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ALGEBRAIC DESCRIPTION OF BOUNDED REAL MATRIXES*

A description is given in algebraic terms of the constraints imposed on a square matrix of rational functions when the matrix is bounded real.

Bounded real matrixes occur in more than one area of system theory, where, by a bounded real matrix, we mean an $n \times n$ matrix $S(s)$ of functions of a complex variable satisfying the following three conditions:[1]

(*a*) All singularities of elements of $S(s)$ lie in the region $\mathrm{Re}\, s < 0$
(*b*) $S^*(s) = S(s^*)$ in the region $\mathrm{Re}\, s > 0$
(*c*) $I - S'(-s)\, S(s)$ is nonnegative definite Hermitian in the region $\mathrm{Re}\, s > 0$.

Examples of bounded real matrixes are provided by the scattering matrix of a passive network,[1] by the inverse of the return difference matrix of an optimal linear control system[2] and, as pointed out in Reference 3, by matrixes occurring in nuclear-particle scattering and electromagnetic-wave scattering. Of particular interest are rational bounded real matrixes (each element being rational in s), which arise where the system with which the matrix is associated is finite-dimensional. In the case of a network, this means that the network is composed of a finite interconnection of resistors, capacitors, inductors, transformers and possibly gyrators.

The family of bounded real rational matrixes is closely related to the family of positive real rational matrixes; recently,[4] a description of the positive real constraint was given in algebraic terms, and this description was applied to solving problems of spectral factorisation, nonlinear-control-system stability, optimal control and passive-network synthesis.[4,5] Consequently, there seems to be justification for presenting a description of the bounded real property in algebraic terms. A full proof and an application of the result to optimal-control theory will be presented at a later date.

The algebraic description is in terms of the matrixes F, G, H, J of a minimal realisation of $S(s)$;[6] i.e. the quadruple $\{F, G, H, J\}$ is such that

$$S(s) = J + H'(sI - F)^{-1}G \qquad (1)$$

and F has minimal dimension [equal to the *degree* of $S(s)$]. Note that explicit procedures are available for constructing minimal realisations from $S(s)$ (see, for example, Reference 5), while $S(s)$ can readily be recovered from a minimal realisation.

The following theorem then characterises bounded real rational matrixes.

Theorem 1

Let $S(s)$ be an m matrix of rational functions with $S(\infty) < \infty$, and let $\{F, G, H, J\}$ be a minimal realisation of $S(s)$. Suppose that the eigenvalues of F all have negative real parts. $S(s)$ is bounded real if, and only if, there exist matrixes $P = P' > 0$ (i.e. P is symmetric and positive definite), $\hat{H}$ and $\hat{J}$; so that

$$PF + F'P = -HH' - \hat{H}\hat{H}' \qquad (2a)$$

$$PG = HJ + \hat{H}\hat{J} \qquad (2b)$$

$$I - J'J = \hat{J}'\hat{J} \qquad (2c)$$

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The significance of H and J is partly described in the following corollary.

Corollary 1

With the same hypothesis as theorem 1, suppose that $S(s)$ is bounded real. With H, J as in eqns. 2, a solution S_1 of the equations

$$I - S'(-s)S(s) = S_1'(-s)S_1(s) \qquad (3)$$

is given by

$$S_1(s) = J + H'(sI - F)^{-1}G \qquad (4)$$

The computation of P, $\hat{H}$ and $\hat{J}$ can proceed via similar techniques to those used in the associated algebraic description of positive real matrixes. The matrix P is the simplest to compute initially, and from it, $\hat{H}$ and $\hat{J}$ can easily be found; while a full discussion will be given elsewhere, we note here corollary 2.

Corollary 2

With the same hypothesis as theorem 1, suppose that $I - J'J$ is nonsingular. $\hat{J}$ and $\hat{H}$ can be eliminated from eqn. 2, to give an equation quadratic in P which may be solved via the procedures outlined in References 4 and 7.

In retrospect, it is not surprising that an algebraic description of the bounded real (or positive real) property should exist. A rational $S(s)$ can always be described by a finite set of parameters, e.g. the poles of the elements and the associated residues, or the elements in the matrixes of a minimal realisation. Consequently, restrictions such as

$$I - S'(-s)S(s) \geqslant 0 \text{ in } \mathrm{Re}\, s > 0 \qquad (5)$$

are highly redundant in the sense that eqn. 5 is an infinite set of conditions (any one obtained by selecting a value of s with positive real part), while $S(s)$ can be described by a finite set of numbers. The replacement of the analytical condition (eqn. 5) by the algebraic conditions of eqn. 2 is thus a reflection of the fact that a finite set of conditions on a finite set of parameters defining a rational $S(s)$ are necessary and sufficient for $S(s)$ to be bounded real.

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