\begin{aligned} Q(k) &\geqslant -u'(k) y(k) \\ &\quad + \sup_{u(l),\ l \in [k+1, \infty)} \left[-\sum_{l=k+1}^{\infty} u'(l) y(l) \right] \\ &= -u'(k) y(k) + Q(k+1), \end{aligned}$$

and (2.1c) is obtained. □

Lemma 2.1 is concerned with positive real transfer function matrices which may or may not be strictly positive real. We shall now strengthen Lemma 2.1 for the case when strict positive realness holds.

Lemma 2.2. *Let $Z(\rho z) - \delta I$ be a positive real, discrete-time, transfer function matrix for some $\rho \in (0, 1)$ and $\delta > 0$, and let S be a linear, time-invariant, discrete-time system with $Z(\cdot)$ as its transfer function matrix. Suppose S is in the zero state at time 0. Then for each k there exists a functional $Q(k)$ of the input $u(l)$, $0 \leqslant l \leqslant k-1$, with the following properties*:

$$Q(k) \geqslant 0 \quad \forall k, \tag{2.3a}$$

$$Q(k) < \infty \quad \text{if } \|u(l)\| < \infty, l \in [0, k-1], \tag{2.3b}$$

$$\begin{aligned} Q(k+1) - Q(k) \leqslant{}& u'(k) y(k) - \delta u'(k) u(k) \\ & -(1-\rho^2) Q(k). \end{aligned} \tag{2.3c}$$

Proof. Define sequences $\bar{u}(k)$, $\bar{y}(k)$ on $[0, \infty)$ by

$$\begin{aligned} \bar{u}(k) &= u(k) \rho^{-k}, \\ \bar{y}(k) &= y(k) \rho^{-k} - \delta u(k) \rho^{-k} \end{aligned} \tag{2.4}$$

and observe that the transfer function matrix mapping $\bar{U}(z)$, the z-transform of the $\bar{u}(\cdot)$ sequence, into $\bar{Y}(z)$, the z-transform of the $\bar{y}(\cdot)$ sequence, is

$$\bar{Z}(z) = Z(\rho z) - \delta I.$$

Since $\bar{Z}(z)$ is positive real, by Lemma 2.1, we know there exists $\bar{Q}(k)$, a function of $\bar{u}(l)$, $l \in [0, k-1]$, such that

$$\bar{Q}(k) \geqslant 0,$$

$$\bar{Q}(k) \leqslant \infty \quad \text{when} \quad \|\bar{u}(l)\| < \infty, l \in [0, k-1],$$

$$\bar{Q}(k+1) - \bar{Q}(k) \leqslant \bar{u}'(k)\bar{y}(k) = \rho^{-2k}[u'(k)y(k) - \delta u'(k)u(k)].$$

Now define $Q(k) = \rho^{2k-2}\bar{Q}(k)$. Then $Q(k)$ can be regarded as a function of $u(k)$, $l \in [0, k-1]$, and (2.3) follow easily. □

The following lemma contains a lower bound on $Q(k)$ which will be important in the sequel.

Lemma 2.3. *Assume the same hypothesis as Lemma 2, and define*

$$y_0(k) = y(k) - Z(\infty)u(k) \tag{2.5}$$

for all k. Then

$$Q(k) \geqslant \tfrac{1}{4}\rho^{-2}y_0'(k)Z^{-1}(\infty)y_0(k). \tag{2.6}$$

Proof. Note first that $Z(\infty)$ must be nonsingular, since $\bar{Z}(z) = Z(\rho z) - \delta I$ is positive real. For the positive real property of $\bar{Z}(z)$ guarantees that

$$\bar{u}'(0)\bar{y}(0) \geqslant 0$$

for all $\bar{y}(0)$, and this quality is precisely $\bar{u}'(0)\bar{Z}(\infty)\bar{u}(0)$.

Now

$$\begin{aligned} Q(k) &= \rho^{2k-2}\bar{Q}(k) \\ &= \rho^{2k-2} \sup_{\bar{u}(l), l \in (k, \infty)} \left[-\sum_{l=k}^{\infty} \bar{u}'(l)\bar{y}(l) \right] \\ &\geqslant \rho^{2k-2} \sup_{\bar{u}(k)} [-\bar{u}'(k)\bar{y}(k)] \\ &= \rho^{-2} \sup_{u(k)} [-u'(k)y(k) + \delta u'(k)u(k)] \\ &\geqslant \rho^{-2} \sup_{u(k)} [-u'(k)y_0(k) - u'(k)Z(\infty)u(k)] \\ &= \tfrac{1}{4}\rho^{-2}y_0'(k)Z^{-1}(\infty)y_0(k). \qquad \square \end{aligned}$$

Notice that $y_0(k)$ depends on $u(l)$, $l \in [0, k-1]$, but not on $u(k)$, i.e. $y_0(k)$ depends on the state at time k. Thus (2.6) relates two functions of the state.

3. Stability of the identification algorithm

In this section, our task is to study the behaviour of the basic algorithm (1.5), with Γ replaced by I. We must make the following assumption:

Assumption 3.1. There exists a time $-K < 0$ and a sequence $u(k)$, $k \in [-K, -1]$, taking the system S with transfer function $Z(z)$ from the zero state at time $-K$ to the state required for initialization at time 0 of the adaptive system (1.5).

If $Z(z)$ is rational, we are simply asserting that the system with transfer function $Z(z)$ is viewed as a state space system which is controllable. Assumption 3.1 allows us to work with an energy function $Q(k)$ for S, defined for $k \geqslant 0$, and satisfying properties (2.3) and (2.6).

Lemma 3.2. *Consider, under assumption* 3.1, *the system* (1.5) *with* $\Gamma = I$, *and define a pseudo-Lyapunov function*

$$W(k) = \phi'(k)\phi(k) = 2\varepsilon Q(k). \tag{3.1}$$

Then

$$W(k+1) - W(k) \leqslant -2\delta\varepsilon[\phi'(k+1)x(k)]^2 - 2\varepsilon(1-\rho^2)Q(k). \tag{3.2}$$

Proof. Using (1.5), we have

$$\begin{aligned} &\phi'(k+1)\phi(k+1) - \phi'(k)\phi(k) \\ &\quad = \phi'(k+1)\phi(k+1) \\ &\qquad - [\phi(k+1) + \varepsilon x(k)Z\{x'(\cdot)\phi(\cdot+1)\}(k)]' \\ &\qquad \cdot [\phi(k+1) + \varepsilon x(k)Z\{x'(\cdot)\phi(\cdot+1)\}(k)] \\ &\quad = -2\varepsilon\phi'(k+1)x(k)Z\{x'(\cdot)\phi(\cdot+1)\}(k) \\ &\qquad - \varepsilon^2\|x(k)Z\{x'(\cdot)\phi(\cdot+1)\}(k)\|^2. \end{aligned} \tag{3.3}$$

Also, using (2.3c), we can write

$$\begin{aligned}2\varepsilon[Q(k+1)-Q(k)] &\leqslant 2\varepsilon\phi'(k+1)x(k)Z\{x'(\cdot)\phi(\cdot+1)\}(k)\\ &\quad -2\delta\varepsilon[\phi'(k+1)x(k)]^2-2\varepsilon(1-\rho^2)Q(k).\end{aligned}\tag{3.4}$$

Adding (3.3) and (3.4) yields

$$\begin{aligned}W(k+1)-W(k) &\leqslant -\varepsilon^2\|x(k)Z\{x'(\cdot)\phi(\cdot+1)\}(k)\|^2\\ &\quad -2\delta\varepsilon[\phi'(k+1)x(k)]^2-2\varepsilon(1-\rho^2)Q(k)\end{aligned}$$

and (3.2) is immediate. □

Lemma 3.2 shows of course that $\phi(\cdot)$ is a bounded sequence. To establish convergence of $\phi(\cdot)$ to zero requires, not surprisingly, more assumptions. Before stating the main result, we need however one more preliminary result.

Lemma 3.3. *There exists a constant α dependent solely on α_2 in* (1.2) *and on $Z(\infty)$ such that for all j*

$$\|\phi(j+1)-\phi(j)\|\leqslant\varepsilon\alpha[\|\phi(j)\|+Q^{1/2}(j)].\tag{3.5}$$

Proof. The fundamental equation (1.5) yields

$$\phi(j+1)=\phi(j)-\varepsilon x(j)Z(\infty)x'(j)\phi(j+1)-\varepsilon x(j)y_0(j)\tag{3.6}$$

or

$$\phi(j+1)=\left[I-\frac{\varepsilon Z(\infty)x(j)x'(j)}{1+\varepsilon x'(j)Z(\infty)x(j)}\right]\phi(j)+\varepsilon\frac{x(j)}{1+\varepsilon x'(j)Z(\infty)x(j)}y_0(j).$$

The result is then immediate on using (2.6). □

We can observe the effect of iteration with this lemma. Thus,

$$\begin{aligned}\|\phi(j+2)-\phi(j)\| &\leqslant\|\phi(j+2)-\phi(j+1)\|+\|\phi(j+1)-\phi(j)\|\\ &\leqslant\varepsilon\alpha[\|\phi(j+1)\|+Q^{1/2}(j+1)]\\ &\quad+\varepsilon\alpha[\|\phi(j)\|+Q^{1/2}(j)]\\ &\leqslant 2\varepsilon\alpha\|\phi(j)\|+\varepsilon\alpha[Q^{1/2}(j)+Q^{1/2}(j+1)]\\ &\quad+\varepsilon^2\alpha^2[\|\phi(j)\|+Q^{1/2}(j)].\end{aligned}$$

More generally, for any finite p,

$$\begin{aligned}\|\phi(j+p+1)-\phi(j)\| &\leqslant p\varepsilon\alpha\|\phi(j)\|\\ &\quad+\varepsilon\alpha[Q^{1/2}(j)+\cdots+Q^{1/2}(j+p)]\\ &\quad+O(\varepsilon^2)[\|\phi(j)\|\\ &\quad+Q^{1/2}(j)+\cdots+Q^{1/2}(j+p-1)].\end{aligned}\tag{3.7}$$

Now we can state the following main result.

Theorem 3.4. *Consider the system* (1.5) *under the assumptions that the persistency of excitation condition* (1.2) *holds, that $\Gamma=I$, that $Z(\rho z)-\delta I$ is positive real for some $\rho\in(0,1)$ and $\delta>0$ and that assumption* 3.1 *holds. With the definition of the pseudo-Lyapunov function $W(k)$ in* (3.1) *there holds for some positive a_1, all k and all ε with $0<\varepsilon\leqslant\varepsilon_0$ for some $\varepsilon_0>0$,*

$$W(k+S)-W(k)\leqslant-\varepsilon a_1W(k).\tag{3.8}$$

Proof. Replace k in (3.2) by $(k+p)$ and use the triangle inequality to get

$$\begin{aligned}W(k+p+1)-W(k+p) &\leqslant-2\delta\varepsilon[\phi'(k+p+1)x(k+p)]^2\\ &\quad-2\varepsilon(1-\rho^2)Q(k+p)\\ &\leqslant-2\delta\varepsilon[\phi'(k)x(k+p)]^2\\ &\quad+2\delta\varepsilon\{[\phi'(k+p+1)-\phi'(k)]x(k+p)\}^2\\ &\quad-2\varepsilon(1-\rho^2)Q(k+p).\end{aligned}\tag{3.9}$$

By (3.7), the second term on the right-hand side is overbounded by

$$2\delta\varepsilon\sup_{j\in[0,\infty)}\|x(j)\|^2\{2(p+1)^2\alpha^2\cdot[\varepsilon^2+o(\varepsilon^2)]\left[\|\phi(k)\|^2+\sup_{j\in[k,k+p]}Q(j)\right]\}.\tag{3.10}$$

Adding up inequalities (3.10) for $p=0,1,\ldots,S-1$ and making use of (1.2) gives

$$\begin{aligned}W(k+S)-W(k) &\leqslant-2\delta\varepsilon\alpha_1\|\phi(k)\|^2\\ &\quad-2\varepsilon(1-\rho^2)\sum_{j=k}^{k+S-1}Q(j)+M(k)\end{aligned}\tag{3.11}$$

where $M(k)$ is overbounded by

$$O(\varepsilon^3)\left[\|\phi(k)\|^2 + \sup_{j\in[k,k+S-1]} Q(j)\right]. \tag{3.12}$$

Together, (3.11) and (3.12) imply that there exists $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leqslant \varepsilon_0$

$$W(k+S) - W(k) \leqslant -\varepsilon\gamma_1\|\phi(k)\|^2 - \varepsilon\gamma_2 Q(k) \tag{3.13}$$

and (3.8) is then immediate. □

Theorem 3.4 is the nearest thing we can obtain to our exponential convergence result which shows that convergence must occur at a rate which is at least exponential. Of course, it does follow from (3.8) that $W(k)$ can be bounded by a decaying exponential, and (3.1) relates $\phi(k)$ and $W(k)$.

From Section 1, it will be recalled that, in the case of equation (1.1) – which can be regarded as a special form of (1.5) with $Z(z)$ constant – we can exhibit a maximum rate of convergence, as illustrated by the lower bound in (1.3). For completeness, we shall illustrate the existence of such a bound for (1.5). We assume the same hypothesis as for Theorem 3.4.

Because $Z(z)$ is discrete positive real, we have for some N, all $x(\cdot)$ and $\phi(\cdot)$,

$$\begin{aligned}\sup_k&|Z\{x'(\cdot)\phi(\cdot+1)\}(k)|\\ &< N\sup_k|x'(k)\phi(k+1)|\\ &< N'\sup_k\left[\phi'(k+1)\phi(k+1)\right]^{1/2}\end{aligned}$$

for some constant N'.

Now consider the special case of (1.5) where $Q(0) = 0$. By Theorem 3.4 and the form of $W(k)$, we have

$$\|\phi(k)\| \leqslant \|\phi(0)\| \quad \text{for all } k$$

and so (1.5) yields

$$\begin{aligned}\|\phi(j) - \phi(0)\| &\leqslant \varepsilon \sup_k\|x(k)\| jN'\|\phi(0)\|\\ &\leqslant \varepsilon jN''\|\phi(0)\|\end{aligned}$$

whence

$$\|\phi(0)\| - \|\phi(j)\| \leqslant \varepsilon jN''\|\phi(0)\|$$

or

$$\|\phi(j)\| \geqslant (1 - \varepsilon jN'')\|\phi(0)\|.$$

From this relation, the lower bound in (1.3) follows with k replaced by S. Thus for at least one initial condition for (1.5), viz. $Q(0) = 0$, the lower bound of (1.3) cannot be bettered. [It is possible to better the bound for another initial condition: $\phi(0) = 0$, $Q(0) \neq 0$.]

Finally, we comment on the behaviour of (1.5) when ε is not necessarily very small. Of course, when $Z(z)$ is rational, the results of [8–10] imply exponential asymptotic stability. Asymptotic stability is easily seen. Thus, Lemma 3.2 yields

$$Q(k) \to 0, \qquad |\phi'(k+1)x(k)| \to 0.$$

Then (3.6) with (2.6) yields

$$\|\phi(j+1) - \phi(j)\| \to 0$$

whence

$$|\phi'(k+l)x(k)| \to 0 \quad \text{for } l - 2, 3, \ldots, S$$

or

$$|\phi'(k+1)x(k-l)| \to 0 \quad \text{for } l = 0, 1, \ldots, S-1.$$

Now (1.2) yields $\|\phi(k)\| \to 0$, i.e. asymptotic stability holds. A strengthening of this argument yields exponential stability: suppose exponential stability does not hold. Then for arbitrarily small η, we can find an initial condition $\phi(0)$, $Q(0)$ and a k such that

$$W(k+S) - W(k) \geqslant -\eta^2 W(k). \tag{3.14}$$

It follows easily that

$$\|\phi(k+p) - \phi(k)\| \leqslant O(\eta)\|\phi(k)\|$$

for $p \in [0, 1, \ldots, S]$. Then (3.9) yields

$$\begin{aligned}W(k&+p+1) - W(k+p)\\ &\leqslant -2\delta\left[\phi'(k)x(k+p)\right]^2\\ &\quad -\varepsilon(1-\rho^2)Q(k+p) + O(\eta^2)\|\phi(k)\|^2.\end{aligned}$$

Adding S such inequalities and using the fact that η is arbitrarily small yields a contradiction of (3.14).

4. Conclusion

In addition to the main result, Theorem 3.4, there are several concepts which need to be emphasised. First, we have dispensed with the rationality of $Z(z)$. This rationality assumption appeared to be crucial in [8–10], for [8–10] studied (1.5) with a Lyapunov function, and this Lyapunov function was constructed by appeal to a state-variable characterization of the discrete positive real property, [11]. Of course, the ideas of Section 2 can be thought of as some type of infinite dimensional generalization of the ideas of [11].

Second, though we have worked here in discrete time, considering this to be more important from the applications viewpoint, there should be little difficulty in obtaining a continuous time result, involving both a small value of the gain parameter and a nonrational positive real transfer function matrix, generalizing thereby the results of [12].

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