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OUTPUT FEEDBACK AND GENERIC STABILIZABILITY*

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Abstract. We consider questions of pole placement and stabilization for generic linear systems with prescribed state, input and output dimensions, where the controller must be implemented by linear memoryless output feedback. We present a criterion, in terms of a special pole placement property, for generic stabilizability and apply this to describe constraints on the dimensions which are consistent with generic stabilizability. We also discuss the rationality and solvability by radicals of stabilizing or pole positioning gains, and we describe how decision algebra can theoretically handle existence questions for generic systems.

Key words. multivariable control, output feedback, stabilizability of multivariable systems, decision algebra, Galois theory, solvability of pole placement equations by radicals

1. Introduction. In this paper, we are concerned with questions of pole assignability and stabilizability for real linear input–output systems

$$\frac{dx}{dt}=Fx+Gu, \qquad y=Hx, \tag{1.1}$$

or

$$x(t+1)=Fx(t)+Gu(t), \qquad y(t)=Hx(t) \tag{1.1'}$$

where we allow constant gains $u=Ky$ as feedback. The equations of pole assignability are real polynomials, and it is natural to attempt to solve these equations by eliminating the unknown variable K. Similar remarks apply to the equations of stabilizability which include, however, algebraic inequalities arising for example from the Routh–Hurwitz criteria. In what follows, we shall use various results from classical algebraic geometry, including elimination theory and the Schubert calculus of enumerative geometry, which apply to the equations of pole placement.

Put geometrically, elimination theory consists in the study of a projection

$$p_1: X\times Y\to K \tag{1.2}$$

restricted to an algebraic, or semialgebraic, set $Z\subset X\times Y$, where X and Y can be taken to real or complex vector spaces, e.g. $X=\mathbb{R}^N$, $Y=\mathbb{R}$. The main problem in elimination theory consists in finding a description of the set

$$p_1(z)=\{x:\exists y \text{ such that } (x,y)\in Z\}$$

in terms of Z. A basic example is given by

$$Z=\{(x,y): x=y^2\}, \tag{1.3}$$

which is algebraic but for which $p_1(Z)$ is only semialgebraic if we take real coefficients.[1]

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[1] Semialgebraic sets are defined in (3.4), § 3.

In relation to the pole assignability question for a prescribed F, G, H, we can identify the entries of K with the space Y and the coefficients of the closed loop characteristic polynomial, call them $p_1, \cdots, p_n$, with X. Then

$$Z = \left\{(p_1, \cdots, p_n, K) \colon \det(sI - F - GKH) = s^n + \sum_{i=1}^{n} p_i s^{n-i}\right\},$$

and pole assignability of a generic closed-loop polynomial holds if and only if $p_1(Z)$ coincides with all of R^n save a proper subvariety.

Among the results we obtain using classical algebraic geometry is: the condition $mp \leqq n$ is necessary for the stabilizability of the generic (F, G, H). This condition is well known to be necessary for pole assignability of the generic (F, G, H), and our result raises the question as to whether or not, in terms of the values m, n, p, these two questions might not be equivalent. As unlikely as this may be, at the time we write there is no counterexample (although there is evidence in this direction for $m = 2$, $n = 9$, and $p = 6$, see [5]). We also show that if a stabilizing gain exists, then such a gain can be found by a rational procedure. On the other hand, we show that if $mp = n$, a rational procedure for finding a gain K which assigns a given characteristic polynomial (assuming such a K exists) does not exist unless $\min(m, p) = 1$, in which case a linear formula can be found. Moreover, using square roots as well as rational operations only helps if $\min(m, p) = \max(m, p) = 2$. This is of course in contrast with pole assignment by state feedback, and answers in the negative a question raised in [1].

We also argue that one can in principle determine by rational calculations whether, given m, n, p, generic F, G, H are pole assignable, generically pole assignable, or stabilizable. We say "in principle" since the number of calculations required is enormous.

We use several tools to prove the results. One of the theorems, due to Tarski–Seidenberg, asserts that if Z is semialgebraic, then $p_1(Z)$ is semialgebraic. This theorem can be used iteratively to reduce the question of the existence of a solution $x \in \mathbb{R}^n$ to a set of semialgebraic equations to the question of existence of a solution to another set of semialgebraic equations in, for example, the unknown $x_1 \in \mathbb{R}$. Such existence can be decided by a rational procedure in the coefficients of the resulting semialgebraic equations. The Tarski–Seidenberg theorem is extremely qualitative, and "worst-case" analysis [7] shows that such a decision procedure takes at least 2^{kn} steps, where $k > 0$ is a constant and n is the length of the input formula.

We also use a classical form of elimination theory, over $\mathbb{C}$: if $Z \subset \mathbb{C}^N \times \mathbb{C}^M$ is defined by equations which are homogeneous in y, then $p_1(Z) \subset \mathbb{C}^N$ is definable by polynomial equations. In particular, $p_1(Z)$ is closed.

A topological form of this elimination theorem also holds over $\mathbb{R}$ and is crucial in showing that (for $mp \leqq n$) the image of the pole-placement map is Euclidean closed in $\mathbb{R}^n$ for the generic system [4]. Our proof of Theorem 1 relies on this result.

We must also use rather explicit elimination arguments which have appeared in the literature. Among these are the works by Willems–Hesselink [23] and, more recently, Morse–Wolovich–Anderson [19] which treat the case $m = p = 2$, and $m = 2$, $p = 3$. These authors, after considerable calculation, obtain a single explicit equation in a single unknown, and it is possible to obtain some quantitative and qualitative results from the form of the equations. Finally, we use the results of Brockett–Byrnes [3] who determined the degree of this equation, for general m, p, using methods of the Schubert calculus. This calculus was developed in the 19th century in order to deduce the degree of the final equation one would obtain in certain problems of enumerative geometry, without going through the elimination theory first. It is a fortunate fact that the return difference equation corresponds to a classical equation of enumerative geometry, enabling one to determine this degree as a function of m and p.

2. Statements of the main results. Let us suppose that (F, G, H) is a triple of matrices which correspond to either a discrete or a continuous time system having m inputs, n states and p outputs. We consider the questions, for m, n, p fixed:

Question 1. Is it true that for all (F, G, H), except perhaps those contained in a proper algebraic set, one can arbitrarily assign the (closed-loop) eigenvalues of $F+GKH$ by suitable choice of output feedback K?

Question 2. Is it true that one can stabilize all (F, G, H), except perhaps those contained in a proper algebraic set, by some output feedback K?

Concerning Question 1, it is known [13], [23] that $mp \geqq n$ is a necessary condition on the parameters m, n, p. In § 3 we derive a stabilizability criterion as a limiting form of the equivalence of generic stabilizability for continuous and for discrete time systems. This can be thought of as an equivalence between generic stabilizability and the generic existence to an output feedback deadbeat control problem for nondegenerate systems (in the sense of [3], [4]):

THEOREM 1. *If* $mp \leqq n$, *the following statements are equivalent*:

(i) m, n, p *are such that the generic* (F, G, H) *is stabilizable*;

(ii) m, n, p *are such that for any nondegenerate* (F, G, H) *there exists a gain* K *such that the closed loop polynomial is* s^n.

This result holds for $mp > n$ as well, with "nondegenerate" replaced by the weaker term "generic". Since we do not need the general result here, we shall only prove it in the case $mp \leqq n$. From Theorem 1 we obtain

THEOREM 2. $mp \geqq n$ *is necessary for generic stability.*

This result of course implies that $mp \geqq n$ is necessary for Question 1 as well, but also raises the question as to whether the answers to Questions 1 and 2 might not agree, as functions of the parameters m, n, and p. On the one hand, if $\max(m, p) \geqq n$ then generically either G or H is of rank n so that one is in the state feedback situation where the answer to Question 1, and therefore to Question 2, is well known to be in the affirmative under the generic hypothesis of reachability. On the other extreme, Theorem 2 shows that for $mp < n$ the answer to both questions is in the negative, so that explicit calculations for $mp \sim n$ are therefore quite interesting. However, aside from a few special cases, our knowledge is incomplete.

Example 1. $(m = p = 2)$. If $n = 4$, it has been shown by Willems–Hesselink [23] that pole placement does not hold for an open subset of (F, G, H). In [3] it is shown that pole placement does not hold unless the transfer function $T(s) = H(sI - F)^{-1}G$ has rank 1. In particular, pole placement does not hold for (F, G, H) in an open, dense set. In [19], necessary and sufficient conditions for generic pole placement for a particular system of this dimension are derived.

Thus, by Kimura's theorem [16] and the Willems–Hesselink counterexample, the answer to Question 1 is yes if, and only if, $n \leqq 3$. In [23] it is asserted that a modification by P. Molander of the techniques in [23] shows that the answer to Question 2 is in the negative if $n = 4$. Thus, the answer to both Questions 1 and 2, if $m = p = 2$, is yes if, and only if, $n \leqq 3$. Since this result is unpublished, in § 4 we present a verification of Molander's conclusion as a corollary to our generic stabilizability criterion. This of course gives another proof of the Willems–Hesselink theorem.

THEOREM 3 (Molander). *There is a nonempty open set of* (*nondegenerate*) 2×2 *systems of degree* 4 *which are not stabilizable by constant output gain feedback.*

Example 2. ($m=2$, $p=2^k-1$). It is known in this case that the answer to Question 1, and therefore to Question 2, is in the affirmative [3] provided $mp \geqq n$. By Theorem 2, the answer to both questions, for these values of m, p, is therefore yes if, and only if, $mp \geqq n$.

Example 3. ($m=2$, $p=4$). At present, one is able to deduce from the results proved in [3] and more refined topological methods that the answer to Question 1, and therefore to Question 2, is in the affirmative whenever $n \leqq 7$. Theorem 2 then asserts that the only case which remains to be analyzed is $n=8$, where it has been conjectured [6] that the answer to Question 1 is in the negative.

We should mention, however, that there are cases (e.g. $m=2, p=6, n=9$) where generic stabilizability is known to hold, but where Question 1 remains unanswered [5].

Until now, we have only discussed the existence of solutions to the problem of pole positioning and stabilization. Equally important is the consideration of what kind of algorithm might exist for finding a gain K which places the poles, or stabilizes the system, provided such a gain exists. In §§ 5 and 6 we analyze each of these questions and prove

THEOREM 4. *Suppose there exists a gain which stabilizes the system* (F, G, H). *Then, one can find such a* K *by an algorithm which is rational in the coefficients of* (F, G, H).

In [1] the question was raised as to whether rational formulae exist for a gain K which places the closed loop characteristic polynomial at $p(s)=s^n+p_1s^{n-1}+\cdots+p_n$. That is, provided such a gain K exists, can one find K as a rational function of $(F, G, H, p_1, \cdots, p_n)$? This holds for the case of state feedback and, in particular, where $\min(m,p)=1$ and $\max(m,p) \geqq n$. In this case, a linear formula for K follows from consideration of the phase-variable canonical form. However, as the equation obtained by Willems–Hesselink (see also [3], [19]) shows for the case $m=p=2, n=4$, there exist precisely 2 gains (possibly a complex conjugate pair) counted with multiplicity which place a given real monic polynomial

$$s^4+p_1s^3+\cdots+p_4.$$

Moreover, the coefficients of such a 2×2 gain K are given by the solution formula for a quadratic equation. Thus, in general, a rational formula does not exist. If $mp=n$, we can give a more precise answer to the question raised in [1]:

THEOREM 5. *If* $mp=n$, *the following statements are equivalent for the generic* (F, G, H) *and monic polynomial* $p(s)$:

(a) *There exists a rational formula, in the coefficients of* $p(s)$ *and entries of* (F, G, H), *for some* K *which places the closed loop polynomial at* $p(s)$;

(b) *There exists a linear formula, in the coefficients of* $p(s)$ *and entries of* (F, G, H), *for such a* K;

(c) $\min(m,p)=1$ *and* $\max(m,p)=n$.

THEOREM 6. *If* $mp=n$, *the following statements are equivalent for the generic* (F, G, H) *and monic polynomial* $p(s)$:

(a) *There exists a formula, involving rational expressions and square roots, for some* K *which places the closed loop polynomial at* $p(s)$;

(b) *Either* $\min(m,p)=1$ *or* $\min(m,p)=\max(m,p)=2$.

Indeed, if $mp=n$ we conjecture that the only cases for which there exists formulae for K involving rational operations and radicals are

(i) $\min(m,p)=1$ and $\max(m,p)=n$; or

(ii) $\min(m,p)=\max(m,p)=2$.

This conjecture appears natural in the light of our techniques (§ 6), which are an application of Galois theory and of the methods used in [3] enabling one to express the number $d_{m,p}$ of (perhaps complex) gains K which place the poles of a given generic (= nondegenerate) system at a given monic polynomial if $mp = n$. In fact

$$d_{m,p} = \frac{1! \cdots (p-1)!(mp)!}{m! \cdots (m+p-1)!}.$$

This agrees with the Willems–Hesselink calculation [20] that $d_{2,2} = 2$, and with the recent calculation made by Morse–Wolovich–Anderson [19] that[2] $d_{2,3} = d_{3,2} = 5$.

Our methods for proving Theorem 4 rely quite heavily on the Tarski–Seidenberg theorem (Prop. 3.2). In the course of the proof we need several other results from "decision algebra". With these results in hand, it only requires modest additional effort to show that the question raised in this paper, i.e. whether or not Questions 1 and 2 are equivalent for any fixed m, n, p triple, can in fact be answered by decision algebra. This is shown in the Appendix.

The actual application of a decision-algebra-based checking procedure is of course extremely impractical to implement, but we should emphasize that, at present, this is the only method which is even in principle capable of answering this equivalence question for arbitrary m, n, p. For this reason, we feel it is worthwhile to give a proof of this statement.

3. Proof of Theorems 1 and 2. We shall begin by proving that, for m, n, p fixed, stabilizability for the generic $(F, G, H) \in \mathbb{R}^{n^2+n(m+p)}$ is equivalent to the property that $(s-\rho)^n$, $\rho \in \mathbb{R}$, may be assigned as the closed loop characteristic polynomial for the generic $(F, G, H) \in \mathbb{R}^{n^2+n(m+p)}$. It is intuitively clear that Question 2 should not distinguish between continuous time and discrete time stabilizability. This follows from the first lemma where $\varepsilon = 1$ and $\rho = 0$:

LEMMA 3.1. *The following statements are equivalent:*

(i) *m, n, p are such that for all (F, G, H)—except perhaps those contained in a proper algebraic set—there exists a stabilizing gain K;*

(ii) *m, n, p are such that for all (F, G, H)—except perhaps those contained in a proper algebraic set—for all real ρ and all $\varepsilon > 0$, there exists a gain K such that the eigenvalues of $F + GKH$ are contained in an ε-disc centered about ρ.*

Proof. We first note that to say (1.1) is stabilizable is to say the system

$$\dot{x} = Fx + Gu, \qquad y = Hx + Ju, \tag{3.1}$$

with J arbitrary but fixed, is stabilizable. For, if K is a stabilizing gain for (1.1), and $I + JK$ is nonsingular, then the gain $u = \bar{K}y$, where $K(I+JK)^{-1}$ stabilizes (3.1). $I + JK$ is singular; we may choose $\bar{K}$ sufficiently close to K so that $\bar{K}$ is a well-defined stabilizing gain for (3.1).

Now consider the conformal transformation

$$\phi(z) = \left(\frac{z-\rho}{\varepsilon+1}\right)\left(\frac{z-\rho}{\varepsilon-1}\right)^{-1}$$

and define the rational matrix valued function

$$V(z) = W(\phi(z)) = \tilde{H}(zI - \tilde{F})^{-1}\tilde{G} + \tilde{J} \tag{3.2}$$

[2] Based on our techniques and those in [12], the authors of [6] have confirmed our conjecture in the case $m = 2$, $p = 3$ by showing that the Galois group of the output feedback problem is the full symmetric group, S_5.

where $W(z)$ is the open loop transfer function,

$$W(z)=H(zI-F)^{-1}G+J. \tag{3.3}$$

Now let $\bar{K}$ be a gain such that the closed-loop poles of

$$W(z)(I+KW(z))^{-1}$$

are at $z_1, \cdots, z_n$. Then, generically, the poles $\phi(z_1), \cdots, \phi(z_n)$ of $V(z)(I+KV(z))^{-1}$ will be finite. Since

$$\operatorname{Re}[z]<0 \quad \text{if and only if} \quad |\phi(z)-\rho|<\varepsilon,$$

$\bar{K}$ stabilizes $W(z)$ with respect to $\operatorname{Re}[z]<0$ if, and only if, it stabilizes $V(z)$ with respect to the ε-disc centered about ρ. We claim that, consequently, a generic (F, G, H, J) is stabilizable with respect to $\operatorname{Re}[z]<0$ if, and only if, a generic $(\tilde{F}, \tilde{G}, \tilde{H}, \tilde{J})$ is stabilizable with respect to the ε-disc $B(\rho;\varepsilon)$. Assuming the claim, by our first observation the "direct part" J may be omitted, and the lemma is proved.[3]

To verify the claim, we first develop $W(z)$ in a Laurent series

$$W(z)=J+\sum_{i=1}^{\infty} L_i z^{-i}$$

and form the $n\times n$, $p\times m$ block Hankel matrix

$$h_w=[L_{i+j-1}].$$

Then $W(z)$ determines, and is determined by, a point in the set

$$\mathcal{H}^n_{m,p}=\{(J, L_1, \cdots, L_{2n}): \operatorname{rank} h_w=n\}.$$

$\mathcal{H}^n_{m,p}$ is, by definition, an open subset of an algebraic set of matrices. Moreover, $\mathcal{H}^n_{m,p}$ is the image of the rational map

$$\Pi: \mathcal{M}\subset \mathbb{R}^{n^2+n(m+p)+mp}\rightarrow \mathcal{H}^n_{m,p}$$

defined on the open dense set $\mathcal{M}$ of minimal systems by

$$\Pi(F, G, H, J)=(J, L_1, \cdots, L_{2n})$$

where of course

$$H(sI-F)^{-1}G+J=J+\sum_{i=1}^{\infty} L_i z^{-i}.$$

Therefore, $\mathcal{H}^n_{m,p}$ is irreducible, as the image of an irreducible algebraic set [21]. In this language, we have:

(i) ϕ induces, via (3.2), a rational map

$$\Phi: \mathcal{H}^n_{m,p}\rightarrow \mathcal{H}^n_{m,p}$$

with singularities on the algebraic set $\{h_W: W \text{ has a pole at } 1\}$, since $V(z)=\Phi(W(z))=W(\phi(z))$ is proper if, and only if, $W(1)$ is finite.

(ii) Image $\Phi=\mathcal{H}^n_{m,p}-\{h_V: V \text{ has a pole at } \varepsilon+\rho\}$ for similar reasons to those in (i).

Furthermore, since stability of minimal systems is an input-output property, if $\mathcal{D}$ is a self-conjugate subset of $\mathbb{C}$, then

(iii) The set

$$U=\{\sigma=(F, G, H, J): \sigma \text{ is stabilizable with respect to } \mathcal{D}\}$$

[3] Argument along these lines has been developed independently by J. C. Willems.

is open and dense in $\mathcal{M}$, if and only if,

$$\Pi(U) \subset \mathcal{H}^n_{m,p}$$

is open and dense in $\mathcal{H}^n_{m,p}$.

The claim then follows from (i), (ii) and (iii). Q.E.D.

Remark. A similar, perhaps well-known, result is that for fixed m, n, p stabilizability is generic if, and only if, for generic (F, G, H) there exists a gain K such that the closed-loop spectrum lies in $\mathrm{Re}[s] < \sigma$ or $\mathrm{Re}[s] > \sigma$, with $\sigma \in \mathbb{R}$ arbitrary.

The next proof relies on the following result which is stated in the notation of (1.2). For f, g polynomials, set:

$$U\{f_i\} = \{x \in \mathbb{R}^n : f_i(x) > 0, \forall i\},$$
$$V\{g_i\} = \{x \in \mathbb{R}^n : g_i(x) \geqq 0, \forall i\}. \tag{3.4}$$

A subset $Z \subset \mathbb{R}^n$ is called *semialgebraic* if it is a finite union of finite intersections of sets of the form (3.4). For example, the algebraic set

$$Z = \{x \in \mathbb{R}^n : g(x) = 0\}$$

is semialgebraic. A subset of the form $U\{f_i\}$ is called a basic open semialgebraic set, and those of the form $V\{g_i\}$ are called *basic closed semialgebraic sets.*

PROPOSITION 3.2. *If $Z \subset X \times Y$ is a semialgebraic set, then $p_1(Z) \subset X$ is a semialgebraic set. Thus, the existence of Y such that*

$$p_1(x_0, y) = x_0$$

can be checked by a finite number of rational operations in x_0.

This theorem is of course a version of the Tarski–Seidenberg theorem. It is worth noting that a recent improvement on this result has been made [8], [9], viz. if it is known that $p_1(Z)$ is Euclidean closed (or open), then $p_1(Z)$ is a finite union of basic closed (or open) semialgebraic sets. Of course, $p_1(Z)$ is not necessarily closed, even if Z is closed.

LEMMA 3.3. *If $mp \leqq n$, then the following statements are equivalent*:

(i) *m, n, p are such that the generic (F, G, H) is stabilizable*;

(ii) *m, n, p are such that for all real ρ and for the generic (F, G, H), there exists a gain K such that the closed loop characteristic polynomial is $(s-\rho)^n$.*

Proof. Statement (ii) obviously implies (i). For the converse, consider the function, for $\sigma = (F, G, H)$,

$$\chi_\sigma : \mathbb{R}^{mp} \to \mathbb{R}^n, \quad \text{defined via } \chi_\sigma(K) = (p_1, \cdots, p_n), \tag{3.5}$$

where

$$s^n + p_1 s^{n-1} + \cdots + p_m = \det(sI - F - GKH).$$

If statement (i) holds, then for each r there exists an open dense subset $U_r \subset \mathbb{R}^{n^2} \times \mathbb{R}^{nm} \times \mathbb{R}^{np} = \mathbb{R}^N$ such that for $(F, G, H) \in U_r$

$$(p_1, \cdots, p_n) \in \text{image}(\chi_\sigma)$$

where the roots of $s^n + p_1 s^{n-1} + \cdots + p_n$ lie in an $1/r$-disc centered about ρ. By the Baire category theorem,

$$U = \bigcap_{r=1}^{\infty} U_r$$

is a dense subset of $\mathbb{R}^N$ such that for $(F, G, H) \in U$,

$$(\bar{p}_1, \cdots, \bar{p}_n) \in \overline{\text{image}(\chi_\sigma)}$$

where

$$s_n + \bar{p}_1 s^{n-1} + \cdots + \bar{p}_n = (s-\rho)^n.$$

Now, according to [4, § 4, Thm.] provided $mp \leqq n$ there exists an open dense subset $W \subset \mathbb{R}^N$—the set of nondegenerate systems—such that image (χ_σ) is Euclidean closed for $(F, G, H) \in W$. Thus, if

$$(F, G, H) \in U = U' \cap W,$$

then

$$(\bar{p}_1, \cdots, \bar{p}_n) \in \text{image}(\chi_\sigma).$$

Now, any real gain K may be regarded as a point in $\mathbb{R}^{mp}$, and we may consider the real algebraic set

$$V^\rho = \{(F, G, H, K): \det(sI - F - GKH) = (s-\rho)^n\} \subset \mathbb{R}^N \times \mathbb{R}^{mp}. \tag{3.6}$$

By the Tarski–Seidenberg theorem (Prop. 3.2),

$$p_1(V^\rho) \subset \mathbb{R}^N,$$

the projection onto the first factor, is a semialgebraic set in $\mathbb{R}^N$; i.e., $p_1(V^\rho)$ is defined by a finite set of equations and inequations as in (3.4). Since

$$U \subset p_1(V^\rho) = \mathbb{R}^N$$

is dense, it follows that $p_1(V^\rho)$ may be defined by algebraic conditions (perhaps disjunctive)

$$f_1(F, G, H) > 0, \quad \cdots, \quad f_r(F, G, H) > 0,$$

from which it follows that $p_1(V^\rho)$ is open and dense. Since $(F, G, H) \in p_1(V^\rho)$ if, and only if, there exists a K such that the closed-loop characteristic polynomial is $(s-\rho)^n$, the lemma is proved. Q.E.D.

For the more precise assertion in part (ii) of Theorem 1, we need the following:

LEMMA 3.4. *For any* $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$, *the subset*

$$V_p = \{\sigma = (F, G, H) \in W: \chi_\sigma(K) = p \text{ for some } K\}$$

is closed in W.

Remark. The corresponding assertion for (F, G, H) minimal can be false. This is quite analogous to the fact that the set

$$\{x \in \mathbb{R}: \exists y \in \mathbb{R} \text{ such that } xy = 1\}$$

is not closed in $\mathbb{R}$, while the set

$$\{x \in \mathbb{R} - \{0\}: \exists y \in \mathbb{R} \text{ such that } xy = 1\}$$

is closed in the open dense subset $W = \mathbb{R} - \{0\} \subset \mathbb{R}$.

Proof. As in [4], we may think of $K \in \mathbb{R}^{mp}$ as a point in Grass $(p, m+p)$—the set of p-planes in $\mathbb{R}^{m+p}$—via the assignment

$$K \mapsto \text{graph}(K) = \{(y, Ky)\} \subset \mathbb{R}^p \oplus \mathbb{R}^m.$$

It is known (see e.g. [4] and references cited therein) that Grass $(p, m+p)$ may be regarded as a compact manifold of dimension mp. Moreover,

$$\text{Grass}\,(p, m+p) = \mathbb{R}^{mp} \cup \sigma(\infty)$$

where $\sigma(\infty)$ is the closed subset defined by

$$\sigma(\infty) = \{\Pi \in \text{Grass}\,(p, m+p) \colon \dim\,(\Pi \cap \mathbb{R}^m) \geqq 1\}.$$

That is, $\Pi \in \sigma(\infty)$ if, and only if, Π is not complementary to U. Thus, $\Pi \notin \sigma(\infty)$ if, and only if,

$$\Pi = \text{graph}\,(K), \quad \text{for some linear } K \colon \mathbb{R}^p \to \mathbb{R}^m.$$

On the other hand, one may regard the monic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_n$$

as a point $(p_1, \cdots, p_n) \in \mathbb{R}^n$ and therefore [4] as a point, via the homogeneous coordinates

$$[p_1, \cdots, p_n, 1] \in \mathbb{RP}^n,$$

in real projective n-space. Of course, $\mathbb{RP}^n = \text{Grass}\,(1, n+1)$ by definition. According to [4, Remarks, p. 103], for nondegenerate σ the map χ_σ extends continuously to a map

$$\chi_\sigma \colon \text{Grass}\,(p, m+p) \to \mathbb{RP}^n,$$

satisfying

$$\chi_\sigma(\Pi) = [p_1, \cdots, p_n, 0] \tag{3.7a}$$

if, and only if,

$$\Pi \in \sigma(\infty). \tag{3.7b}$$

Matters being so, consider the continuous function

$$\chi \colon W \times \text{Grass}\,(p, m+p) \to \mathbb{RP}^n$$

defined via

$$\chi(F, G, H, \Pi) = \chi(\sigma, \Pi) = \chi_\sigma(\Pi).$$

Therefore, if $\bar{p} = [1, 0, \cdots, 0, 1]$ corresponds to $\bar{p}(s) = s^n$,

$$Z = \chi^{-1}(\bar{p}) \subset W \times \text{Grass}\,(p, m+p)$$

is a closed subset. Since Grass $(p, m+p)$ is compact,

$$p_1(Z) \subset W$$

is closed and, by virtue of (3.6),

$$p_1(Z) = \{\sigma = (F, G, H) \colon \chi_\sigma(K) = \bar{p}, \text{ for some } K\} = V_{\bar{p}}. \qquad \text{Q.E.D.}$$

On the other hand, $U \cap W \subset V_{\bar{p}}$ is dense in W by the Baire category theorem, and therefore

$$V_{\bar{p}} = W,$$

from which (ii), and Theorem 1, follow. Q.E.D.

We now turn to a proof of Theorem 2. Clearly, it suffices to consider the case $mp \leqq n$; thus, the preceding lemmata and Theorem 1 are applicable.

Consider, then, the algebraic set of nilpotent $n \times n$ real matrices

$$\mathcal{N} = \{N: N^k = 0, \text{ for some } k\}$$

and the algebraic set $V = V^0$ obtained by setting $\rho = 0$ in (3.6). We define the polynomial mapping

$$\Phi: \mathcal{N} \times \mathbb{R}^{nm} \times \mathbb{R}^{np} \times \mathbb{R}^{mp} \to \mathbb{R}^{n^2} \times \mathbb{R}^{nm} \times \mathbb{R}^{np} \tag{3.8}$$

via

$$\Phi(N, G, H, K) = (N - GKH, G, H).$$

From Theorem 1, we have:

LEMMA 3.5. *If $mp \leqq n$ and if the generic system is stabilizable, then the image of Φ contains an open, dense set.*

Denote by $\mathcal{N}_{\mathbb{C}}$ the algebraic set of nilpotent $n \times n$ complex matrices. It is known (see e.g. [17], [20]) that $\mathcal{N}_{\mathbb{C}}$ is an irreducible algebraic set. Therefore there exists an open dense subset U of $\mathcal{N}_{\mathbb{C}}$ which is itself a complex manifold and therefore has a dimension. Indeed [17], [20],

$$\dim_{\mathbb{C}} (U) = n^2 - n.$$

The points of U are called simple, and one of the thorny points in real algebraic geometry [18] is that in general an irreducible real algebraic set $V_{\mathbb{R}}$ may contain none of the simple points of $V_{\mathbb{C}}$. This, for example, is the reason for the failure of the Hilbert Nullstellensatz over $\mathbb{R}$, and the best-known example of this phenomenon is

$$W_{\mathbb{R}} = \{(x, y): x^2 + y^2 = 0\}.$$

If $V_{\mathbb{R}}$ contains a simple point of $V_{\mathbb{C}}$, then for example $\dim_{\mathbb{R}} (V_{\mathbb{R}})$ is defined as above and

$$\dim_{\mathbb{R}} (V_{\mathbb{R}}) = \dim_{\mathbb{C}} (V_{\mathbb{C}}). \tag{3.9}$$

It is an elementary computation to check that the real matrix

$$N = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \cdot & & 0 \\ & & \cdot & \cdot & \\ 0 & & & \cdot & 1 \\ & & & & 0 \end{bmatrix}$$

is a simple point of $\mathcal{N}_{\mathbb{C}}$. Thus, $\dim_{\mathbb{R}} N$ satisfies (3.9). We will now give a self-contained proof of

LEMMA 3.6. $\dim_{\mathbb{C}} (\mathcal{N}) = n^2 - n$.

Proof. Since the matrix N consists of a single Jordan block, the dimension of the centralizer

$$Z(N) = \{T \in GL(n, \mathbb{C}): TN = NT\}$$

is n, according to the Frobenius dimension formula ([15, Vol. II, Thm. 19, p. 111]). Now consider the orbit of N under $GL(n, \mathbb{C})$:

$$\mathcal{O}(N) = \{TNT^{-1}: T \in GL(n, \mathbb{C})\} \simeq \frac{GL(n, \mathbb{C})}{Z(N)}.$$

In particular,

$$\dim_{\mathbb{C}} \mathcal{O}(N) = \dim GL(n, \mathbb{C}) - \dim Z(N) = n^2 - n.$$

We claim $\overline{\mathcal{O}(N)} = \mathcal{N}$, from which follows:

(i) $\mathcal{N}$ is irreducible, since $\mathcal{O}(N)$ is irreducible; and

(ii) $\dim_{\mathbb{C}} \mathcal{O}(N) = \dim_{\mathbb{C}} (\mathcal{N})$, by the closed orbit lemma [14] and (3.9).

Following [20], note that if N_i is any nilpotent Jordan canonical form, then clearly there is a 1-parameter diagonal subgroup $T_\lambda \in GL(n, \mathbb{C})$ such that

$$\lim_{\lambda \to \infty} T_\lambda N T_\lambda^{-1} = N_i.$$

Therefore, $\overline{\mathcal{O}(N)} = \mathcal{N}$. Q.E.D.

Now suppose that m, n and p are such that the generic system is stabilizable, and $mp \leqq n$. By Lemma 3.5 and [21, Thm. 7, p. 60] one has

$$\dim \mathcal{N}_n + n(m+p) + mp \geqq n^2 + n(m+p). \tag{3.10}$$

In the light of Lemma 3.6 and (3.10),

$$n^2 - n + mp \geqq n^2,$$

yielding

$$mp \geqq n.$$

In conclusion, if $mp \leqq n$ then $mp = n$ is necessary for generic stabilizability, whence Theorem 2. Q.E.D.

4. Proof of Theorem 3. In the proof of Theorem 1 (cf. Lemma 3.2), we made use of certain facts concerning $p \times m$ systems of degree n which also allow us to show, together with Theorem 1, that for $n = 4$, $m = p = 2$, generic stabilizability is not possible. Specifically:

(i) if $mp \leqq n$, then the class W of nondegenerate systems is open and dense in $\mathbb{R}^{n^2} \times \mathbb{R}^{mn} \times \mathbb{R}^{np}$; and

(ii) for any monic polynomial $p(s)$ of degree n, the set

$$V_p = \{(F, G, H) \in W : \det(sI - F - GKH) = p(s), \text{ for some } K\}$$

is closed in W (Lemma 3.4).

In light of Theorem 1, if $\bar{p}(s) = s^n$, then generic stabilizability implies that $V_{\bar{p}}$ is dense and closed in W, hence coincides with W. Therefore, to find one nondegenerate system for which $\bar{p}(s)$ is not assignable as a closed-loop polynomial is to prove that stabilizability is not generic.

We shall now give a "frequency domain" criterion [3] (which can be taken as a definition, compare [4]) for nondegeneracy. If $T(s)$ is the transfer function

$$T(s) = H(sI - F)^{-1} G \tag{4.1}$$

of the system, denote by $t_i(s)$ the ith column of the $(p+m) \times m$ matrix

$$\mathcal{T}(s) = \begin{bmatrix} T(s) \\ I \end{bmatrix}.$$

If $\phi(y, u)$ is a complex linear functional on $\mathbb{C}^p \oplus \mathbb{C}^m$, then we can form the scalar rational function

$$\phi(t_i(s)) \quad \text{for } i = 1, \cdots, m.$$

Now suppose $\Phi=\{\phi_1,\cdots,\phi_p\}$ is any linearly independent set of linear functionals on $(m+p)$-space, and form the determinant

$$\Phi(s)=\det[\phi_i(t_j(s))]. \tag{4.2}$$

(F, G, H) is said to be nondegenerate provided

$$\Phi(s)\not\equiv 0 \quad \text{in } s \tag{4.3}$$

for any choice of Φ.

Remarks 1. If (F, G, H) is scalar, then (F, G, H) is nondegenerate since (4.2)–(4.3) reduces, for $\phi(u, y)=au+by$, to $ag(s)+b\not\equiv 0$ in s.

2. The zeros of the set $\Phi=\{\phi_1,\cdots,\phi_m\}$ defines a p-plane in (u, y)-space which is the graph either of a linear function $u=Ky$, i.e. a finite constant gain, or of a linear relation between u and y, i.e. an infinite constant gain. The zeros of (4.2) are then, modulo pole-zero cancellation, the closed-loop poles at this gain, and (4.3) just asks that these zeros be finite in number, i.e. that the root-locus map χ be defined and continuous at this gain.

Example 4. Suppose $m=p=2$, $n=4$ and consider

$$G=\begin{bmatrix} I \\ 0\end{bmatrix}, \qquad H=[I \quad 0]$$

and

$$F=\begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & -1\\ 0 & -1 & 0 & 0\end{bmatrix}.$$

We claim (F, G, H) is nondegenerate; to this end we compute (clearing denominators)

$$\mathcal{T}(s)=\begin{bmatrix} s^3-1 & -s\\ s & s^3\\ s^4+s-1 & 0\\ 0 & s^4+s-1\end{bmatrix}$$

and consider 2 linear functionals

$$\phi_1(y, u)=a_1y_1+a_2y_2+a_3u_1+a_4u_2, \qquad \phi_2(u, y)=b_1y_1+b_2y_2+b_3u_1+b_4u_2.$$

Thus,

$$\Phi(s)=\det[\phi_i(t_j(s))]=\det\begin{bmatrix}\alpha_{11}(s) & \alpha_{12}(s)\\ \alpha_{21}(s) & \alpha_{22}(s)\end{bmatrix} \tag{4.4}$$

where

$$\begin{aligned}
\alpha_{11}(s)&=a_3s^4+a_1s^3+(a_2+a_3)s-a_3-a_1,\\
\alpha_{12}(s)&=a_4s^4+a_2s^3+(a_4-a_1)s+a_4,\\
\alpha_{21}(s)&=b_3s^4+b_1s^3+(b_2-b_3)s+b_3-b_1,\\
\alpha_{22}(s)&=b_4s^4+b_2s^3+(b_4-b_1)s+b_4.
\end{aligned} \tag{4.5}$$

Now, (4.4) vanishes just in case there exists c_s—a priori depending on s—such that

$$c_s\alpha_{11}(s)=\alpha_{21}(s), \qquad c_s\alpha_{12}(s)=\alpha_{22}(s) \tag{4.6}$$

for all but finitely many $s \in \mathbb{C}$. Comparing coefficients shows that c_s is constant for all but finitely many, and hence all, s and therefore an inspection of (4.5)–(4.6) shows that

$$c\phi_1 = \phi_2,$$

contradicting linear independence of the functionals ϕ_i.

Recall that in the proof of Lemma 3.3, the fact that image (χ) is closed for all nondegenerate (F, G, H) was used rather crucially. If

$$K = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

it is readily verified that

$$-\det(sI - F - GKH) = s^4 + (-x_1 - x_4)s^3 + (x_1x_4 - x_2x_3)s^2 + (1 - x_2 + x_3)s + (1 + x_1). \tag{4.7}$$

By the quadratic formula, it is easily verified that image (χ) is a closed semialgebraic set. Furthermore, if (4.7) is to be s^4, we require

$$x_1 + x_4 = x_1x_4 - x_2x_3 = 1 - x_2 + x_3 = 1 + x_1 = 0, \tag{4.7'}$$

whence

$$x_2x_3 = -1, \qquad x_2 = 1 + x_3,$$

whence

$$x_3^2 + x_3 + 1 = 0. \tag{4.8}$$

This equation (4.8) cannot be satisfied by any real x_3, i.e. there is no real gain producing closed-loop poles at $s = 0$. Since (F, G, H) is nondegenerate, our previous remarks imply Theorem 3, thereby verifying Molander's conclusion.

5. Proof of Theorem 4. In addition to the Tarski–Seidenberg theorem (Prop. 3.2), we shall also need a somewhat different result from decision algebra, which deals with the question of describing the set of x_0 for which $p_1^{-1}(x_0) \cap Z = \{x_0\} \times Y$, i.e. for which $(x_0, y) \in Z$ for all y. In the course of deriving this result we also will state the Tarski–Seidenberg theorem in what is perhaps a more familiar form [15, Vol. III], [22].

Notational conventions are as follows: x, y, z denote collections of indeterminates, with each of x, y, z considered to be shorthand for a number of indeterminates $x_1, \cdots, x_n$ etc. Particular real values taken by these quantities will be denoted by $\hat{x}, \hat{y}, \hat{z}$; p, q, r, s, perhaps with subscripts will denote polynomials in x, y, z with real coefficients. We shall regard $p(x, y) = 0$ or $q(x, y) \geqq 0$ as examples of equations or inequations, (i.e. descriptions of problems for which solutions are sought, should they exist), and we shall regard $p(\hat{x}, \hat{y}) = 0$ or $q(\hat{x}, \hat{y}) \geqq 0$ as examples of equalities or inequalities (i.e. statements of fact that can be verified by arithmetic, and which show that $\hat{x}, \hat{y}$ are solutions of $p(x, y) = 0$ or $q(x, y) \geqq 0$).

We shall reserve script letters $\mathcal{S}$, $\mathcal{T}$, etc. to denote collections of a finite number of equations and inequations or inequalities and inequalities of the following type. $\mathcal{S}(x)$ is an abbreviation for:

either $\{p_{i1}(x) = 0$ and $q_{j1}(x) > 0$ and $r_{k1}(x) \neq 0$ and $s_{l1}(x) \geqq 0\}$
or $\{p_{i2}(x) = 0$ and $q_{j2}(x) > 0$ and $r_{k2}(x) \neq 0$ and $s_{l2}(x) \geqq 0\}$
or
⋮
or $\{p_{it}(x) = 0$ and $q_{jt}(x) > 0$ and $r_{kt}(x) \neq 0$ and $s_{lt}(x) \geqq 0\}$,

where it is understood that $p_{i\alpha}(x)=0$ is shorthand for $p_{1\alpha}(x)=0$ and $P_{2\alpha}(x)=0$ and $\cdots$ and $p_{i_\alpha\alpha}(x)=0$, and similarly for $q_{j\alpha}$ etc. Naturally, $\mathcal{S}(\hat{x})$ is an abbreviation for the associated set of equalities and inequalities. We can talk of the problem of solving $\mathcal{S}(x)$ and of $\mathcal{S}(\hat{x})$ holding, or of $\hat{x}$ being a solution of $\mathcal{S}(x)$.

The above type of $\mathcal{S}(x)$ is more or less standard in decision algebra. However, we shall sometimes use a simple modification. Each $s_{\alpha\beta}\geqq 0$ is a disjunction: $s_{\alpha\beta}>0$ or $s_{\alpha\beta}=0$. This means that any $\mathcal{S}(x)$ and thus any $\mathcal{S}(\hat{x})$ can be rewritten to exclude inequations or inequalities of the $\geqq$ type.

LEMMA 5.1. *The statement $\mathcal{S}(\hat{x})$ does not hold is equivalent to a statement $\bar{\mathcal{S}}(\hat{x})$ holds where $\bar{\mathcal{S}}(x)$, termed the negator of $\mathcal{S}$, is itself a collection of equations and inequations of the standard form.*

Proof. "$\mathcal{S}(\hat{x})$ holds" is a disjunction ("or" statement) of conjunctions ("and" statements) of formulas of the type $p(\hat{x})=0$, $q(\hat{x})>0$, $r(\hat{x})\neq 0$ and $s(\hat{x})\geqq 0$. Hence "$\mathcal{S}(\hat{x})$ does not hold" is a conjunction of disjunctions of negations of these formulas, i.e. of $p(\hat{x})\neq 0$, $-q(\hat{x})\geqq 0$, $r(\hat{x})=0$ and $-s(\hat{x})>0$. Any conjunction of disjunctions can be rearranged as a disjunction of conjunctions, and in this way, $\bar{\mathcal{S}}(x)$ is defined.

Obviously, $\bar{\bar{\mathcal{S}}}=\mathcal{S}$.

Next, we recall the main result of decision algebra, the Tarski–Seidenberg theorem. We break it into two parts.

PROPOSITION 5.2. (A) *Consider an equation/inequation set $\mathcal{S}(x, y)$. Then one can determine by a finite number of rational calculations a second such set $\mathcal{T}(y)$ such that $\mathcal{T}(\hat{y})$ holds if and only if there exists at least one $\hat{x}$ such that $\mathcal{S}(\hat{x}, \hat{y})$ holds.*

(B) *The solvability of any equation/inequation set $\mathcal{T}(y)$ is determinable by a finite number of rational calculations.*

We remark that the set $\mathcal{T}(y)$ in part A may be empty: this would imply that there are no pairs $\hat{x}$, $\hat{y}$ for which $\mathcal{S}(\hat{x}, \hat{y})$ holds.

PROPOSITION 5.3. *Consider an equation/inequation set $\mathcal{S}(x, y)$. Then the set of values $\hat{y}$ of y such that for all $\hat{x}$, $\mathcal{S}(\hat{x}, \hat{y})$ holds, is definable by an equation/inequation set $\mathcal{T}(y)$.*

Proof. Let $\bar{\mathcal{S}}(x, y)$ be the negator of $\mathcal{S}(x, y)$, existing by Lemma 5.1. By Proposition 5.2A, we can find $\bar{\mathcal{T}}(y)$ such that $\bar{\mathcal{T}}(\hat{y})$ holds if and only if there exists at least one $\hat{x}$ such that $\bar{\mathcal{S}}(\hat{x}, \hat{y})$ holds. Let $\mathcal{T}$ be the negator of $\bar{\mathcal{T}}$. Then $\mathcal{T}(\hat{y})$ holds if and only if there exists no $\hat{x}$ such that $\bar{\mathcal{S}}(\hat{x}, \hat{y})$ holds, i.e. if and only if for all $\hat{x}$, $\mathcal{S}(\hat{x}, \hat{y})$ holds.

The following algorithm, in conjunction with Propositions 5.2 and 5.3, gives a proof of Theorem 4. We find it convenient to break this into two parts.

Part I. Find a cube containing a stabilizing gain K.

I.1. Choose $N>0$, and consider the semialgebraic set

$$Z\subset \mathbb{R}^{mp-1}\times\mathbb{R}$$

defined by $Z=Z_1\cap Z_2$, with

$$Z_1=\{K: \det(sI-F-GKH) \text{ is Hurwitz}\},$$

$$Z_2=\{K: \sum (k_{ij})^2<N\}.$$

From the Routh–Hurwitz criterion, it follows that Z_1 is a (basic open) semialgebraic set and it is then clear that Z is semialgebraic. Using Proposition 5.2 inductively, we can decide by rational operations whether there exists a gain $K\in Z$. If $Z\neq\varnothing$, go to Step II.1. Otherwise, go to Step I.2.

I.2. Replace N by $2N$ and go to Step I.1. Since a stabilizing gain exists by hypothesis, we will eventually move to Part II.

Part II. Find a cube contained in the set of stabilizing gains K.
II.1. We suppose there is a stabilizing gain K in the cube $\|K\|<N$. Using Proposition 5.3 inductively, we can decide by rational operations whether all such K are stabilizing. If so, choose any K such that $\|K\|<N$. If not, go to Step II.2.
II.2. Divide the cube into 2^{mp} cubes with sides of length N. Return to Step I.1 with this list of cubes.

This algorithm will stop at some stage, since the set of stabilizing gains is open and therefore contains a cube of sufficiently small size. Q.E.D.

Example 5. One might ask whether one can bound the number of steps in this program simply in terms of m, n, p. The answer is no, as we now illustrate. Consider the open loop system with transfer function

$$w(s)=\frac{1}{s^3+as^2+bs}$$

where $a, b>0$. For negative feedback with gain k, the closed loop characteristic polynomial is s^3+as^2+bs+k and therefore is Hurwitz if, and only if, $k\in(0, ab)$. It follows that the size of a cube (here, an interval) contained in the open set of stabilizing gains can be made arbitrarily small by suitable choice of ab. In turn, the number of steps in Part II of the algorithm can be made arbitrarily large, though for fixed (a, b) it is of course finite.

6. Proof of Theorems 5 and 6. Since we have already demonstrated the existence of linear formulae for the appropriate values of m, n, p, it is enough to show that these are the only values for which such formulae can exist. Moreover, it suffices to prove this last assertion over $\mathbb{C}=R(\sqrt{-1})$. Consider the closed loop characteristic coefficient map χ, defined in (3.4), extended to gains with complex coefficients

$$\chi_{\mathbb{C}}: \mathbb{C}^{mp}\to\mathbb{C}^n \tag{6.1}$$

where (F, G, H) is understood to be a generic, but fixed, system with $n=mp$. We first analyze the question as to whether there exists a formula for $(k_{ij})\in\chi^{-1}(p)$ which is rational in the coordinates of $p=(p_l)\in\mathbb{C}^n$. Thus, we consider the field K_1 of all rational expressions (or functions) in the p_l, and the field K_2 of all rational functions in the (k_{ij}):

$$K_1=\mathbb{C}(p_l),\qquad K_2=\mathbb{C}(k_{ij}). \tag{6.2}$$

Since $\chi_{\mathbb{C}}$ is polynomial, if $f\in K_1$ then $f\circ\chi_{\mathbb{C}}\in K_2$. For generic (F, G, H), image $\chi_{\mathbb{C}}$ contains an open set [13] so that

$$f\circ\chi_{\mathbb{C}}=0\Rightarrow f=0. \tag{6.3}$$

By virtue of (6.3), we can think of K_1 as a subfield of K_2, i.e.

$$K_1\simeq\chi_{\mathbb{C}}^*K_1\subset K_2 \tag{6.4}$$

where $\chi_{\mathbb{C}}^*f=f\circ\chi_{\mathbb{C}}$, and an easy dimension argument shows that (6.4) is a finite field extension. That is, K_2, as a vector space over the field of scalars K_1, is finite dimensional. For example, to say rational formulae for $(k_{ij})\in\chi_{\mathbb{C}}^{-1}(p_l)$ exist is to say the dimension of this vector space

$$\delta=[K_2:K_1]=\dim_{K_1}(K_2) \tag{6.5}$$

is equal to 1, i.e. $K_1=K_2$. We shall now give a formula for δ, in terms of m, p. In [4] it was shown that $\chi_{\mathbb{C}}$ is proper and it follows from the proof in [4] that

$$R_1\simeq\chi_{\mathbb{C}}^*R_1\subset R_2$$

is an integral ring extension, where

$$R_1 = \mathbb{C}[p_l], \qquad R_2 = \mathbb{C}[k_{ij}].$$

In this case (since the field $\mathbb{C}$ has characteristic zero), δ is given by the number d of solutions, counted with multiplicity, to the equation [18, pp. 116–117]

$$\chi_{\mathbb{C}}(K) = p.$$

On the other hand, d has been computed using methods of the Schubert calculus in [3] to be

$$d = \frac{1! \cdots (p-1)!(mp)!}{m! \cdots (m+p-1)!}. \tag{6.6}$$

Thus, Theorem 5 follows from the following elementary observation:

LEMMA 6.1. *In* (6.6), $d = 1 \Leftrightarrow \min(m, p) = 1$.

As for Theorem 6, from the explicit form of the solution to the pole-placement equations, derived via elimination methods by Willems–Hesselink [23], it is clear that (over $\mathbb{R}$ or $\mathbb{C}$) quadratic formulae and rational expressions are sufficient to express K as a function of $(p_1, \cdots, p_n)$ for generic (F, G, H) when $m = p = 2$, and $n = 4$. We shall now prove that, except for the linear cases $\min(m, p) = 1$, this is the only case when formulae—involving square roots and rational operations—for K in terms of $(p_1, \cdots, p_n)$ exist.

To this end, we consider a Galois extension

$$K_1 \subset K, \tag{6.7}$$

that is, a minimal normal extension of $K_1 = \mathbb{C}(p)$ which contains all of the roots to the equation

$$\chi_{\mathbb{C}}(K) = (p). \tag{6.8}$$

If a solution expressible by square roots and rational operations alone exists, then

$$\delta' = [K : K_1]$$

is a power of 2 [2]. On the other hand, by Artin's theorem of the primitive element [2], we may regard $K_2 \subset K$ and therefore

$$\delta = [K_2 : K_1] \quad \text{divides } [K : K_1],$$

from which it follows that

$$\delta = d_{m,p} = 2^r, \quad \text{for some } r.$$

Theorem 6 therefore follows from the following result:

LEMMA 6.2. *If* $\min(m, p) \geqq 2$ *and* $m + p \geqq 5$, *then* $d_{m,p}$ *is divisible by an odd prime.*

Remark. The proof we present here is based on an application of the strong form of Bertrand's postulate [11, p. 373] shown to us by W. H. Gustafson.

Proof. By the strong form of Bertrand's postulate, there is a prime q satisfying

$$m + p - 1 < q < 2(m + p) - 4, \tag{6.9}$$

under the hypothesis $m + p \geqq 5$. Clearly, q does not divide the denominator of $d_{m,p}$. On the other hand, if $\min(m, p) \geqq 2$, then

$$mp > q,$$

so that q divides the numerator of $d_{m,p}$. Hence, $q | d_{m,p}$. Q.E.D.

Appendix. "In principle" answers to Questions 1 and 2 by decision algebra. In [1], indications of the applicability of decision algebra to problems of systems theory were given. In particular, it was shown that one can determine, at least in principle, by rational operations whether a given system (F, G, H) can be stabilized. We shall extend these results to show that one can answer Questions 1 and 2 by rational operations using decision theoretic techniques, but we emphasize that such results are very qualitative. In fact, a "worst-case" analysis [7] shows that any decision procedure takes at least 2^{k^n} steps, where $k>0$ is a constant and n is the length of the input formula.

However, in the absence of any other technique which allows one, for example, even in principle to distinguish between Questions 1 and 2, we thought it worthwhile to point out that this is a question which can be answered by the Tarski–Seidenberg theory. An interesting special case is whether or not we can place poles for generic 2×2 systems with McMillan degree 8. One does know that there exist 14 complex solutions to the pole-placement equations, but at present one does not know whether any of these are real.

The new ingredient here is the consideration of the generic system (F, G, H) rather than a particular choice of system (F_0, G_0, H_0), and we shall need to present some further results from decision algebra. The notation is as in § 5.

LEMMA A.1. *Consider an equation/inequation set $\mathcal{S}(x, y, z)$. Then there exists a set $\mathcal{T}(y)$ such that $\mathcal{T}(\hat{y})$ holds if and only if for all $\hat{z}$, there exists $\hat{x}$ depending on $\hat{y}$, $\hat{z}$ with $\mathcal{S}(\hat{x}, \hat{y}, \hat{z})$ holding.*

Proof. By Proposition 5.1′, there exists $\mathcal{R}(y, z)$ such that $\mathcal{R}(\hat{y}, \hat{z})$ holds if and only if $\mathcal{S}(x, \hat{y}, \hat{z})$ is solvable, i.e. if and only if there exists at least one $\hat{x}$, depending on $\hat{y}$ and $\hat{z}$, such that $\mathcal{S}(\hat{x}, \hat{y}, \hat{z})$ holds. By Proposition 5.3, there exists $\mathcal{T}(y)$ such that $\mathcal{T}(\hat{y})$ holds if and only if $\mathcal{R}(\hat{y}, \hat{z})$ holds for all $\hat{z}$. Then clearly, $\mathcal{T}(\hat{y})$ holds if and only if, for all $\hat{z}$, there exists $\hat{x}$ such that $\mathcal{S}(\hat{x}, \hat{y}, \hat{z})$ holds.

In Proposition 5.3 and Lemma A.1, the set $\mathcal{T}(y)$ may be empty. The following lemma replaces the "all $\hat{x}$" in Proposition 5.3 by "almost all", and in this sense may enable one to get a practical result when the $\mathcal{T}(y)$ of this proposition is empty.

LEMMA A.2. *Consider an equation/inequation set $\mathcal{S}(x, y)$. Then there exists an equation/inequation set $\mathcal{T}(y)$ such that $\mathcal{T}(\hat{y})$ holds if and only if $\mathcal{S}(\hat{x}, \hat{y})$ holds for all $\hat{x}$ save a set contained in a proper variety depending on $\hat{y}$.*

Proof. Given a polynomial $p(x, y)$, it is clear that there exists a possibly empty $\mathcal{P}(y)$ such that $\mathcal{P}(\hat{y})$ holds if and only if $p(x, \hat{y})$ is the zero polynomial, i.e. $p(\hat{x}, \hat{y})=0$ for all $\hat{x}$. Further, if $p(\hat{x}, \hat{y})=0$ for all $\hat{x}$ save those lying in a proper variety, $p(\hat{x}, \hat{y})=0$ for all $\hat{x}$.

Given a polynomial $r(x, y)$, it is clear that there exists $\mathcal{R}(y)$ such that $\mathcal{R}(\hat{y})$ holds if and only if $r(x, \hat{y})\neq 0$ is solved by all x save those on a proper variety depending on $\hat{y}$.

Given a polynomial $s(x, y)$, it is clear that there exists $\bar{\mathcal{S}}(y)$ such that $\bar{\mathcal{S}}(y)$ holds if and only if $s(\hat{x}, \hat{y})<0$ for some $\hat{x}$. Hence $\mathcal{S}(\hat{y})$ holds if and only if $s(\hat{x}, \hat{y})\geqq 0$ for all $\hat{x}$. Further, if $s(\hat{x}, \hat{y})\geqq 0$ for all $\hat{x}$ save those in a proper variety, $s(\hat{x}, \hat{y})\geqq 0$ for all $\hat{x}$.

Given a polynomial $q(x, y)$, it is clear that there exists $\mathcal{Q}_1(\hat{y})$ such that $q(\hat{x}, \hat{y})\geqq 0$ for all $\hat{x}$ and $\mathcal{Q}_2(\hat{y})$ such that $q(\hat{x}, \hat{y})\neq 0$ for all $\hat{x}$ save those in a proper variety. Let $\mathcal{Q}(y)$ denote the conjunction of $\mathcal{Q}_1(y)$ and $\mathcal{Q}_2(y)$. Then $\mathcal{Q}(\hat{y})$ holds if and only if $q(\hat{x}, \hat{y})>0$ for all $\hat{x}$ save those in a proper variety depending on $\hat{y}$.

Suppose now that $\mathcal{S}(x, y)$ is the disjunction of equation/inequation sets $\mathcal{S}_i(x, y)$ where each $\mathcal{S}_i(x, y)$ is a conjunction of

$$p_{\alpha i}(x, y)=0, \quad q_{\beta i}(x, y)>0, \quad r_{\gamma i}(x, y)\neq 0, \quad s_{\delta i}(x, y)\geqq 0.$$

By the discussion above, it is clear that there exists $\mathcal{T}_i(y)$ such that $\mathcal{T}_i(\hat{y})$ holds if and only if $\mathcal{S}_i(\hat{x}, \hat{y})$ holds for all $\hat{x}$ save those in a proper variety depending on $\hat{y}$. $\mathcal{T}(y)$ is obtained as the disjunction of the $\mathcal{T}_i(y)$.

Now consider the system (1.1), subject to output feedback $u = Ky$. The coefficients of the closed-loop characteristic polynomial, as a function of K, give rise to the polynomial mapping (3.4)

$$\chi : \mathbb{R}^{mp} \to \mathbb{R}^n,$$

and we wrote $\chi_{(F,G,H)}$ to emphasize the dependence on the open loop system (1.1). Then, Question 1 asks whether $\chi_{(F,G,H)}$ is surjective for the generic (F, G, H) and we claim that this question can be answered within the scope of decision algebra. To this end, let $X = \mathbb{R}^{mp}$, $Y = \mathbb{R}^{n^2+nm+np}$ and $Z = \mathbb{R}^n$, so that $(K, (F, G, H), (p_i)) \in X \times Y \times Z$, and consider the algebraic subset $W \subset X \times Y \times Z$ defined by the equations

$$\text{(A.1)} \qquad \mathcal{S}(x, y, z): \chi_{(F,G,H)}(K) = (p_i).$$

By Lemma A.1, there exists an equation/inequation set $\mathcal{T}$ in $y = \{F, G, H\}$ such that $\mathcal{T}(\hat{y})$ holds if and only if for all $\hat{z}$, i.e. for all p_i, there exists $\hat{x}$, i.e. a value of K, such that $\mathcal{S}(\hat{x}, \hat{y}, \hat{z})$ holds, i.e. such that (A.1) holds.

Let $\bar{\mathcal{T}}(y)$ denote the negator of T, and write $\bar{T}(y)$ as a disjuntion of conjunctions $\bar{\mathcal{T}}_i$. As observed in § 4, we can assume without loss of generality that each $\bar{T}_i$ contains equations $p_{\alpha i}(y) = 0$, and inequations $q_{\beta i}(y) > 0$ and $r_{\gamma i}(y) \neq 0$, *without* inequations of the type $s_{\delta i}(y) \geqq 0$. We can determine (see Prop. 5.2B) whether any $\bar{\mathcal{T}}_i$ defines an empty set of solutions; if so, we discard it.

Now with $\bar{\mathcal{T}}_i$ of the form just noted, and with each possessing a solution, we can readily answer Question 1.

If $\bar{\mathcal{T}}_i(\hat{F}, \hat{G}, \hat{H})$ holds for any i, pole positionability for all α_i via choice of K is not possible, and conversely. It follows that if each $\bar{\mathcal{T}}_i$ includes one or more equalities, then the set of $\hat{F}$, $\hat{G}$, $\hat{H}$ for which pole positionability is not possible lies within a proper variety, and that for almost all $\hat{F}$, $\hat{G}$, $\hat{H}$, pole positionability for all p_i can be achieved.

On the other hand, if $\bar{\mathcal{T}}_i$ contains no equalities then it is clear that there exists a neighborhood of any one solution of $\bar{\mathcal{T}}_i(y)$ which consists entirely of solutions. (The fact that $\bar{\mathcal{T}}_i$ contains no inequations of the type $s_{\delta i}(y) \geqq 0$ is crucial.) In this case, it cannot be true that for almost all $\hat{F}$, $\hat{G}$, $\hat{H}$, pole positionability can be achieved for all p_i.

This analysis of the $\bar{\mathcal{T}}_i$ answers Question 1.

Now one can also ask whether image (χ) is almost all of $\mathbb{R}^n$, for almost all (F, G, H). Let is identify K with x and F, G, H and the p_i with y. Equations (A.1) yield a collection $\mathcal{S}(x, y)$ of polynomial equations. By Proposition 5.2A, there exists $\mathcal{T}(y) = \mathcal{T}(F, G, H, p_i)$ such that $\mathcal{T}(\hat{y})$ holds if and only if $\mathcal{S}(x, \hat{y})$ is solvable. Using arguments like those above, it is easy to check whether or not the set of $\hat{y}$ for which $\bar{\mathcal{T}}(\hat{y})$ is true is contained in a proper variety. If it is, then and only then will it be true that for almost all F, G, H, the map χ is almost onto R^n.

We shall now turn to an analysis of Question 2.

If the closed-loop characteristic polynomial has all roots in the half plane $\text{Re}[s] < 0$, certain polynomial inequalities in the p_i obtainable from the Hurwitz determinants (see [4]) must hold, and conversely. Accordingly, we have

$$\text{(A.2)} \qquad \begin{aligned} &p_i(F, G, H, K) = p_i, && i = 1, \cdots, n, \\ &q_j(p_i) > 0, && j = 1, \cdots, n. \end{aligned}$$

Identify K and p with x and F, G, H, with y. Regard (A.2) as an equation/inequation set $\mathscr{S}(x, y)$. By Tarski–Seidenberg–A, there exists $\mathscr{T}(y)$ such that $\mathscr{T}(\hat{y}) = \mathscr{T}(\hat{F}, \hat{G}, \hat{H})$ holds if and only if (A.2) can be satisfied by some K, p_i. If the set of $\hat{y}$ such that $\bar{\mathscr{T}}(\hat{y})$ holds is contained in a proper variety, then and only then Question 2 has an affirmative answer. The discussion of Question 1 described how one could check whether the set of $\hat{y}$, such that $\bar{\mathscr{T}}(\hat{y})$ holds, is or is not contained in a proper variety.

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