Markov Parameter Characterization of the Strict Positive Real Problem

N. HAMADA AND B. D. O. ANDERSON

Abstract —**Suppose that a rational function $Z(s)$ is defined by a Laurent series, the coefficients of which are known. Several criteria are given in terms of these coefficients (the Markov parameters of $Z(s)$) to ensure that $\operatorname{Re} Z(j\omega) > 0$ for all real ω. The criteria are defined by using a Cauchy index formulation of the ratio of two rational functions, and they are of three types—involving a Routh-like table with first two rows initialized using the coefficients, and Hurwitz and Bezout matrices with entries which are the coefficients themselves, or integral expressions in the coefficients. The matrix positive real property is also investigated.**

I. Introduction

Let $Z(s)$ be a real rational function of s. As is well known, $Z(s)$ is termed positive real [1], if (1) all poles of $Z(s)$ either lie in $\operatorname{Re}[s] < 0$, or lie on $\operatorname{Re}[s] = 0$ and are simple with positive residues, (2) $\operatorname{Re} Z(j\omega) \geqslant 0$ for all real ω, with $j\omega$ not a pole of $Z(s)$.

In this paper, we examine the question of how the positive real nature of $Z(s)$ can be characterized in terms of the Markov parameters [2] of $Z(s)$. These are the constants γ_i in the expansion

$$Z(s) = \gamma_{-1}s + \gamma_0 + \gamma_1 s^{-1} + \gamma_2 s^{-2} + \cdots \tag{1.1}$$

(without loss of generality, we can assume $\gamma_{-1} = 0$. For if not, $Z(s)$ is positive real if and only if $Z(s) - \gamma_{-1}s$ is positive real and $\gamma_{-1} > 0$, [1]).

Our main interest is in rational $Z(s)$. For such $Z(s)$, we recall that one can express the positive real property in several ways. For example, suppose that one knows $p(\cdot)$ and $q(\cdot)$ such that $Z(s) = q(s)/p(s)$; one can first identify the greatest common factor of $p(s)$ and $p(-s)$. Call it $p_1(s)$ and let $p(s) = p_1(s)p_2(s)$. Then find $q_1(s), q_2(s)$ such that $Z = (q_1/p_1) + (q_2/p_2)$. One then checks that the poles and zeros of q_1/p_1 interlace on the $j\omega$-axis (various tests are available), that $p_2(s)$ has all zeros in the left half plane and that $p_2(-s)q_2(s) + p_2(s)q_2(-s)$ is nonnegative for all $s = j\omega$, ω real; again various tests are available [3]. Alternatively, one can characterize the positive reality via a minimal state-variable realization of $Z(s)$ [4], although it is somewhat difficult to test this condition.

The various characterizations and tests have a variety of uses—e.g., for developing further theory, or as a basis for checking the positive real property. The test advanced here may on occasion be useful for checking the positive real property, depending how $Z(s)$ is characterized in the first instance.

Recent developments in the theory of Hankel norm approximations suggest that the descriptions available for $Z(s)$ may well be via the Markov parameters with a simple integer, the order of $Z(s)$, rather than via numerator and denominator polynomials of $Z(s)$.

Of course, knowing the γ_i and the degree of the denominator of a rational $Z(s)$, one could compute the numerator and denominator of $Z(s)$, [2], and then check positive realness via a standard procedure [3]. We aim here for a more direct procedure.

Manuscript received December 30, 1982; revised September 29, 1983 and December 7, 1983.

N. Hamada is with the Department of Electrical Engineering, Keio University, Yokohama, Japan.

B. D. O. Anderson is with the Department of Systems Engineering, Research School of Physical Sciences, The Australian National University, Canberra, Australia.

0098-4094/84/0900-0814$01.00

(The precise meaning of "direct" will be defined by the tests themselves.)

We shall also be rather more concerned with examining strict positive realness, where all poles of $Z(s)$ must be in $\mathrm{Re}[s]<0$ and $\mathrm{Re}\, Z(j\omega)>0$ for all real ω. Strict positive realness tends to be more important in applications [5]–[7].

In [8], a test based on the γ_i is provided for checking that the denominator zeros of a rational $Z(s)$ all lie in $\mathrm{Re}[s]<0$. In applying this test, one should either know an a priori bound on the degree of the denominator polynomial or be prepared to evaluate the rank of an infinite Hankel matrix—this is a standard matrix in the well-known realization problem, requiring the determination of F, g, h, k such that $k+h'(sI-F)^{-1}g=\Sigma\gamma_i s^{-i}$.

In view of the existence of the test for the pole restriction on $Z(s)$ and our concentration on strict positive realness, we shall confine our attention for the most part to characterizing the properties of the γ_i which will ensure that $\mathrm{Re}\, Z(j\omega)>0$ for all real ω.

As it turns out, the test we give for this property is independent of the location of the poles of $Z(s)$.

II. Reformulation of the Positivity Criterion

Suppose it is known that

$$Z(s)=\frac{q(s)}{p(s)} \tag{2.1}$$

for some real polynomials q, p with also $Z(\infty)<\infty$. Then

$$\begin{aligned} R(\omega) &\triangleq \mathrm{Re}\, Z(j\omega) \\ &= \frac{q(j\omega)p(-j\omega)+q(-j\omega)p(j\omega)}{2p(j\omega)p(-j\omega)}=\frac{f(\omega)}{d(\omega)} \end{aligned} \tag{2.2}$$

where $f(\cdot)$ and $d(\cdot)$ have the obvious definitions.

The main result on which the positivity test depends is as follows. (Notice that $R'(\omega)/R(\omega)$, which appears in the lemma statement, is rational and $'$ means differentiation with respect to ω.)

Lemma 2.1: With $Z(s)$ as in (2.1) and with $Z(\infty)<\infty$, $\mathrm{Re}\, Z(j\omega)>0$ for all real ω except where $j\omega$ is a pole of $Z(s)$ if and only if

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{R'(\omega)}{R(\omega)}=0, \qquad R(a)>0 \tag{2.3}$$

for some arbitrary real a. (The symbol

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} x(\omega)$$

for rational x denotes the Cauchy index of $x(\cdot)$ over $(-\infty,+\infty)$, see Appendix and [2].)

Proof: First, suppose that $Z(s)$ has no purely imaginary poles. Then in (2.2), $d(\omega)>0$ for all ω. Then $\mathrm{Re}\, Z(j\omega)>0$ for all real ω if and only if [2]

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{f'(\omega)}{f(\omega)}=0, \qquad f(a)>0 \tag{2.4}$$

for some arbitrary a. Now

$$\begin{aligned} \mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{R'(\omega)}{R(\omega)} &= \mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{[f'(\omega)d(\omega)-d'(\omega)f(\omega)]/d^2(\omega)}{f(\omega)/d(\omega)} \\ &= \mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{f'(\omega)}{f(\omega)}-\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{d'(\omega)}{d(\omega)}. \end{aligned} \tag{2.5}$$

Since $d(\omega)\neq 0$ for all real ω,

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty}[d'(\omega)/d(\omega)]=0$$

see [2]; thus the equality in (2.4) is equivalent to the equality in (2.3). The same is clearly true of the inequality.

Now suppose $Z(s)$ has k pure imaginary poles. The number of distinct real zeros of $d(\omega)=2p(-j\omega)p(j\omega)$ is equal to the number of distinct pure imaginary zeros of $p(j\omega)$. Thus [2],

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{d'(\omega)}{d(\omega)}=k.$$

Also, $f(\omega)=q(j\omega)p(-j\omega)+q(-j\omega)p(j\omega)$ clearly has a zero at any pure imaginary zeros of $p(j\omega)$ (which is also a pure imaginary zero of $p(-j\omega)$). If $f(\omega)$ has a zero at any other value of ω, then $\mathrm{Re}\, Z(j\omega)$ will be zero, and conversely. So to have $\mathrm{Re}\, Z(j\omega)>0$ for all ω other than those for which $j\omega$ is a pole of $Z(s)$, it is necessary and sufficient that $f(\omega)$ have precisely k distinct real zeros, i.e.,

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{f'(\omega)}{f(\omega)}=k.$$

Use of (2.5), which holds irrespective of where the zeros of $p(s)$ are located then yields the main result.

Remarks 1: Using the Markov coefficients, it is easy to verify whether or not $R(a)>0$ for any a. For if $Z(s)$ is as in (1.1), we have

$$R(\omega)=\gamma_0-\gamma_2\omega^{-2}+\gamma_4\omega^{-4}-\gamma_6\omega^{-6}+\cdots \tag{2.6}$$

and the following alternative conditions guarantee $R(\Omega)>0$ for sufficiently large Ω.

$$\begin{aligned} &\gamma_0>0 \quad \text{or} \quad \gamma_0=0, \quad \gamma_2<0 \\ &\text{or} \quad \gamma_0=0, \quad \gamma_2=0, \quad \gamma_4>0, \cdots \end{aligned} \tag{2.7}$$

Remarks 2: There is a corresponding expression for $R'(\omega)$ as

$$R'(\omega)=2\gamma_2\omega^{-3}-4\gamma_4\omega^{-5}+6\gamma_6\omega^{-7}-\cdots \tag{2.8}$$

and so

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{R'(\omega)}{R(\omega)}=\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{\gamma_2\omega^{-3}-2\gamma_4\omega^{-5}+3\gamma_6\omega^{-7}-\cdots}{\gamma_0-\gamma_2\omega^{-2}+\gamma_4\omega^{-4}-\cdots} \tag{2.9}$$

where it is understood that the numerator and denominator series are separately summed for suitably large ω to give a rational function and then the rational functions are used to define the Cauchy index.

In the sequel, we shall study the evaluation of Cauchy indexes for expressions of the form of (2.9).

III. Sturm Sequence Method

Now suppose that we are given two Laurent series

$$\begin{aligned} R_1(\omega) &= \alpha_0+\alpha_1\omega^{-1}+\alpha_2\omega^{-2}+\cdots \\ R_2(\omega) &= \beta_0+\beta_1\omega^{-1}+\beta_2\omega^{-2}+\cdots \end{aligned} \tag{3.1}$$

with R_1, R_2 both known to be rational, suppose that R_2/R_1 is proper, i.e., finite at $\omega=\infty$. We shall be interested in the evaluation of the Cauchy index

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{R_2(\omega)}{R_1(\omega)}=\mathop{\mathrm{I}}_{-\infty}^{+\infty}\frac{\beta_0+\beta_1\omega^{-1}+\cdots}{\alpha_0+\alpha_1\omega^{-1}+\cdots}. \tag{3.2}$$

Without loss of generality, we may assume $\alpha_0\neq 0$ (else replace $R_1(\omega)$, $R_2(\omega)$ by $\omega^k R_1(\omega)$ and $\omega^k R_2(\omega)$ for an appropriate value of k).

A computational approach to the index evaluation will now be derived. Suppose that

$$R_i(\omega)=\frac{h_i(\omega)}{\gamma(\omega)}, \qquad i=1,2 \tag{3.3}$$

for some polynomials h_1, h_2, and γ. (Note: one or both of the pairs h_1, γ, and h_2, γ may not be coprime.) Via a Euclidean algorithm, define $h_3, h_4, \cdots$, through

$$h_{i+1}(\omega)=q_i(\omega)h_i(\omega)-h_{i-1}(\omega) \tag{3.4}$$

until $h_{N+2}(\omega)=0$. Then [2]

$$\mathop{\mathrm{I}}_a^b \frac{R_2(\omega)}{R_1(\omega)}=\mathop{\mathrm{I}}_a^b \frac{h_2(\omega)}{h_1(\omega)}=V_h(a)-V_h(b) \tag{3.5}$$

when $V_h(a)$ denotes the number of variations in sign of $\{h_1(\omega), h_2(\omega), \cdots, h_{N+1}(\omega)\}$ evaluated at a.

Now introduce rational functions

$$R_i(\omega)=\frac{h_i(\omega)}{\gamma(\omega)}, \qquad i=3,4,\cdots \tag{3.6}$$

and assume that $\gamma(a)\gamma(b)\neq 0$. Then clearly,

$$R_{i+1}(\omega)=q_i(\omega)R_i(\omega)-R_{i-1}(\omega) \tag{3.7}$$

and

$$\mathop{\mathrm{I}}_a^b \frac{R_2(\omega)}{R_1(\omega)}=V_R(a)-V_R(b). \tag{3.8}$$

Now suppose that $\gamma(\omega)$, $h_i(\omega)$ have degree n and n_i. Then each $R_i(\omega)$ has the form

$$R_i(\omega)=\alpha_{i,0}\omega^{-(n-n_i)}+\alpha_{i,1}\omega^{-(n-n_i)-1}+\cdots \tag{3.9}$$

if (3.1) and (3.7) are used to compute the successive R_i.

Now observe *that the computation of the* $q_i(\omega)$ *does not need the* $h_i(\omega)$: *it is enough to use the Laurent series* (3.1) *and* (3.9). The key to selecting $q_i(\omega)$ knowing $R_i(\omega)$ and $R_{i-1}(\omega)$ is to ensure that $q_i(\omega)R_i(\omega)-R_{i-1}(\omega)$ has a Laurent series the first term in which is of higher power in ω^{-1} than $R_i(\omega)$. Also, the computations end after a finite number N of iterations. In fact, if one knows that R_2/R_1 has denominator degree not greater than N', then we have $N'\geqslant N$.

Our interest is in

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} R_2(\omega)/R_1(\omega).$$

Together, (3.8) and (3.9) imply

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)}=V\{(-1)^{n_1}\alpha_{1,0},(-1)^{n_2}\alpha_{2,0},\cdots(-1)^{n_{N+1}}\alpha_{N+1,0}\}$$
$$-V\{\alpha_{1,0},\alpha_{2,0},\cdots,\alpha_{N+1,0}\} \tag{3.10}$$

where $V\{\cdots\}$ means the number of variations in sign of series in $\{\cdots\}$. These observations now allow us to establish a Routh table-like method for evaluating the Cauchy index (2.9). In case $\gamma_0\neq 0$ in (2.9), observe that $R'(\omega)/R(\omega)$ has no singularity at $\omega=0$, so that

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R'(\omega)}{R(\omega)}=\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{\omega^2R'(\omega)}{R(\omega)}=\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{\gamma_2\omega^{-1}+2\gamma_4\omega^{-3}+\cdots}{\gamma_0-\gamma_2\omega^{-2}+\gamma_4\omega^{-4}-\cdots} \tag{3.11}$$

Set $a_i=\gamma_{2i}$, $b_i=i\gamma_{2i}$ and

$$R_1(\omega)=R(\omega)=a_1-a_2\omega^{-2}+a_3\omega^{-4}-\cdots \tag{3.12a}$$
$$R_2(\omega)=\omega^2R'(\omega)=b_1\omega^{-1}-b_2\omega^{-3}+b_3\omega^{-5}-\cdots \tag{3.12b}$$

In case $\gamma_0=0$, there exists an even power of ω, ω^{2k} say, such that $\omega^{2k}R(\omega)$ has a Laurent series with leading coefficient nonzero. Then we set

$$R_1(\omega)=\omega^{2k}R(\omega)=a_1-a_2\omega^{-2}+a_3\omega^{-4}-\cdots \tag{3.13a}$$
$$R_2(\omega)=\omega^{2k}R'(\omega)=b_1\omega^{-1}-b_2\omega^{-3}+b_3\omega^{-5}-\cdots. \tag{3.13b}$$

(Thus if $\gamma_0=0$, $\gamma_2\neq 0$, $a_i=\gamma_{2i}$, $b_i=i\gamma_{2i}$.)

We now form the following table:

row					
1	a_1	a_2	a_3	a_4	$\cdots$
2	b_1	b_2	b_3	b_4	$\cdots$
3	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$N+1$	p_1	p_2	p_3		
$N+2$	0	0	0		

where $c_i, d_i, \cdots$ are computed by following the Routh-table pattern:

$$c_i=\frac{-1}{b_1}\begin{vmatrix} a_1 & a_{i+1}\\ b_1 & b_{i+1}\end{vmatrix} \qquad d_i=\frac{-1}{c_1}\begin{vmatrix} b_1 & b_{i+1}\\ c_1 & c_{i+1}\end{vmatrix}$$

and so on. Then

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R'(\omega)}{R(\omega)}=\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)}=N-2V(a_1,b_1,c_1,\cdots,p_1). \tag{3.14}$$

This Routh test has a corresponding Hurwitz test. It is obtained just as in the conventional case [2].

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)}=N-2\{V(a_1,\Delta_1,\Delta_3,\cdots)+V(1,\Delta_2,\Delta_4,\cdots)\} \tag{3.15}$$

where Δ_i denotes the ith Hurwitz determinant, that is the ith principal minor of

$$\begin{vmatrix} b_1 & b_2 & & \cdots\\ a_1 & a_2 & & \cdots\\ 0 & b_1 & b_2 & \cdots\\ 0 & a_1 & a_2 & \cdots\end{vmatrix}$$

N is defined by $\Delta_N\neq 0$ and $\Delta_{N+i}=0$ for all $i>0$.

It is helpful to write the relations between the degree of various quantities to understand the limit on N. Notice first that the denominators of $Z(s)$, $R(\omega)$, and $R'(\omega)$ are $p(s)$, $p(j\omega)p(-j\omega)$, and $p^2(j\omega)p^2(-j\omega)$, respectively. Also, the denominator of $R'(\omega)/R(\omega)$ is

$$[q(j\omega)p(-j\omega)+q(-j\omega)p(j\omega)]p(-j\omega)p(j\omega).$$

If $p(j\omega)$ has degree δ, then the degree of this denominator is 4δ if $\deg q=\deg p$ or possibly lesser in general. This is the maximum value of N encountered in the above algorithm.

IV. Bezoutian Approach

We shall base our development on the following result.

Lemma 4.1 [9], [10]: Let $h_1(x)$, $h_2(x)$ be real polynomials with $\deg h_1 = n$, $\deg h_2 = n$. Define an $n \times n$ matrix $C = (c_{ij})$ by

$$\frac{h_2(y)h_1(x) - h_1(y)h_2(x)}{x-y} = \sum_{ij=1}^{n} c_{ij}x^{n-i}y^{n-j}. \quad (4.1)$$

Then

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{h_2(x)}{h_1(x)} = \text{signature } C \quad (4.2a)$$

$$N \triangleq n - \deg[\text{g.c.d. of } h_1(x), h_2(x)] = \text{rank } C. \quad (4.2b)$$

Now let $R_1(\omega)$, $R_2(\omega)$ be two real proper rational functions with $R_2(\omega)/R_1(\omega)$ proper and

$$R_1(\omega) = \alpha_0 + \alpha_1\omega^{-1} + \cdots$$
$$R_2(\omega) = \beta_0 + \beta_1\omega^{-1} + \cdots. \quad (4.3)$$

Since we are interested in the Cauchy index of R_2/R_1, we can, as earlier, assume $\alpha_0 \neq 0$ without loss of generality. Define the Bezoutian matrix of $R_1(\omega)$, $R_2(\omega)$ to be $D = (d_{ij})$ where

$$\frac{R_1(x)R_2(y) - R_1(y)R_2(x)}{x-y} = \sum_{ij=1}^{\infty} d_{ij}x^{-i}y^{-j}. \quad (4.4)$$

We shall now relate D to C. Suppose that

$$R_i(\omega) = \frac{h_i(\omega)}{\gamma(\omega)}, \qquad i = 1,2 \quad (4.5)$$

for polynomials $h_i(\omega)$ and $\gamma(\omega)$. We have $n = \deg h_1 = \deg \gamma \geqslant \deg h_2$.

Then, (4.1), (4.4), and (4.5) yield

$$\sum_{ij=1}^{\infty} d_{ij}x^{-i}y^{-j} = \frac{1}{\gamma(x)\gamma(y)} \sum_{ij=1}^{n} c_{ij}x^{n-i}y^{n-j}$$
$$= \frac{\pi'(y)}{\gamma(y)} C \frac{\pi(x)}{\gamma(x)} \quad (4.6)$$

where

$$\pi(x) = [x^{n-1}, x^{n-2}, \cdots, x, 1]'.$$

Now define $\hat{\gamma}_i$ by

$$\frac{1}{\gamma(x)} = \hat{\gamma}_0 x^{-n} + \hat{\gamma}_1 x^{-(n+1)} + \hat{\gamma}_2 x^{-(n+2)} + \cdots$$

where $\hat{\gamma}_0 \neq 0$. Then

$$\frac{1}{\gamma(x)}\pi(x) = \begin{bmatrix} \hat{\gamma}_0 & \hat{\gamma}_1 & \hat{\gamma}_2 & \cdots \\ 0 & \hat{\gamma}_0 & \hat{\gamma}_1 & \\ 0 & 0 & \hat{\gamma}_0 & \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} x^{-1} \\ x^{-2} \\ x^{-3} \\ \vdots \end{bmatrix}$$
$$\triangleq [\Gamma_0 \quad \Gamma_1] x_\infty$$

where Γ_0 is square and nonsingular, and $\Gamma_0, \Gamma_1, x_\infty$ have obvious definitions.

Combining this with (4.6) shows that

$$D = \begin{bmatrix} \Gamma_0' \\ \Gamma_1' \end{bmatrix} C [\Gamma_0 \quad \Gamma_1] \quad (4.7)$$

and so

$$\text{rank } D = \text{rank } C, \qquad \text{signature } D = \text{signature } C. \quad (4.8)$$

With this observation and Lemma 4.1, we have, therefore, established the following result:

Theorem 4.2: Let $R_1(\omega)$, $R_2(\omega)$ be two real rational proper functions with $R_2(\omega)/R_1(\omega)$ proper and $R_1(\infty) \neq 0$. Then with $D = (d_{ij})$ defined by (4.4),

$$\text{rank } D = \text{degree of denominator of } R_2/R_1 \text{ after cancellations} \quad (4.9a)$$

$$\text{signature } D = \mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)}. \quad (4.9b)$$

The application to the positivity problem of Section II is clear.

Remark: Let $\hat{D}$ denote the $n \times n$ top left corner of D. Then (4.7) and the nonsingularity of Γ_0 imply

$$\text{rank } \hat{D} = \text{rank } D \qquad \text{signature } D = \text{signature } \hat{D}. \quad (4.10)$$

This will be of convenience in using (4.9b). Let D_k denote the determinant of the $k \times k$ top left corner of D. Then if rank $D = N$ and none of $D_1, D_2, \cdots, D_{N-1}$ is zero,

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} = N - 2V\{1, D_1, \cdots, D_N\}. \quad (4.11)$$

If some of the D_i are zero, more general rules for the computation of the signature of D must be used, see [11, ch. X].

The key to the importance of these equations is that the entries of D are readily computable from the Markov parameters α_i, β_i associated with the $R_i(\omega)$ in (4.3). Since in fact we aim to use the ideas for the positivity problem of Section II, let us suppose, as in the last section, that

$$R_1(\omega) = a_1 - a_2\omega^{-2} + a_3\omega^{-4} \cdots$$
$$R_2(\omega) = b_1\omega^{-1} - b_2\omega^{-3} + b_3\omega^{-5} - \cdots. \quad (4.12)$$

Then multiplying both sides of (4.4) by $(x-y)$ and equating like powers gives

$$\begin{aligned} d_{i+1,j} - d_{i,j+1} &= (-1)^{(i/2)} a_{(i/2)+1} (-1)^{(j-1)/2} b_{(j+1)/2} \quad (i\text{: even, } j\text{: odd}) \\ &= -(-1)^{j/2} a_{(j/2)+1} (-1)^{(i-1)/2} b_{(i+1)/2} \quad (i\text{: odd, } j\text{: even}) \\ &= 0, \quad \text{otherwise.} \end{aligned} \quad (4.13)$$

From the third equation, it follows that for $i + j$ odd, $d_{ij} = 0$ and this means that $\hat{D}$, modulo column and row reordering, may be expressed as the direct sum of smaller matrices (just as in the reduced Hermite criterion for the continuous stability problem [7]). That is

$$M'\hat{D}M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad (4.14)$$

and

$$A = [d_{2k+1,2l+1}], \qquad B = [d_{2k,2l}] \quad (4.15)$$

with

$$d_{2k+1,2l+1} = (k, l+1) + (k-1, l+2) + \cdots + (0, k+l+1), \quad k \leqslant l$$

$$(i,j) \triangleq (-1)^{i+j+1}(a_{i+1}b_j - b_i a_{j+1})$$

$$d_{2k,2l} = \langle k-1, l\rangle + \langle k-2, l+1\rangle + \cdots + \langle 0, k+l-1\rangle, \quad k \leqslant l$$

$$\langle i,j\rangle \triangleq (-1)^{i+j}(a_{i+1}b_{j+1} - a_{j+1}b_{i+1}). \quad (4.16)$$

If now $R_1(\omega)$, $R_2(\omega)$ are associated with

$$Z(s) = \sum \gamma_i s^{-i}$$

and $\gamma_0 = 0$, where $\gamma_2 \neq 0$, then $a_i = \gamma_{2i}$ and $b_i = i\gamma_{2i}$ (as stated in Section II).

Of course,

$$\text{signature } \hat{D} = \text{signature } A + \text{signature } B \tag{4.17}$$

and so the evaluation of

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)}$$

is made easier.

If A_i, B_i denote the ith principal minor of A, B and if all such principal minors are nonzero up to $l = \text{rank } A$, $m = \text{rank } B$ then

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} = \text{rank } A - 2V(1, A_1, \cdots, A_l) + \text{rank } B - 2V(1, B_1, \cdots, B_m). \tag{4.18}$$

V. Positivity Tests for Matrix Transfer Functions

Suppose that

$$Z(s) = G_0 + G_1 s^{-1} + G_2 s^{-2} + \cdots \tag{5.1}$$

is a real rational matrix. Suppose we wish to check, using the G_i in a direct way, whether $Z(j\omega) + Z'(-j\omega)$ is positive definite for all real ω other than those for which $j\omega$ is a pole of an entry of $Z(s)$. In principle this can be done in the following way. Form

$$\begin{aligned} R(\omega) &= Z(j\omega) + Z'(-j\omega) \\ &= G_0 - G_2\omega^{-2} + G_4\omega^{-4} - G_6\omega^{-6} + \cdots \end{aligned} \tag{5.2}$$

and

$$\gamma(\omega) = \gamma_0 - \gamma_2\omega^{-2} + \gamma_4\omega^{-4} + \cdots$$

where

$$\gamma(\omega) = \det R(\omega).$$

Notice that the Markov parameters of $\gamma(\omega)$ can (admittedly with computational difficulty) be formed from those of $R(\omega)$. Also, $\gamma(\omega)$ is rational. Then, as is easily seen [12], $R(\omega)$ is positive definite for all ω if and only if $\gamma(\omega) > 0$ for all ω and $R(a)$ is positive definite for an arbitrary real a. (An appropriate modification applies if $Z(s)$ has a $j\omega$-axis pole.) The earlier method can be applied to checking $\gamma(\omega) > 0$. The checking of $R(\Omega) > 0$ for large Ω may be more messy than in the scalar case, but in principle is not difficult.

VI. Conclusion

We have stated three separate tests for strict positivity of $\operatorname{Re} Z(j\omega)$ for all ω other than those for which $j\omega$ is a pole of $Z(s)$, in terms of the Markov coefficients associated with $Z(s)$. The Routh test is initialized by the coefficients and involves mainly, but not exclusively, integral operation to build up a Routh-type table. The other two tests, Hurwitz and Bezout, involve the coefficients in an entirely integral fashion.

It will be recalled that the test is one for $\operatorname{Re} Z(j\omega) > 0$ rather than $\operatorname{Re} Z(j\omega) \geqslant 0$. The extension to cover positive realness which is not strict could be achieved theoretically, but would doubtless be intricate.

Appendix [2]

1) Definition of Cauchy Index

The Cauchy index of a rational function $R(x)$ between a and b which is denoted by

$$\mathop{\mathrm{I}}_{a}^{b} R(x)$$

is the difference between the numbers of jumps of $R(x)$ from $-\infty$ to $+\infty$ and the number of jumps from $+\infty$ to $-\infty$ as x moves from a to b.

2) Using the Cauchy index, we can count the numbers of distinct real roots of a polynomial $f(x)$ in the interval (a, b). If $f(x) = a_0(x - z_1)^{m_1}(x - z_2)^{m_2} \cdots (x - z_p)^{m_p} g(x)$ has p distinct real roots $z_1, z_2, \cdots, z_p$ with multiplicities $m_1, m_2, \cdots, m_p$, respectively, and $g(x)$ is polynomial with no real roots, then

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{p} \frac{m_i}{x - z_i} + R_1(x)$$

where $R_1(x)$ is a rational function without real poles.

Therefore, the number of distinct real roots of $f(x)$ in the interval (a, b) is equal to

$$\mathop{\mathrm{I}}_{a}^{b} \frac{f'(x)}{f(x)}.$$

3) Sturm's methods for computing the Cauchy index

$$\mathop{\mathrm{I}}_{a}^{b} \frac{f_2(x)}{f_1(x)}.$$

A Sturm chain in the interval (a, b) is a sequence of real polynomials

$$f_1(x), f_2(x), \cdots, f_m(x)$$

which has the following properties 1) and 2):

1) For $a < x < b$, if $f_k(x) = 0$ then $f_{k-1}(x) f_{k+1}(x) < 0$
2) $f_m(x) \neq 0$ for $a < x < b$.

A Sturm chain beginning with $f_1(x), f_2(x)$ can be constructed by means of the Euclidean algorithm [2, p. 176].

If $f_1(x), f_2(x), \cdots, f_m(x)$ is a Sturm chain in (a, b) then

$$\mathop{\mathrm{I}}_{a}^{b} \frac{f_2(x)}{f_1(x)} = V(a) - V(b)$$

where $V(x)$ is the number of variations of sign in the chain evaluated at x for $x \in (a, b)$, and $V(a) = \lim_{x \downarrow a} V(x)$, $V(b) = \lim_{x \uparrow b} V(x)$.

References

[1] L. Weinberg, *Network Analysis and Synthesis*. New York: McGraw-Hill, 1962.

[2] F. R. Gantmacher, *The Theory of Matrices*, Vol. 2. New York: Chelsea, 1959.

[3] D. D. Siljak, "New algebraic criteria for positive realness," *J. Franklin Inst.*, vol. 291, no. 2, Feb. 1971.

[4] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*. Englewood Cliffs, NJ: Prentice-Hall, 1973.

[5] V. M. Popov, *Hyperstability of Control Systems*. New York: Springer-Verlag, 1973.

[6] D. D. Siljak, "Algebraic criterion for absolute stability, optimality, and passivity of dynamical systems," *Proc. Inst. Elect. Eng.*, vol. 117, no. 10, pp. 2033–2036, Oct. 1970.

[7] E. I. Jury, *Inners and Stability of Dynamic Systems*. New York: Wiley, 1974.

[8] S. Hosoe, Y. Kaneda, and M. Ito, "Stability considerations on the minimal realized systems from Markov sequences," Research Rep. of Automatic Control Lab., Faculty of Engineering, Nagoya University, vol. 26, June 1979.
[9] B. D. O. Anderson, "On the computation of the Cauchy index," *Quart. Appl. Math.*, vol. 29, no. 4, pp. 577–582, Jan. 1972.
[10] A. S. Householder, "Bezoutians, elimination and localization," *SIAM. Rev.*, vol. 12, 1970.
[11] F. R. Gantmacher, *The Theory of Matrices*, vol. 1. New York: Chelsea, 1959.
[12] D. D. Siljak, "Stability criteria for two-variable polynomials," *IEEE Trans. Circuits Syst.*, vol. CAS-22, Mar. 1975.