Markov Parameter Characterization of the Strict Positive Real Problem

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Abstract — Suppose that a rational function \( Z(s) \) is defined by a Laurent series, the coefficients of which are known. Several criteria are given in terms of these coefficients (the Markov parameters of \( Z(s) \)) to ensure that \( \text{Re} Z(j\omega) > 0 \) for all real \( \omega \). The criteria are defined by using a Cauchy index formulation of the ratio of two rational functions, and they are of three types— involving a Routh-like table with first two rows initialized using the coefficients, and Hurwitz and Bezout matrices with entries which are the coefficients themselves, or integral expressions in the coefficients. The matrix positive real property is also investigated.

I. INTRODUCTION

Let \( Z(s) \) be a real rational function of \( s \). As is well known, \( Z(s) \) is termed positive real [1], if (1) all poles of \( Z(s) \) either lie in \( \text{Re}[s] < 0 \), or lie on \( \text{Re}[s] = 0 \) and are simple with positive residues, (2) \( \text{Re} Z(j\omega) > 0 \) for all real \( \omega \), with \( j\omega \) not a pole of \( Z(s) \).

In this paper, we examine the question of how the positive real nature of \( Z(s) \) can be characterized in terms of the Markov parameters \( [Z] \) of \( Z(s) \). These are the constants \( \gamma_i \) in the expansion

\[
Z(s) = \gamma_{-1}s + \gamma_0 + \gamma_1s^{-1} + \gamma_2s^{-2} + \cdots \quad (1.1)
\]

(without loss of generality, we can assume \( \gamma_{-1} = 0 \). For if not, \( Z(s) \) is positive real if and only if \( Z(s) - \gamma_{-1}s \) is positive real and \( \gamma_{-1} > 0 \), [1]).

Our main interest is in rational \( Z(s) \). For such \( Z(s) \), we recall that one can express the positive real property in several ways. For example, suppose that one knows \( p(\cdot) \) and \( q(\cdot) \) such that \( Z(s) = q(s)/p(s) \); one can first identify the greatest common factor of \( p(s) \) and \( p(-s) \). Call it \( p_1(s) \) and let \( p(s) = p_1(s)p_2(s) \). Then find \( q_1(s), q_2(s) \) such that \( Z = (q_1/p_1) + (q_2/p_2) \). One then checks that the poles and zeros of \( q_1/p_1 \) interlace on the \( jo\)-axis (various tests are available), that \( p_2(s) \) has all zeros in the left half plane and that \( p_2(-s)q_2(s) + p_2(s)q_2(-s) \) is nonnegative for all \( s = j\omega, \omega \) real; again various tests are available [3]. Alternatively, one can characterize the positive reality via a minimal state-variable realization of \( Z(s) \) [4], although it is somewhat difficult to test this condition.

The various characterizations and tests have a variety of uses— e.g., for developing further theory, or as a basis for checking the positive real property. The test advanced here may on occasion be useful for checking the positive real property, depending how \( Z(s) \) is characterized in the first instance.

Recent developments in the theory of Hankel norm approximations suggest that the descriptions available for \( Z(s) \) may well be via the Markov parameters with a simple integer, the order of \( Z(s) \), rather than via numerator and denominator polynomials of \( Z(s) \).

Of course, knowing the \( \gamma_i \) and the degree of the denominator of a rational \( Z(s) \), one could compute the numerator and denominator of \( Z(s) \), [2], and then check positive realness via a standard procedure [3]. We aim here for a more direct procedure.

Manuscript received December 30, 1982; revised September 29, 1983 and December 7, 1983.

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The precise meaning of “direct” will be defined by the tests themselves.

We shall also be rather more concerned with examining strict positive realness, where all poles of \( Z(s) \) must be in \( \text{Re}[s] < 0 \) and \( \text{Re} \, Z(j\omega) > 0 \) for all real \( \omega \). Strict positive realness tends to be more important in applications for some denominator zeros of a rational function. It is therefore necessary to consider the evaluation of the rank of an infinite Hankel matrix, which will ensure that \( \text{Re} \, Z(j\omega) > 0 \) for all real \( \omega \).

As it turns out, the test we shall be interested in is a standard one, which will ensure that \( \text{Re} \, Z(j\omega) > 0 \) for all real \( \omega \), except where \( j\omega \) is a pole of \( Z(s) \) and so on.

In [8], a test based on the \( \gamma \) used for checking the denominator zeros of a rational \( Z(s) \) all lie in \( \text{Re}[s] < 0 \). In applying this test, one should either know an a priori bound on the degree of the denominator polynomial or be prepared to evaluate the rank of an infinite Hankel matrix — this is a standard matrix in the well-known realization problem, requiring the determination of \( F, \gamma, f, k \) such that \( k \) is a real solution.

In view of the existence of the test for the pole restriction on \( Z(s) \) and our concentration on strict positive realness, we shall confine our attention for the most part to characterizing the properties of the \( \gamma \), which will ensure that \( \text{Re} \, Z(j\omega) > 0 \) for all real \( \omega \).

As it turns out, the test we give for this property is independent of the location of the roots of \( Z(s) \).

II. REFORMULATION OF THE POSITIVITY CRITERION

Suppose it is known that
\[
Z(s) = \frac{g(s)}{p(s)}
\] (2.1)
for some real polynomials \( g, p \) with also \( Z(\infty) < \infty \).

Then
\[
R(\omega) \triangleq \text{Re} \, Z(j\omega) = \frac{q(j\omega)p(-j\omega) + q(-j\omega)p(j\omega)}{2p(j\omega)p(-j\omega)} \frac{f(\omega)}{d(\omega)}
\] (2.2)
for all \( \omega \) other than those for which \( j\omega \) is a pole of \( Z(s) \) and only if
\[
\int_{-\infty}^{+\infty} R(\omega) \, d\omega = 0, \quad R(\omega) > 0
\] (2.3)
for some arbitrary real \( a \). (The symbol
\[
\int_{-\infty}^{+\infty} x(\omega)
\] for rational \( x \) denotes the Cauchy index of \( x(\cdot) \) over \((-\infty, +\infty)\), see Appendix [2].)

Proof: First, suppose that \( Z(s) \) has no purely imaginary roots. Then in (2.2), \( d(\omega) > 0 \) for all \( \omega \). Then \( \text{Re} \, Z(j\omega) > 0 \) for all real \( \omega \) if and only if [2]
\[
\int_{-\infty}^{+\infty} \frac{f'(\omega)}{f(\omega)} \, d\omega = 0, \quad f(a) > 0
\] (2.4)
for some arbitrary \( a \). Now
\[
\int_{-\infty}^{+\infty} R(\omega) \, d\omega = \int_{-\infty}^{+\infty} \left[ \frac{f'(\omega)d(\omega)}{f(\omega)} - \frac{d'(\omega)f(\omega)}{d^2(\omega)} \right] \, d\omega
\]
\[
= \int_{-\infty}^{+\infty} \frac{f'(\omega)}{f(\omega)} - \int_{-\infty}^{+\infty} \frac{d'(\omega)}{d(\omega)}
\] (2.5)
Since \( d(\omega) \neq 0 \) for all real \( \omega \),
\[
\int_{-\infty}^{+\infty} \left[ \frac{d'(\omega)}{d(\omega)} \right] = 0
\] see [2]; thus the equality in (2.4) is equivalent to the equality in (2.3). The same is clearly true of the inequality.

Now suppose \( Z(s) \) has \( k \) pure imaginary poles. The number of distinct real zeros of \( d(\omega) = 2p(-j\omega)p(j\omega) \) is equal to the number of distinct pure imaginary zeros of \( p(j\omega) \). Thus [2],
\[
\int_{-\infty}^{+\infty} \frac{d'(\omega)}{d(\omega)} = k.
\]
Also, \( f(\omega) = q(j\omega)p(-j\omega) + q(-j\omega)p(j\omega) \) clearly has a zero at any pure imaginary zeros of \( p(j\omega) \) (which is also a pure imaginary zero of \( p(-j\omega) \)). If \( f(\omega) \) has a zero at any value of \( \omega \), then \( \text{Re} \, Z(j\omega) \) will be zero, and conversely. So to have \( \text{Re} \, Z(j\omega) > 0 \) for all \( \omega \) other than those for which \( j\omega \) is a pole of \( Z(s) \), it is necessary and sufficient that \( f(\omega) \) have precisely \( k \) distinct real zeros, i.e.,
\[
\int_{-\infty}^{+\infty} \frac{f'(\omega)}{f(\omega)} = k.
\]
Use of (2.5), which holds irrespective of where the zeros of \( p(s) \) are located then yields the main result.

Remarks 1: Using the Markov coefficients, it is easy to verify whether or not \( R(a) > 0 \) for any \( a \). If \( Z(s) \) is as in (1.1), we have
\[
R(\omega) = \gamma_0 - y_2 \omega^{-2} + y_4 \omega^{-4} - y_6 \omega^{-6} + \cdots
\] (2.6)
and the following alternative conditions guarantee \( R(\Omega) > 0 \) for sufficiently large \( \Omega \).
\[
\gamma_0 > 0 \quad \text{or} \quad \gamma_0 = 0, \quad y_2 < 0
\]
or \( \gamma_0 = 0, \quad y_2 = 0, \quad y_4 > 0, \cdots \) (2.7)

Remarks 2: There is a corresponding expression for \( R'(\omega) \) as
\[
R'(\omega) = 2y_2 \omega^{-3} - 4y_4 \omega^{-5} + 6y_6 \omega^{-7} - \cdots
\] (2.8)
and so
\[
\int_{-\infty}^{+\infty} R'(\omega) \, d\omega = \int_{-\infty}^{+\infty} \frac{y_2 \omega^{-3} - 2y_4 \omega^{-5} + \gamma_6 \omega^{-7} - \cdots}{y_0 - y_2 \omega^{-2} + y_4 \omega^{-4} - \cdots}
\] (2.9)
where it is understood that the numerator and denominator series are separately summed for suitably large \( \omega \) to give a rational function and then the rational functions are used to define the Cauchy index.

In the sequel, we shall study the evaluation of Cauchy indexes for expressions of the form of (2.9).

III. STURM SEQUENCE METHOD

Now suppose that we are given two Laurent series
\[
R_1(\omega) = \alpha_0 + \alpha_1 \omega^{-1} + \alpha_2 \omega^{-2} + \cdots
\]
\[
R_2(\omega) = \beta_0 + \beta_1 \omega^{-1} + \beta_2 \omega^{-2} + \cdots
\] (3.1)
with \( R_1, R_2 \) both known to be rational, suppose that \( R_2/R_1 \) is proper, i.e., finite at \( \omega = \infty \). We shall be interested in the evaluation of the Cauchy index
\[
\int_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} \, d\omega = \int_{-\infty}^{+\infty} \frac{\beta_0 + \beta_1 \omega^{-1} + \cdots}{\alpha_0 + \alpha_1 \omega^{-1} + \cdots}
\] (3.2)
Without loss of generality, we may assume \( \alpha_0 \neq 0 \) (else replace \( R_1(\omega), R_2(\omega) \) by \( \omega^3 R_1(\omega) \) and \( \omega^3 R_2(\omega) \) for an appropriate value of \( k \).
A computational approach to the index evaluation will now be derived. Suppose that
\[ R_i(\omega) = \frac{h_i(\omega)}{\gamma(\omega)}, \quad i = 1, 2, \ldots \tag{3.3} \]
for some polynomials \( h_1, h_2, \gamma \). (Note: one or both of the pairs \( h_i, \gamma \), and \( h_2, \gamma \) may not be coprime.) Via a Euclidean algorithm, define \( h_3, h_4, \ldots \), through
\[ h_{i+1}(\omega) = q_i(\omega)h_i(\omega) - h_{i-1}(\omega) \tag{3.4} \]
until \( h_{N+2}(\omega) = 0 \). Then [2]
\[ \int_{-\infty}^{\infty} \frac{R_2(\omega)}{R_1(\omega)} d\omega = \int_{-\infty}^{\infty} \frac{h_2(\omega)}{h_1(\omega)} d\omega = V_h(a) - V_h(b) \tag{3.5} \]
when \( V_h(a) \) denotes the number of variations in sign of \((h_1(\omega), h_2(\omega), \ldots, h_{N+1}(\omega))\) evaluated at \( a \).

Now introduce rational functions
\[ R_i(\omega) = \frac{h_i(\omega)}{\gamma(\omega)}, \quad i = 3, 4, \ldots \tag{3.6} \]
and assume that \( \gamma(a) \gamma(b) \neq 0 \). Then clearly,
\[ R_{i+1}(\omega) = q_i(\omega)R_i(\omega) - R_{i-1}(\omega) \tag{3.7} \]
and
\[ \int_{-\infty}^{\infty} \frac{R_2(\omega)}{R_1(\omega)} d\omega = V_h(a) - V_h(b) \tag{3.8} \]
Now suppose that \( \gamma(\omega), h_1(\omega) \) have degree \( n \) and \( n_1 \). Then each \( R_i(\omega) \) has the form
\[ R_i(\omega) = a_{i,0}\omega^{-(n-n_i)} + a_{i,1}\omega^{-(n-n_i)-1} + \cdots \tag{3.9} \]
if (3.1) and (3.7) are used to successively \( R_i \).

Now observe that the computation of the \( q_i(\omega) \) does not need the \( h_i(\omega) \): it is enough to use the Laurent series (3.1) and (3.9). The key to selecting \( q_i(\omega) \) knowing \( R_i(\omega) \) and \( R_{i-1}(\omega) \) is to ensure that \( q_i(\omega)R_i(\omega) - R_{i-1}(\omega) \) has a Laurent series with leading power in \( \omega \) which is of higher power in \( \omega \) than \( R_i(\omega) \). Also, the computations end after a finite number \( N \) of iterations. In fact, if one knows that \( R_2/R_1 \) has denominator degree not greater than \( N' \), then we have \( N \geq N' \).

Our interest is in
\[ \int_{-\infty}^{\infty} \frac{R_2(\omega)}{R_1(\omega)} d\omega. \]
Together, (3.8) and (3.9) imply
\[ \int_{-\infty}^{\infty} \frac{R_2(\omega)}{R_1(\omega)} d\omega = V\{-1\}a_{1,0}, \ldots, a_{n+1}a_{n+1,0}\}
\[ -V\{a_{1,0}, a_{2,0}, \ldots, a_{n+1,0}\} \tag{3.10} \]
where \( V\{\ldots\} \) means the number of variations in sign of series in \{\ldots\}. These observations now allow us to establish a Routh-table-like method for evaluating the Cauchy index (2.9). In case \( y_0 \neq 0 \) in (2.9), observe that \( R'(\omega)/R(\omega) \) has no singularity at \( \omega = 0 \), so that
\[ \int_{-\infty}^{\infty} \frac{R'(\omega)}{R(\omega)} d\omega = \int_{-\infty}^{\infty} \frac{\omega^2R'(\omega)}{R(\omega)} d\omega = \int_{-\infty}^{\infty} \frac{\omega^2}{\gamma \omega - \gamma_0 - \gamma_2 + \gamma_4} d\omega \tag{3.11} \]
where \( \gamma_0, \gamma_2, \gamma_4 \) are computed by following the Routh-table pattern:
\[ c_1 = \frac{-1}{b_1} b_2 \]
\[ d_1 = -1 \]
and so on. Then
\[ \int_{-\infty}^{\infty} \frac{R'(\omega)}{R(\omega)} d\omega = N - 2 \{V(a_1, a_2, a_3, \ldots) + V(1, a_1, a_2, \ldots)\} \tag{3.15} \]
where \( \Delta_i \) denotes the \( i \)th Hurwitz determinant, that is the \( i \)th principal minor of
\[ \begin{vmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \\ 0 & b_1 & a_2 \\ a_1 & a_2 & \cdots \\ \end{vmatrix} \]

It is useful to write the relations between the degree of various quantities to understand the limit on \( N \). Notice first that the denominators of \( Z(s), R(\omega), \) and \( R'(\omega) \) are \( p(s), p(j\omega)p(-j\omega), \) and \( p^2(j\omega)p^2(-j\omega), \) respectively. Also, the denominator of \( R'(\omega)/R(\omega) \) is
\[ [q(j\omega)p(-j\omega) + q(-j\omega)p(j\omega)]p(-j\omega)p(j\omega). \]
If \( p(j\omega) \) has degree \( \delta \), then the degree of this denominator is \( 4\delta \) if \( \deg q = \deg p \) or possibly lesser in general. This is the maximum value of \( N \) encountered in the above algorithm.
IV. BEZOUTIAN APPROACH

We shall base our development on the following result.

Lemma 4.1 [9, 10]: Let \( h_1(x), h_2(x) \) be real polynomials with\( \deg h_1 = n, \deg h_2 = n \). Define an \( n \times n \) matrix \( C = (c_{ij}) \) by

\[
\frac{h_2(y)h_1(x) - h_1(y)h_2(x)}{x - y} = \sum_{\ell=1}^{n} c_{ij} x^{-\ell} y^{n-j}.
\]

Then

\[
\int_{-\infty}^{\infty} \frac{h_2(x)}{h_1(x)} = \text{signature } C \tag{4.2a}
\]

\[
N \triangleq n - \deg \{ \text{g.c.d. of } h_1(x), h_2(x) \} = \text{rank } C. \tag{4.2b}
\]

Now let \( R_1(\omega), R_2(\omega) \) be two real proper rational functions with \( R_2(\omega) / R_1(\omega) \) proper and

\[
R_1(\omega) = \alpha_0 + \alpha_1 \omega^{-1} + \cdots \quad R_2(\omega) = \beta_0 + \beta_1 \omega^{-1} + \cdots \tag{4.3}
\]

Since we are interested in the Cauchy index of \( R_2 / R_1 \), we can, as earlier, assume \( \alpha_0 \neq 0 \) without loss of generality. Define the Bezoutian matrix of \( R_1(\omega), R_2(\omega) \) to be \( D \) where

\[
\frac{R_1(\omega)R_2(y) - R_2(\omega)R_1(y)}{x - y} = \sum_{\ell=1}^{\infty} d_{ij} x^{-\ell} y^{n-j}. \tag{4.4}
\]

We shall now relate \( D \) to \( C \). Suppose that

\[
R_i(\omega) = \frac{h_i(\omega)}{\gamma(\omega)}, \quad i = 1, 2 \tag{4.5}
\]

for polynomials \( h_i(\omega) \) and \( \gamma(\omega) \). We have \( n = \deg h_1 = \deg \gamma \geq \deg h_2 \).

Then, (4.1), (4.4), and (4.5) yield

\[
\sum_{\ell=1}^{\infty} d_{ij} x^{-\ell} y^{n-j} = \frac{1}{\gamma(x)\gamma(y)} \sum_{\ell=1}^{n} c_{ij} x^{-\ell} y^{n-j} = \pi(x) \pi(x) \tag{4.6}
\]

where

\[
\pi(x) = [x^{-1}, x^{-2}, \ldots, x, 1]'. \tag{4.6}
\]

Now define \( \bar{\gamma} \) by

\[
\frac{1}{\gamma(x)} = \gamma_0 x^{-n} + \gamma_1 x^{-n+1} + \gamma_2 x^{-n+2} + \cdots
\]

where \( \gamma_0 \neq 0 \). Then

\[
\frac{1}{\gamma(x)} \pi(x) = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots \end{bmatrix} \begin{bmatrix} x^{-1} \\ x^{-2} \\ \vdots \\ x \end{bmatrix} = [\Gamma_0 \ 1 \ 0 \ \Gamma_1] x_{\infty}
\]

where \( \Gamma_0 \) is square and nonsingular, and \( \Gamma_0, \Gamma_1, x_{\infty} \) have obvious definitions.

Combining this with (4.6) shows that

\[
D = [\Gamma_0 \ 1 \ 0 \ \Gamma_1] C [\Gamma_0 \ 1 \ 0 \ \Gamma_1] \tag{4.7}
\]

and so

\[
\text{rank } D = \text{rank } C, \quad \text{signature } D = \text{signature } C. \tag{4.8}
\]

With this observation and Lemma 4.1, we have, therefore, established the following result:

Theorem 4.2: Let \( R_1(\omega), R_2(\omega) \) be two real rational proper functions with \( R_2(\omega)/R_1(\omega) \) proper and \( \text{Det } R_1(\omega) \neq 0 \). Then with \( D = (d_{ij}) \) defined by (4.4),

\[
\text{rank } D = \text{degree of denominator of } R_2 / R_1 \text{ after cancellations}
\]

\[
\text{signature } D = \prod_{i=1}^{n} \frac{R_2(\omega)}{R_1(\omega)}. \tag{4.9a}
\]

The application to the positivity problem of Section II is clear. Remark: Let \( D \) denote the \( n \times n \) top left corner of \( D \). Then

\[
\text{rank } D = \text{rank } \hat{D} \quad \text{signature } D = \text{signature } \hat{D}. \tag{4.10}
\]

This will be of convenience in using (4.9b). Let \( D_k \) denote the determinant of the \( k \times k \) top left corner of \( D \). Then if rank \( D = N \) and none of \( D_1, D_2, \ldots, D_{N-1} \) is zero,

\[
\prod_{i=1}^{n} \frac{R_2(\omega)}{R_1(\omega)} = N - 2V(1, D_1, \ldots, D_N). \tag{4.11}
\]

If some of the \( D_i \) are zero, more general rules for the computation of the signature of \( D \) must be used, see [11, ch. X].

The key to the importance of these equations is that the entries of \( D \) are readily computable from the Markov parameters \( a_i, b_i \) associated with the \( R_i(\omega) \) in (4.3). Since in fact we aim to use the ideas for the positivity problem of Section II, let us suppose, as in the last section, that

\[
R_1(\omega) = a_1 - a_2 \omega^{-1} + a_3 \omega^{-2} \ldots
\]

\[
R_2(\omega) = b_1 \omega^{-1} - b_2 \omega^{-2} + b_3 \omega^{-3} \ldots. \tag{4.12}
\]

Then multiplying both sides of (4.4) by \((x - y)\) and equating like powers gives

\[
d_{i+j, i-j+1} - d_{i-j, i+j+1} = (-1)^{(i+j)/2} a_{(i+j)+1} (-1)^{(i-j)/2} b_{(i+j)+1} \quad (i: \text{even, } j: \text{odd})
\]

\[
-(-1)^{(i-j)/2} a_{(i-j)+1} (-1)^{(i+j)/2} b_{(i-j)+1} \quad (i: \text{odd, } j: \text{even})
\]

\[
-0, \quad \text{otherwise}. \tag{4.13}
\]

From the third equation, it follows that for \( i + j \) odd, \( d_{ij} = 0 \) and this means that \( D \), modulo column and row reordering, may be expressed as the direct sum of smaller matrices (just as in the reduced Hermite criterion for the continuous stability problem [7]). That is

\[
M^T \hat{D} M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{4.14}
\]

and

\[
A = [d_{2k+1,2l+1}], \quad B = [d_{2k,2l}] \tag{4.15}
\]

with

\[
d_{2k+1,2l+1} = (k, l+1) + (k-1, l+2) + \cdots + (0, k + l+1), \quad k \leq l
\]

\[
(i, j) \triangleq (-1)^{(i+j)/2} a_{i+j+1} b_{j-i+1}
\]

\[
d_{2k,2l} = (k-1, l) + (k-2, l+1) + \cdots + (0, k + l-1), \quad k \leq l
\]

\[
(i, j) \triangleq (-1)^{(i+j)/2} a_{i+j+1} b_{j-i+1} - a_{j+i+1} b_{i-j+1}. \tag{4.16}
\]
If now $R_1(\omega)$, $R_2(\omega)$ are associated with
\[ Z(s) = \sum \gamma_i s^{-i} \]
and $\gamma_0 = 0$, where $\gamma_2 \neq 0$, then $a_i = \gamma_2 i$ and $b_i = i \gamma_2 i$ (as stated in Section II).
Of course,
\[ \text{signature} D = \text{signature} A + \text{signature} B \quad (4.17) \]
and so the evaluation of
\[ \int_{-\infty}^{+\infty} R_2(\omega) \quad - \int_{-\infty}^{+\infty} R_1(\omega) \]
is made easier.

If $A_i$, $B_i$ denote the $i$th principal minor of $A$, $B$ and if all such principal minors are nonzero up to $l - \text{rank} A$, $m = \text{rank} B$ then
\[ \int_{-\infty}^{+\infty} R_2(\omega) = \text{rank} A - 2V(1, A_1, \ldots, A_l) \]
+ \text{rank} B - 2V(1, B_1, \ldots, B_m) \quad (4.18) \]

V. POSITIVITY TESTS FOR MATRIX TRANSFER FUNCTIONS

Suppose that
\[ Z(s) = G_0 + G_1 s^{-1} + G_2 s^{-2} + \cdots \quad (5.1) \]
is a real rational matrix. Suppose we wish to check, using the $G_i$ in a direct way, whether $Z(j\omega) + Z'(-j\omega)$ is positive definite for all real $\omega$ other than those for which $j\omega$ is a pole of an entry of $Z(s)$. In principle this can be done in the following way. Form
\[ R(\omega) = Z(j\omega) + Z'(-j\omega) \]
\[ = G_0 - G_2 \omega^{-2} + G_4 \omega^{-4} - G_6 \omega^{-6} + \cdots \quad (5.2) \]
and
\[ \gamma(\omega) = \gamma_0 - \gamma_2 \omega^{-2} + \gamma_4 \omega^{-4} + \cdots \]
where
\[ \gamma(\omega) = \text{det} R(\omega). \]

Notice that the Markov parameters of $\gamma(\omega)$ can (admittedly with computational difficulty) be formed from those of $R(\omega)$. Also, $\gamma(\omega)$ is rational. Then, as is easily seen [12], $R(\omega)$ is positive definite for all $\omega$ if and only if $\gamma(\omega) > 0$ for all $\omega$ and $R(a)$ is positive definite for an arbitrary real $a$. (An appropriate modification applies if $Z(s)$ has a $j\omega$-axis pole.) The earlier method can be applied to checking $\gamma(\omega) > 0$. The checking of $R(\Omega) > 0$ for large $\Omega$ may be more messy than in the scalar case, but in principle is not difficult.

VI. CONCLUSION

We have stated three separate tests for strict positivity of $\text{Re} Z(j\omega)$ for all $\omega$ other than those for which $j\omega$ is a pole of $Z(s)$, in terms of the Markov coefficients associated with $Z(s)$. The Routh test is initialized by the coefficients and involves mainly, but not exclusively, integral operation to build up a Routh-type table. The other two tests, Hurwitz and Bezout, involve the coefficients in an entirely integral fashion.

It will be recalled that the test is one for $\text{Re} Z(j\omega) > 0$ rather than $\text{Re} Z(j\omega) > 0$. The extension to cover positive realness which is not strict could be achieved theoretically, but would doubtless be intricate.

APPENDIX [2]

1) Definition of Cauchy Index
The Cauchy index of a rational function $R(x)$ between $a$ and $b$ which is denoted by
\[ \int_a^b \frac{f(x)}{f'(x)} \]
is the difference between the numbers of jumps of $R(x)$ from $-\infty$ to $+\infty$ and the number of jumps from $+\infty$ to $-\infty$ as $x$ moves from $a$ to $b$.

2) Using the Cauchy index, we can count the numbers of distinct real roots of a polynomial $f(x)$ in the interval $(a, b)$. If $f(x) = a_0(x - z_1)^{m_1}(x - z_2)^{m_2} \cdots (x - z_p)^{m_p} g(x)$ has $p$ distinct real roots $z_1, z_2, \ldots, z_p$ with multiplicities $m_1, m_2, \ldots, m_p$, respectively, and $g(x)$ is polynomial with no real roots, then
\[ \int_a^b \frac{f'(x)}{f(x)} = \sum_{i=1}^{p} \frac{m_i}{x - z_i} + R_1(x) \]
where $R_1(x)$ is a rational function without real poles.

Therefore, the number of distinct real roots of $f(x)$ in the interval $(a, b)$ is equal to
\[ \int_a^b \frac{f'(x)}{f(x)} \]

3) Sturm's methods for computing the Cauchy index
\[ \int_a^b \frac{f_1(x)}{f_2(x)} \]
A Sturm chain in the interval $(a, b)$ is a sequence of real polynomials
\[ f_1(x), f_2(x), \ldots, f_m(x) \]
which has the following properties 1) and 2):
1) For $a < x < b$ then $f_k(x) = 0$ then $f_{k+1}(x) < 0$ and $f_k(x) = 0$ for $a < x < b$.
A Sturm chain beginning with $f_1(x), f_2(x)$ can be constructed by means of the Euclidean algorithm [2, p. 176].

If $f_1(x), f_2(x), \ldots, f_m(x)$ is a Sturm chain in $(a, b)$ then
\[ \int_a^b \frac{f_2(x)}{f_1(x)} = V(a) - V(b) \]
where $V(x)$ is the number of variations of sign in the chain evaluated at $x$ for $x \in (a, b)$, and $V(a) = \lim_{x \rightarrow a^+} V(x)$, $V(b) = \lim_{x \rightarrow b^+} V(x)$.

REFERENCES


