Using a standard result stated in [8, Sect. III], if the process described by (2.5) has a stable inverse, along the subsequence $\{K_s\}$ we have

$$|Y(K_s)| > \alpha_1 \|X(K_i)\| - \alpha_2 \qquad \text{with } \infty > \alpha_1 > 0 \text{ and } \infty > \alpha_2 \geqslant 0. \tag{A.3}$$

Hence, using (A.1) and (A.3), we have

$$|Y(K_s)| > \alpha_1 \|X(K_s - r - 1)\| - \alpha_2. \tag{A.4}$$

Since $\lim_{K_i \to \infty} \|X(K_i)\| = \infty$, using (A.3), we have that $\lim_{K_s \to \infty} |Y(K_s)| = \infty$. This implies that $\lim_{K_s \to \infty} \|X(K_s - r - 1)\| = \infty$ given that $|Y(K_s)| \leqslant \|\theta\| \|X(K_s - r - 1)\|$ and $\|\theta\|$ is bounded. Therefore, the subsequence $\{K_s\}$ verifies properties a) and b) of assumption 4), which proves this lemma in the noise-free case.

For the case in which $\Delta(K) \neq 0$, an analogous approach can be taken by considering an augmented process where $\Delta(K)$ will at the same time be an additional input and an additional output. Since $\{|\Delta(K)|\}$ is bounded, the augmented process can also be considered as a stable inverse one. Repeating the above reasoning for the augmented process, and using the previously considered standard result for the multivariable case, the complete proof of this lemma is derived.

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d-Step Lattice-Ladder Predictor in Terms of One-Step Lattice-Ladder Predictors

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***Abstract*—In this note, it is shown that a d-step lattice-ladder predictor may be expressed in terms of one-step lattice-ladder predictors.**

I. Introduction

In various applications, it is desirable to process a discrete-time time series so that at any instant of time, the value of the series d steps ahead is being predicted; see, e.g. [1] and [2]. There is an extensive literature on this problem for $d = 1$ (see, e.g., [3]–[10]), and rather less for the case $d > 1$ [1], [2], [11]. In this short note, we record formulas which may be useful in developing d-step-ahead estimates and their associated errors. Since there are attractive structures for the one-step case, the advantages of a number of these structures can be carried over to the $d > 1$ case.

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II. Relations Involving Errors

Let y_t be a stationary, second-order, zero-mean vector process. We shall then make use of the following notation. With d, n positive integers,

$$\hat{y}^n_{t+d|t} = E[y_{t+d} | y_t, \cdots, y_{t-n+1}] \tag{2.1a}$$

$$\tilde{y}^n_{t+d|t} = y_{t-d} - \hat{y}^n_{t-d|t} \tag{2.1b}$$

$$\hat{y}^n_{t-n-d-1|t-n+1} = E[y_{t-n-d-1} | y_{t-n-1}, y_{t-n+2}, \cdots, y_t] \tag{2.1c}$$

$$\tilde{y}^n_{t-n-d+1;t-n-1} = y_{t-n-d+1} - \hat{y}^n_{t-n-d+1;t-n+1}. \tag{2.1d}$$

Of course, we are defining d-step-ahead and d-step-backward estimators and the associated errors. With these definitions, we can now state some important relations.

Lemma 1: With quantities as defined above,

$$\begin{aligned}\tilde{y}^n_{t+d \cdot t} &= \tilde{y}^{n-d-1}_{t-d|t-d-1} + \sum_{i=1}^{d-1} E\left[\tilde{y}^{n+d-i-1}_{t+d|t+d-i-1} \middle| y^{n+d-i-1}_{t+d-i|t+d-i-1}\right] \\ &= \tilde{y}^{n+d-1}_{t-d|t+d-1} + \sum_{i=1}^{d-1} E\left[y_{t+d} \middle| \tilde{y}^{n+d-i-1}_{t+d-i|t+d-i-1}\right]\end{aligned} \tag{2.2a}$$

$$\begin{aligned}&\tilde{y}^n_{t-n-d-1|t-n+1} \\ &\quad = \tilde{y}^{n-d-1}_{t-n-d+1|t-n-d+2} \\ &\qquad + \sum_{i=1}^{d} E\left[y^{n-d-i-1}_{t-n-d+1|t-n-d-2-i} \middle| \tilde{y}^{n-d-i-1}_{t-n-d-1-i|t-n-d+2+i}\right] \\ &\quad = \tilde{y}^{n+d-1}_{t-n-d+1|t-n-d-2} \\ &\qquad + \sum_{i=1}^{d-1} E\left[y_{t-n-d+1} \middle| \tilde{y}^{n-d-i-1}_{t-n-d-1-i|t-n-d+2+i}\right].\end{aligned} \tag{2.2b}$$

Remark: Equation (2.2) enables expressing the d-step forward and backward prediction error as a linear combination of a one-step forward and backward error, respectively. The particular weighting coefficient, which came from evaluating the conditional expectations in (2.2), depends on the particular covariance data.

The proof of (2.2a) and (2.2b) follows by iteration on d of [11, eq. (3.18) and (3.21)]. Hence, the details of the proof are omitted.

III. Evaluation of the Weighting Coefficients

The result below evaluates the particular weighting coefficients to appear in (2.2). We make the following notational definition. The A_{ni}, $i = 1, \cdots, n$ define the one-step-ahead predictor:

$$\hat{y}^n_{t-1|t} = -A_{n1} y_t - A_{n2} y_{t-1} - \cdots - A_{nn} y_{t-n+1}. \tag{3.1}$$

The A^d_{ni}, $i = 1, \cdots, n$ define the d-step-ahead predictor:

$$\hat{y}^n_{t+d|t} = -A^d_{n1} y_t - A^d_{n2} y_{t-1} - \cdots - A^d_{nn} y_{t-n+1}. \tag{3.2}$$

Notice that these definitions imply that

$$\tilde{y}^n_{t+1|t} = [I \quad A_{n1} \quad \cdots \quad A_{nn}][y'_{t+1} \quad y'_t \quad \cdots \quad y'_{t-n+1}]' \tag{3.3}$$

$$\tilde{y}^n_{t+d|t} = \Big[I \quad \underbrace{0 \quad \cdots \quad 0}_{d-1} \quad A^d_{n1} \quad \cdots \quad A^d_{nn}\Big] \cdot \left[y'_{t+d} \quad \cdots \quad y'_t \quad \cdots \quad y'_{t-n+1}\right]'. \tag{3.4}$$

Also, with $R_i = E[y_t y'_{t-i}]$, denote by $\mathbb{R}_n$ the $(n \times n)$ block Toeplitz matrix with an ij block entry R_{i-j}. Define π_n as the error covariance matrix for the one-step ahead predictor using n time measurements. Then there holds the important relation

$$[I \quad A_{n1} \quad \cdots \quad A_{nn}] \mathbb{R}_n = [\pi_n \quad 0 \quad \cdots \quad 0]. \tag{3.5}$$

Similar definitions can, of course, be made for the backward predictors.

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The conventional Levinson algorithm computes the quantities A_{ni} (and associated backward quantities B_{ni}), together with π_n. A formula for $\mathbb{R}_n^{-1}$ is available using the A_{ni} and B_{ni}. Our interest is to set out relations between the d-step-ahead predictor coefficients A^d_{ni} and the one-step-ahead coefficients, so that if the latter are known, the former can be found easily.

Lemma 2: In terms of one-step predictor coefficients, define the quantities $K^{f,i}_{n,d}$, $i=1,\cdots,d-1$ by

$$\begin{bmatrix} K^{f,1}_{n,d} & \cdots & K^{f,d-1}_{n,d}\end{bmatrix}\begin{bmatrix} 1 & A_{n+d-2,1} & \cdots & A_{n+d-2,d-2} \\ 0 & I & \cdots & A_{n+d-3,d-3} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & I\end{bmatrix} = -\begin{bmatrix} A_{n+d-1,1} & \cdots & A_{n+d-1,d-1}\end{bmatrix}. \quad (3.6)$$

Then

$$\begin{bmatrix} I & \underbrace{0 \cdots 0}_{d-1} & A^d_{n1} & \cdots & A^d_{nn}\end{bmatrix} = \begin{bmatrix} I & A_{n+d-1,1} & \cdots & A_{n+d-1,n+d-1}\end{bmatrix} + K^{f,1}_{n,d}\begin{bmatrix}0 & I & A_{n+d-2,1} & \cdots & A_{n+d-2,n+d-2}\end{bmatrix} + \cdots + K^{f,d-1}_{n,d}\begin{bmatrix}\underbrace{0\ 0\ \cdots\ 0}_{d-1} & I & A_{n1} & \cdots & A_{nn}\end{bmatrix} \quad (3.7)$$

and

$$\hat{y}^n_{t+d|t} = \tilde{y}^{n+d-1}_{t+d|t+d-1} + \sum_{i=1}^{d-1} K^{f,i}_{n,d}\tilde{y}^{n+d-i-1}_{t+d-i|t+d-i-1}. \quad (3.8)$$

Proof: Let $\phi(t) = [y'_{t-d}\ \ y'_{t-d-1}\ \ \cdots\ \ y'_{t-n+1}]'$. Now (2.2a) can be written as

$$\tilde{y}^n_{t+d|t} = \tilde{y}^{n+d-1}_{t+d|t+d-1} + \bar{K}^{f,1}_{n,d}\tilde{y}^{n+d-2}_{t+d-1|t+d-2} + \cdots + \bar{K}^{f,d-1}_{n,d}\tilde{y}^n_{t+1|t} \quad (3.9)$$

where

$$\bar{K}^{f,i}_{n,d} = E\left\{y_{t-d}\left(\tilde{y}^{n+d-i-1}_{t+d-i|t+d-i-1}\right)'\right\} \cdot \left[E\left\{\tilde{y}^{n+d-i-1}_{t+d-i|t+d-i-1}\left(\tilde{y}^{n+d-i-1}_{t+d-i|t+d-i-1}\right)'\right\}\right]^{-1}. \quad (3.10)$$

Now use the definition of $\phi(t)$, together with (3.3) and (3.4), to rewrite (3.9) as

$$\begin{bmatrix} I & \underbrace{0 \cdots 0}_{d-1} & A^d_{n1} & \cdots & A^d_{nn}\end{bmatrix}\phi(t) = \begin{bmatrix} I & A_{n+d-1,1} & \cdots & A_{n+d-1,n+d-1}\end{bmatrix}\phi(t) + \bar{K}^{f,1}_{n,d}\begin{bmatrix}0 & I & A_{n+d-2,1} & \cdots & A_{n+d-2,n+d-2}\end{bmatrix}\phi(t) + \cdots + \bar{K}^{f,d-1}_{n,d}\begin{bmatrix}\underbrace{0\ 0\ \cdots\ 0}_{d-1} & I & A_{n1} & \cdots & A_{nn}\end{bmatrix}\phi(t).$$

Postmultiply by $\phi'(t)$, take the expectation, and then postmultiply by $\mathbb{R}^{-1}_{n+d-1}$. There results

$$\begin{bmatrix} I & \underbrace{0 \cdots 0}_{d-1} & A^d_{n1} & \cdots & A^d_{nn}\end{bmatrix} = \begin{bmatrix} I & A_{n+d-1,1} & \cdots & A_{n+d-1,n+d-1}\end{bmatrix} + \bar{K}^{f,1}_{n,d}\begin{bmatrix}0 & I & A_{n+d-2,1} & \cdots & A_{n+d-2,n+d-2}\end{bmatrix} + \cdots + \bar{K}^{f,d-1}_{n,d}\begin{bmatrix}\underbrace{0\ 0\ \cdots\ 0}_{d-1} & I & A_{n1} & \cdots & A_{nn}\end{bmatrix}. \quad (3.11)$$

Equating columns 2 through d on each side yields

$$\begin{bmatrix} \bar{K}^{f,1}_{n,d} & \cdots & \bar{K}^{f,d-1}_{n,d}\end{bmatrix}\begin{bmatrix} I & A_{n+d-2,1} & \cdots & A_{n+d-2,d-2} \\ 0 & I & \cdots & A_{n+d-3,d-3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I\end{bmatrix} = -\begin{bmatrix} A_{n+d-1,1} & \cdots & A_{n+d-1,d-1}\end{bmatrix}. \quad (3.12)$$

We see that the $K^{f,i}_{n,d}$ of (3.6) and the $\bar{K}^{f,i}_{n,d}$ of (3.12) are the same. Then (3.7) and (3.8) follow from (3.11) and (3.9).

To conclude this section, we note an alternative construction for the $K^{fi}_{n,d}$. Recalling (3.10), we have

$$K^{f,i}_{n,d} = R_i + \sum_{j=1}^{n+d-i-1}\left[R_{i+j}A'_{n+d-i-1,j}\right]\pi^{-1}_{n+d-i-1}. \quad (3.13)$$

IV. Conclusions

In this note, it is shown that the quantities including the prediction error and the reflection coefficients of a d-step lattice-ladder prediction algorithm may be expressed in terms of more standard quantities, viz. the coefficients of the forward and backward polynomials and the variance of the prediction errors of one-step lattice-ladder prediction.

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The All-Pass Property of Optimal Open-Loop Tracking Systems

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***Abstract*—The structure of the optimal open-loop linear model-following system is investigated. It is shown that if the given plant is asymptotically stable but has zeros in the right half-plane, the stable optimal system contains an all-pass network whose transference possesses unity singular values on the imaginary axis. In the special case of optimal tracking, it is shown that the resulting optimal transfer function matrix of the system is equal to the all-pass transfer function matrix which is normalized to be the identity matrix at the zero frequency.**

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