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Adaptive systems and time varying plants

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The behaviour of four adaptive algorithms—LMS, equation error, output error identification and the Goodwin–Ramadge–Caines adaptive control algorithm—are examined in the face of plant parameter variations. The key point to note is that persistency of excitation conditions in the time invariant parameter case guarantees exponential convergence. Then standard and newly developed Lyapunov stability results allow us to guarantee tracking error and parameter error boundedness when the plant parameters are actually time varying.

1. Introduction

Most hard results in adaptive identification and control presuppose a linear plant, of known finite order, and with time invariant parameters. A number also assume that no noise is present. There is obviously a need to establish that these results extend in some way to plants where these idealizations are not exactly met, but may be approximately met. It seems particularly important to consider plants with time varying parameters, since such plants often provide the rationale for the additional complexity of adaptive methods.

In this paper, we recall and make modest developments to those results of stability theory which are robust, i.e. results which can be mildly modified when idealizations made in deriving the original results are only approximately true. Then we illustrate application of these ideas to time varying plants which are the subject of an adaptive identification or control algorithm.

The key to obtaining a robust stability result is to have uniform asymptotic stability or exponential stability. (These are equivalent in the linear case, while the second implies, but is not implied by, the first in the non-linear case.) In § 2, we describe how systems which are exponentially stable can be perturbed with the retention of some form of stability. In §§ 3, 4 and 5 we discuss the application of such ideas to equation error (including LMS) identification, output error identification, and adaptive control respectively. Of the various conclusions drawn, the most general is that the various adaptive algorithms described will behave robustly, given satisfaction of a persistency of excitation condition on inputs (for the identification problem or reference trajectory for the adaptive control problem), since it is just such a condition which guarantees exponential stability of the idealized algorithm.

The calculations are done principally to show robustness of the adaptive control algorithm in the face of plant parameter variation. We must stress however, that *the same methods with remarkably little change allow examination of the effects of measurement noise, plant non-linearity, and undermodelling of the plant order*. All one has to do is to obtain true state variable equations for an

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idealized plant, assume excitation conditions which guarantee uniform or exponential stability, and finally, show that the non-ideal character of the plant corresponds to one of the standard variations to the equations considered by the robust Lyapunov stability results.

2. Stability results

The first result basically concludes that if a certain equation is exponentially stable for all fixed values of a parameter in a compact set, it will remain exponentially stable if the parameter is allowed to vary slowly.

Theorem 2.1

Consider the system

$$x_{k+1}=f(x_k, k, p_k, u_k) \tag{2.1}$$

where k is the time index and all other vectors are real and of arbitrary dimension. Suppose that u_k is the output of the linear system, parametrized by p_k, with prespecified input $\{v_k\}$

$$\alpha_{k+1}=A(p_k)\alpha_k+B(p_k)v_k, \quad u_k=C(p_k)\alpha_k+D(p_k)v_k \tag{2.2}$$

Suppose that for all fixed $p\in\mathscr{P}$, a compact set, $A(p)$ has all eigenvalues in $|z|<1$, α_0 is bounded, and (2.1) is exponentially stable. Suppose further that f is Lipschitz in its first, third and fourth variables. Then there exists a Δ such that if $p_k\in\mathscr{P}$ and $|p_{k+1}-p_k|<\Delta$ for all k, (2.1) remains exponentially stable.

A related theorem to the above can be found in Desoer and Vidyasagar (1975, p. 126), where a continuous time system $\dot{x}=f(x, p_k)$ is considered. The generalization to a specific time dependence through other than p_k, i.e. through the second and fourth variables in (2.1) is relatively straightforward, as is the conversion to discrete time and so we omit formal proof of the theorem. We comment that though the definition of exponential stability usually connotes global exponential stability, the theorem is equally valid if x_0 is restricted to lying in some finite ball.

Note that various specializations are easily obtained; for example, if $A=B=C=0$ and $D=I$, the dependence of u_k on p_k can be removed.

The second result we quote relates exponential stability for an undriven system to bounded-input–bounded-state (BIBS) stability for a driven system. Noting that exponential stability implies uniform asymptotic stability, we have from Hahn (1967):

Theorem 2.2

Suppose the system (2.1), with u_k generated as in (2.2), is exponentially stable. *Consider the following modification to* (2.1)

$$x_{k+1}=f(x_k, k, p_k, u_k)+g(x_k, k) \tag{2.3}$$

Then for each $\epsilon>0$, there exists δ_1 and δ_2 such that $|x_k|<\epsilon$ for all $k\geqslant 0$ provided that $|x_0|<\delta_1$ and $|g(x_k, k)|<\delta_2$.

Finally, we need a new result, which extends Theorem 2.2 by effectively removing the magnitude constraint on the initial state, but at the expense of a tighter stability assumption for (2.1). To simplify notation, we shall work with

$$x_{k+1}=f(x_k, k) \tag{2.4}$$

observing that the time dependence may result from p_k, u_k, as in the statement of Theorem 2.1.

Theorem 2.3

Suppose that (2.4) is exponentially stable for all $\|x_0\| \leqslant R$ and that f is C^3 in x_k. Let $\tilde{f}(\cdot, \cdot, \cdot)$ be a function such that

$$\tilde{f}(x_k, k, 0) = f(x_k, k)$$

and

$$\|\tilde{f}(x_k, k, w_k) - f(x_k, k)\| \leqslant M\|w_k\|$$

for all k and all x_k resulting from (2.4) with $\|x_0\| \leqslant R$, and some constant M. Then for arbitrary $R_1 \epsilon (0, R)$ and arbitrary $\epsilon > 0$ there exists $\delta(R_1, \epsilon)$ such that $\|x_0\| \leqslant R_1$ and $\|w_k\| < \delta$ imply that the solution of

$$x_{k+1} = \tilde{f}(x_k, k, w_k) \tag{2.5}$$

satisfies $\|x_k\| \leqslant R' + \epsilon$, where R' is defined as

$$R' = \sup_k \{\|x_k\| : \|x_0\| < R_1, x_k \text{ defined by (2.4) for } k \geqslant 0\} \tag{2.6}$$

Further, for some N

$$\limsup_{k \to \infty} \|x_k\| < N \sup_{k \geqslant 0} \|w_k\| \tag{2.7}$$

For a proof, see Appendices.

There are two points to note regarding this result. First (and this point comes out in the proof of the result), the larger the ball in which the initial state can be, the potentially smaller the bound on the perturbation w_k must be, in order to secure boundedness of $\|x_k\|$. Second, (2.7) states that the smaller the bound on the perturbation, the smaller will be the bound on $\|x_k\|$ after initial condition effects have died away.

An important and well known special case of Theorem 2.3 applies to linear $\tilde{f}(x_k, k, w_k) = A_k x_k + B_k w_k$, with A_k and B_k bounded. Exponential convergence of $x_{k+1} = A_k x_k$ implies BIBS behaviour of $x_{k+1} = A_k x_k + B_k w_k$ (Willems 1970).

3. LMS and equation error systems

For background on LMS and equation error schemes, the reader can consult any one of a number of references, for example, Widrow *et al.* (1976), Landau (1974), Mendel (1973). The basic equation (with normalization used in the LMS case) becomes

$$x_{k+1} = \left[I - \frac{\phi_k \phi'_k}{\sigma^2 + \phi'_k \phi_k} \right] x_k \tag{3.1}$$

where x_k is the vector of errors between the true parameter and the current parameter estimate, and ϕ_k is a vector including values of the input and output sequences of the system being modelled ; the vector ϕ_k is independent of x_k. The main result on the exponential stability of (3.1) is that if the system input satisfies a persistency and richness condition and if certain other more technical conditions hold, (3.1) is exponentially stable (Widrow 1976, Landau 1974, Mendel 1973, Anderson and Johnson 1982).

In a number of references, certain variations of (3.1) are examined. These variations stem typically from time variations of the true parameter vector (i.e. the system being identified is not stationary), from corruption of the measurements by noise, and from errors in the modelling assumptions. Examples of the latter are an assumption that the plant has a finite impulse response when it does not, or that the plant has a transfer function with denominator degree 5 when the degree is actually 6 and so on. Time variation of the true parameter vector leads to (3.1) being modified as, for example, in Bitmead and Anderson (1980)

$$x_{k+1}=\left[I-\frac{\phi_k\phi'_k}{\sigma^2+\phi'_k\,\phi_k}\right]x_k-(w_{k+1}{}^*-w_k{}^*) \tag{3.2}$$

where $w_k{}^*$ is the true parameter vector at time k.

In an LMS scheme, the vector ϕ_k comprises plant inputs alone, and accordingly does not involve $w_k{}^*$. It follows that if the input satisfies a persistently exciting condition, so that (3.1) is exponentially stable, then (3.2) is BIBS with $w_{k+1}{}^*-w_k{}^*$ regarded as the input. (See the comment at the end of the last section.) With (3.2), x_k will not approach zero, but a small l_∞ bound on $w_{k+1}{}^*-w_k{}^*$, i.e. a small rate of time variation, will imply a small l_∞ bound for x_k, i.e. a small error in estimating the parameter. A trivial calculation shows that if the transition matrix of (3.1) obeys for some $K>0$, $\alpha<1$

$$\|\Phi(k,k_0)\|\leqslant K\alpha^{k-k_0},\quad k\geqslant k_0$$

then in (3.2)

$$\limsup_{k\to\infty}\|x_k\|\leqslant\frac{K}{1-\alpha}\|w_{k+1}{}^*-w_k{}^*\|_\infty \tag{3.3}$$

In a similar manner, one may consider in more detail the other variations in the modelling assumptions. *The key idea in each case is to relate the exponential stability of* (3.1) *to the BIBS stability of a related equation.*

In an equation error scheme, the solution is slightly more complicated. The vector ϕ_k includes both plant inputs and outputs. Therefore, for a given input sequence, ϕ_k will depend on the plant parameters, and varying $w_k{}^*$ will give a different ϕ_k. How is this resolved? It is known that when w^* is constant and the plant is asymptotically stable, a persistently exciting condition on the plant input (which is independent of w^*) implies exponential stability of x_k. It follows by Theorem 2.1 that if $w_k{}^*$ is sufficiently slowly varying in a compact set and such that all points in this set correspond to stable plants, and if the plant input satisfies the persistently exciting condition, (3.1)—where now ϕ_k is computed using the varying $w_k{}^*$—will be exponentially stable and again, (3.2) will be BIBS.

The formal connection with Theorem 2.1 is as follows: the quantities u_k, ϕ_k and $w_k{}^*$ in relation to (3.1) correspond to v_k, u_k and p_k in relation to (2.1) and (2.2). The quantity x_k in (3.1) corresponds to x_k in (2.1) and (3.1) corresponds to (2.1) without explicit dependence on k and p_k.

Note that it is *not* guaranteed that (3.2) will be BIBS for arbitrary fast variation of $w_k{}^*$ (even if the persistently exciting condition still holds); this is in contrast to what happens in the LMS scheme.

4. Output error identification

In Bitmead and Anderson (1980), the equations for output error identification are written in the following form

$$\begin{bmatrix} e_{k+1} \\ \tilde{\theta}_{k+1} \end{bmatrix} = \begin{bmatrix} A - \dfrac{x'_k \, \Gamma x_k b h'}{1 + dx'_k \, \Gamma x_k} & \dfrac{bx'_k}{1 + d'_k \, \Gamma x_k} \\ \dfrac{-\Gamma x_k h'}{1 + dx'_k \, \Gamma x_k} & I - \dfrac{d \Gamma x_k x'_k}{1 + dx'_k \, \Gamma x_k} \end{bmatrix} \begin{bmatrix} e_k \\ \tilde{\theta}_k \end{bmatrix} \tag{4.1}$$

where $d + h'(zI - A)^{-1}b$ is strictly positive real, Γ is positive definite, e_k is a vector containing the error between the true system and the identifier outputs at a number of instants of time, $\tilde{\theta}_k$ is a vector containing the errors between the true system parameters and the identifiers estimates of these parameters, and x_k is a vector containing the identifier input and output at a number of instants of time.

To cast (4.1) in the form allowing application of the ideas of § 2, x_k may be rewritten (see Johnson and Anderson 1982) as

$$x_k^{\mathrm{T}} = [y_{k-1} \cdots y_{k-n} \,\vdots\, u_{k-1} \cdots u_{k-m}] - [e_k \,\vdots\, 0] \tag{4.2}$$

where u_k is the true system and identifier input at time k, and y_k is the true system output at time k. Then the coefficient matrix in (4.1) is recognized as also involving e_k. Also (4.1) is in the form of (2.1), with the correspondences

Equation (2.1)	*Equation* (4.1)
x_k	$[e_k^{\mathrm{T}} \ \tilde{\theta}_k^{\mathrm{T}}]^{\mathrm{T}}$
p_k	parameters of system being identified
u_k	$[y_{k-1} \cdots y_{k-n} \ u_{k-1} \cdots u_{k-m}]^{\mathrm{T}}$

Obviously, (4.1) is non-linear.

In Anderson and Johnson (1982), it was proved that under a persistency of excitation condition on the true system input and certain other technical conditions, (4.1) is exponentially stable for all initial conditions in an arbitrarily large ball. Now consider the effect of plant parameter variations ; (4.1) is affected in several ways—first, $[y_{k-1} \cdots y_{k-n} \ u_{k-1} \cdots u_{k-m}]$ is affected by plant parameter variations, analogously to the dependence in (2.2) of u_k on p_k Secondly, A and b in (4.1) (which depend on the parameters of the true system) are affected by the plant parameter variations, analogously to the dependence in (2.1) of f on p_k. These two variations to (4.1) would allow us to retain the conclusion of exponential stability in the face of slow enough variations using Theorem 2.1. However, there is a third adjustment to (4.1) : we must add to the right hand side

$$\begin{bmatrix} 0 \\ p_k - p_{k+1} \end{bmatrix}$$

(analogously to the equation error situation). Since this quantity is smaller the slower the rate of plant parameter variation, we invoke Theorem 2.3 to conclude finally that for sufficiently slow plant parameter variation, we can guarantee boundedness of e_k and $\tilde{\theta}_k$, with moreover

$$\limsup_{k\to\infty} \|[e_k^{\mathrm{T}} \ \tilde{\theta}_k^{\mathrm{T}}]\| \leqslant N \sup_{k\geqslant 0} \|p_k - p_{k+1}\| \tag{4.3}$$

for some N.

We note that this argument is quite similar to a detailed argument in Johnson and Anderson (1982) discussing reduced order modelling.

5. Adaptive control

To illustrate the application of the ideas of § 2 we shall work out in detail the result for plant parameter variation in the case of an adaptive control problem requiring the following of a reference trajectory. The basic conclusion should be clear ; if the adaptive algorithm is exponentially convergent with no plant parameter variation, then there should be a bounded error when plant parameter variation is present. We shall work in terms of one of the Goodwin–Ramadge–Caines algorithms (Goodwin *et al.* 1980) ; the technique should however be transferable without conceptual difficulty to other algorithms.

The first, *and by far the major part*, of the task, is to set up a true state variable description of the adaptive control algorithm, which does not seem to have been done before. The second part is to appeal to the robustness results of § 2 to extend convergence results for time invariant plants to time varying plants ; the application of these results of course presupposes the availability of state variable equations.

Review of the Goodwin–Ramadge–Caines algorithm

Suppose the plant has a delay of d units, with equation

$$y_t = -a_1 y_{t-1} \dots - a_n y_{t-n} + b_d u_{t-d} + \dots + b_m u_{t-m} \tag{5.1}$$

For the moment, assume the plant is time invariant. Now by a standard device, we can find a polynomial $F(z^{-1}) = 1 + f_1 z^{-1} + \dots + f_{d-1} z^{-(d-1)}$ such that for some α_i

$$F(z^{-1})(1 + a_1 z^{-1} + \dots + a_n z^{-n}) = 1 + \alpha_d z^{-d} + \dots + \alpha_{n+d-1} z^{-(n+d-1)} \tag{5.2}$$

and then with β_i defined by

$$F(z^{-1}) \sum_{j=d}^{m} b_j z^{-j} = \sum_{j=d}^{m+d-1} \beta_j z^{-j} \tag{5.3}$$

we obtain

$$y_{t+d} = \phi'_t \, \theta_0 \tag{5.4}$$

where, as usual

$$\theta_0 = [-\alpha_d \dots -\alpha_{n+d-1} \ \beta_d \dots \beta_{m+d-1}]'$$

and

$$\phi_t = [y_t \dots y_{t-n+1} \ u_t \dots u_{t-m+1}]'$$

Also let $\hat{\theta}_t = [-\hat{\alpha}_{d,t} \dots -\hat{\alpha}_{n+d-1,t} \ \hat{\beta}_{d,t} \dots \hat{\beta}_{m+d-1,t}]'$ denote an estimate of θ_0, available at time t *just after* the reception of y_t and *just before* the generation of u_t. Suppose that in this estimate, $\hat{\beta}_{d,t}$ is guaranteed non-zero. There are two key steps in the algorithm. The first is to use the equation $y_{t+d}^* = \phi_t' \, \hat{\theta}_t$

as a definition for u_t, i.e.

$$u_t = \frac{1}{\hat{\beta}_{d,t}} \left[y_{t+d}^* + \sum_{j=0}^{n-1} \hat{\alpha}_{d+j,t} y_{t-j} - \sum_{j=1}^{m-1} \hat{\beta}_{d+j,t} u_{t-j} \right]$$
$$= \frac{1}{\hat{\beta}_{d,t}} \left[y_{t+d}^* + \sum_{j=1}^{n-1} \hat{\alpha}_{d+j,t} y_{t-j} - \sum_{j=1}^{m-1} \hat{\beta}_{d+j,t} u_{t-j} + \hat{\alpha}_{d,t} \left(- \sum_{j=1}^{n} a_j y_{t-j} + \sum_{j=d}^{m} b_j u_{t-j} \right) \right] \tag{5.5}$$

with the second equality following from (5.1).

The second step of the algorithm is to update the estimate of $\hat{\theta}_t$ upon reception of y_{t+1}. One sets

$$\hat{\theta}_{t+1} = \hat{\theta}_t + \frac{\nu_t \phi_{t-d+1}}{1 + \|\phi_{t-d+1}\|^2} (y_{t+1} - \phi'_{t-d+1} \hat{\theta}_t)$$

where ν_t is chosen according to certain rules which ensure that $\hat{\beta}_{d,t+1} \neq 0$ (Goodwin *et al.* 1980). The rules make ν_t a discontinuous function of ϕ_{t-d+1}, $\hat{\theta}_t$ and $y_{t+1} - \phi'_{t-d+1} \hat{\theta}_t = \phi'_{t-d+1}(\theta_0 - \hat{\theta}_t)$; we shall assume the definition is adjusted to make ν_t a continuously differentiable function of its arguments. With $\tilde{\theta}_t = \hat{\theta}_t - \theta_0$, this means that

$$\tilde{\theta}_{t+1} = \tilde{\theta}_t - \frac{\nu_t(\tilde{\theta}_t, \theta_0, \phi_{t-d+1}) \phi_{t-d+1}}{1 + \|\phi_{t-d+1}\|^2} \phi'_{t-d+1} \tilde{\theta}_t \tag{5.6}$$

with ν_t smooth ; also the rules for ν_t ensure that for some $\epsilon > 0$, $\epsilon < \nu_t < 1 - \epsilon$.

Obtaining the state-variable equation for time-invariant plants

Equation (5.6) is part of the state variable equation set. Because ϕ_{t-d+1} involves u_{t-d+1}, which depends on earlier values of $\hat{\theta}_{t-d+1}$ and earlier u_i than u_{t-d+1} (which in their turn depend on earlier $\hat{\theta}_i$), (5.6) above is not acceptable as a full state variable description of the system. (Nor of course is it a linear equation.)

Let $\bar{\phi}_t(\theta_0)$ be the vector ϕ_t which would be obtained if we were to have $\hat{\theta}_t = \theta_0$ for all t, i.e. if no adaption were necessary, and set

$$e_t = \phi_t - \bar{\phi}_t(\theta_0) \tag{5.7}$$

Obviously, one of the adaptive control goals is to have $\phi_t - \bar{\phi}_t$ tend to zero implying that the inputs and outputs of the adapting plant should approach those of a completely adapted plant. Then, as shown in the Appendices, we can write

$$e_t = \bar{A}(\theta_0) e_{t-1} + i_{n+1} [f(\tilde{\theta}_t + \theta_0) - f(\theta_0)]' [\bar{\phi}_{t-1}(\theta_0) + e_{t-1}] + i_{n+1} [g(\tilde{\theta}_t + \theta_0) - g(\theta_0)] y_{t+d}^* \tag{5.8}$$

and

$$\bar{\phi}_t = \bar{A}(\theta_0) \bar{\phi}_{t-1} + i_{n+1} \frac{1}{\beta_d} y_{t+d}^* \tag{5.9}$$

In these two equations, $\bar{A}(\theta_0)$ is a constant matrix, i_{n+1} denotes a unit vector with 1 in position $n+1$, and f and g are certain continuously differentiable functions of their arguments, so long as $\hat{\beta}_{t,d} \neq 0$. With θ_0 fixed, $\{y_{t+d}^*\}$ prespecified and an arbitrary but fixed initial condition on $\bar{\phi}_t$, $\{\bar{\phi}_t\}$ is a fixed sequence.

The relevant equations we need are (5.8) and (5.6) rewritten as

$$\tilde{\theta}_{t+1} = \tilde{\theta}_t - \frac{\nu_t(\tilde{\theta}_t, \theta_0, \bar{\phi}_{t-d+1} + e_{t-d+1})(\bar{\phi}_{t-d+1} + e_{t-d+1})(\bar{\phi}_{t-d+1} + e_{t-d+1})'}{1 + \|\bar{\phi}_{t-d+1} + e_{t-d+1}\|^2} \tilde{\theta}_t \quad (5.10)$$

Together, (5.8) and (5.10) specify, for some m, n

$$e_t = m(\tilde{\theta}_t, e_{t-1}, t), \quad \tilde{\theta}_{t+1} = n(\tilde{\theta}_t, e_{t-d+1}, t)$$

From these, we can get a proper state variable equation set with state vector $[e'_{t-d+2} \; e'_{t-d+3} \ldots e'_t \; \tilde{\theta}'_{t+1}]'$

$$\begin{bmatrix} e_{t-d+2} \\ e_{t-d+3} \\ \vdots \\ e_{t-1} \\ e_t \\ \tilde{\theta}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \ldots 0 & 0 \\ 0 & 0 & 1 \ldots 0 & 0 \\ & & \vdots & \\ 0 & 0 & 0 \ldots 1 & 0 \\ 0 & 0 & 0 \ldots 0 & 0 \\ 0 & 0 & 0 \ldots 0 & 0 \end{bmatrix} \begin{bmatrix} e_{t-d+1} \\ e_{t-d+2} \\ \vdots \\ e_{t-2} \\ e_{t-1} \\ \tilde{\theta}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ m(\tilde{\theta}_t, e_{t-1}, t) \\ n(\tilde{\theta}_t, e_{t-d+1}, t) \end{bmatrix} \quad (5.11)$$

Remarks concerning the state-variable equations

We note the following points.

(1) In Anderson and Johnson (1982), exponential convergence of $\tilde{\theta}_t$ and e_t to zero has been established for initial conditions in an arbitrarily large but finite ball under the assumptions that y_t^* is persistently exciting (as defined in Anderson and Johnson (1982)), that the plant (5.1) is minimum phase, and that there is no common zero of $z^n + a_1 z^{n-1} + \ldots + a_n$ and $z^m + b_1 z^{m-1} + \ldots + b_m$.

(2) The right hand sides of (5.8) and (5.10) have continuous derivatives with respect to any variable appearing of arbitrary order, provided $\alpha(\cdot, \cdot, \cdot)$ is smooth.

Equations given variation of the plant parameters

Suppose now that the plant parameters are time varying with (5.1) replaced by

$$y_t = -\sum_{j=1}^{n} a_{j,t} y_{t-j} + \sum_{j=d}^{m} b_{j,t} u_{t-j} \quad (5.12)$$

with a similar replacement for (5.4). Then with

$$\theta_{0,t} = [-\alpha_{d,t} \ldots -\alpha_{n+d-1,t} \; \beta_{d,t} \ldots \beta_{m+d-1,t}]'$$

we obtain easily the following variations to (5.8) and (5.10)

$$e_t = \bar{A}(\theta_{0,t})e_{t-1} + i_{n+1}[f(\tilde{\theta}_t + \theta_{0,t}) - f(\theta_{0,t})][\hat{\phi}_{t-1} + e_{t-1}] + i_{n+1}[g(\tilde{\theta}_t + \theta_{0,t}) - g(\theta_{0,t})]y_{t+d}^* \quad (5.13)$$

$$\tilde{\theta}_{t+1} = \tilde{\theta}_t - \frac{\nu_t(\tilde{\theta}_t, \theta_{0,t}, \hat{\phi}_{t-d+1} + e_{t-d+1}) \times (\hat{\phi}_{t-d+1} + e_{t-d+1})(\hat{\phi}_{t-d+1} + e_{t-d+1})'}{1 + \|\hat{\phi}_{t-d+1} + e_{t-d+1}\|^2} \tilde{\theta}_t + (\theta_{0,t} - \theta_{0,t+1}) \quad (5.14)$$

with an obvious consequential variation to (5.11). The quantity $\hat{\phi}_{t-d+1}$ is not the same as $\bar{\phi}_{t-d+1}$; the latter quantity depends on y_{t+d}^* and the particular plant parameters—see (5.9). We have in fact

$$\hat{\phi}_t = \bar{A}(\theta_{0,t})\hat{\phi}_{t-1} + i_{n+1} \frac{1}{\beta_{d,t}} y_{t+d}^* \quad (5.15)$$

The changes from the pair (5.8) and (5.10) exhibited in (5.13) and (5.14) are of three kinds: (i) $\bar{\phi}_{t-d+1}$ is replaced by $\hat{\phi}_{t-d+1}$, (ii) certain formerly time invariant parameters have become time varying and (iii) an additive input term, viz. $\theta_{0,t} - \theta_{0,t+1}$, appears.

As was argued for the output error identification problem, for sufficiently slow plant parameter variation, we are guaranteed that e_k and $\tilde{\theta}_k$, the output tracking and parameter estimation errors, will remain bounded, with

$$\limsup_{k\to\infty} \|[e_k^{\mathrm{T}} \ \ \theta_k^{\mathrm{T}}]\| \leqslant N \sup_{k\geqslant 0} \|\theta_{0,k} - \theta_{0,k+1}\| \quad (5.16)$$

for some constant N. Further, the larger the initial error, the slower the plant parameter rate of variation may need to be for (5.16) to hold.

6. Conclusion

We have stated several robust Lyapunov stability results. Next, having pointed out that a number of idealized adaptive algorithms are exponentially stable given some form of persistency of excitation condition, we have shown the capability of such algorithms to perform robustly in the case of variations from the idealizations, and in particular, we have looked at time variation in the plant parameters, especially for adaptive control.

An important idea is that, in general, the less well initialized an adaptive scheme is, the less tolerance there will be of variations from the idealizations.

7. Appendices

7.1. *Proof of Theorem* 2.3

The proof relies on the following lemma.

Lemma A 1

Under the same hypotheses as Theorem 2.3, the linearization of (2.4) around an arbitrary trajectory commencing in $\|x_0\| < R$ is exponentially stable.

Proof

Rewrite (2.4) as $x_{k+1} = A_k x_k + B(x_k, h)x_k$ where A_k is independent of x_k and $\|B(x_k, h)\| < M\|x_k\|$ for some M. Because (2.4) is exponentially stable, it is easily seen that $x_{k+1}=A_k x_k$ is exponentially stable. Now let $x_0+\lambda_0, x_1+\lambda_1, \ldots$ denote a trajectory of (2.3) which is neighbouring to $x_0, x_1, \ldots$. Then

$$\begin{aligned}\lambda_{k+1} &= A_k\lambda_k + B(x_k+\lambda_k, k)(x_k+\lambda_k) - B(x_k, k)x_k \\ &= A_k\lambda_k + B(x_k+\lambda_k, k)\lambda_k + [B(x_k+\lambda_k, k) - B(x_k, k]x_k \end{aligned} \qquad \text{(A 1)}$$

Since $x_k+\lambda_k \to 0$, $B(x_k+\lambda_k, k)\to 0$. Since f is C^3, B is C^1 in its first argument, and so

$$\|B(x_k+\lambda_k, k) - B(x_k, k)\| \leqslant M\|\lambda_k\|$$

for some constant M; since x_k tends to zero, the linearization of (A 1) is accordingly of the form

$$\lambda_{k+1} = A_k\lambda_k + B_k\lambda_k + C_k\lambda_k$$

where B_k tends to zero and C_k tends to zero as k tends to infinity. Accordingly, it is exponentially stable since $x_{k+1}=A_k x_k$ has this property (Willems 1970).

Proof of Theorem 2.3

For each x_0 with $\|x_0\|<R$, the linearization of (2.5) around the trajectory $w_k\equiv 0$ is exponentially stable. By Theorem 2.2 there exists $\delta_1(\epsilon, x_0)$ and $\delta_2(\epsilon, x_0)$ such that if (2.5) has an initial condition in a ball of centre x_0 and radius $\delta_1(\epsilon, x_0)$ and if $\|w_k\|<\delta_2(\epsilon, x_0)$, then x_k deviates by at most ϵ from the trajectory with initial condition x_0 and $w_k\equiv 0$. Since the ball $\|x_0\|<R_1$ is a bounded, open set, there exists a finite covering by the balls of radius δ_1. Let $x_0{}^1, \ldots, x_0{}^r$ be the centres of such balls; choose $\bar{\delta}_2 = \min_i \delta_2(\epsilon, x_0{}^i)$. Then for all x_0 with $\|x_0\|<R$, and with $\|w_k\|<\bar{\delta}_2$, (2.5) has a trajectory that is within ϵ of a trajectory of (2.4).

Equation (2.7) follows by using a result on linearization in Desoer and Vidyasagar (1975, pp. 128–131).

7.2. *Equations for ϕ_t*

If $\hat{\theta}_t=\theta_0$ for all t, then (5.5) holds with the correct values substituted for all the estimates. Then (5.1) and this modified (5.5) imply, with

$$\bar{A} = \left[\begin{array}{ccccc|ccccc}
-a_1 & -a_2 & & & -a_n & 0 & \ldots & 0 & b_d & \ldots\ b_m \\
1 & 0 & & & & 0 & \ldots & 0 & 0 & \ldots\ 0 \\
\vdots & & & & & \vdots & & \vdots & \vdots & \vdots \\
\vdots & 1 & & & & \vdots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & & & & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & & 1 & 0 & 0 & \ldots & 0 & 0 & \ldots\ 0 \\
\hline
\dfrac{\alpha_{d+1}-\alpha_d a_1}{\beta_d} & \dfrac{\alpha_{d+2}-\alpha_d a_2}{\beta_d} & \ldots & \dfrac{\alpha_{d+n-1}-\alpha_d a_{n-1}}{\beta_d} & \dfrac{-\alpha_d a_n}{\beta_d} & \dfrac{-\beta_{d+1}}{\beta_d} & \ldots & \dfrac{-\beta_{2d-1}}{\beta_d} & \dfrac{-\beta_{2d}+\alpha_d b_d}{\beta_d} & \ldots\ \dfrac{\alpha_d b_m}{\beta_d} \\
\vdots & \vdots & & & & & & & & \vdots \\
\vdots & \vdots & & & & 1 & & & & \vdots \\
\vdots & \vdots & & & & & \ddots & & & \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & & 1 & 0
\end{array}\right]$$

and

$$\bar{\phi}_t = [y_t \dots y_{t-n+1} \; u_t \dots u_{t-m+1}]'$$

that

$$\bar{\phi}_t = \bar{A}\bar{\phi}_{t-1} + i_{n+1}\frac{y_{t+d}{}^*}{\beta_d} \tag{5.9}$$

The update equation for ϕ_t is the same, except that the $(n+1)$th row of $\bar{A}$ is varied by the inclusion of estimates of the α_i, β_j, see (5.5). Denote the $(n+1)$th row of $\bar{A}$ by $f(\theta_0)$ and $\beta_d{}^{-1}$ by $g(\theta_0)$. Then

$$\phi_t = \bar{A}\phi_{t-1} + i_{n+1}[f(\tilde{\theta}_t + \theta_0) - f(\theta_0)]'\phi_{t-1} + i_{n+1}g(\tilde{\theta}_t + \theta_0)y_{t+d}{}^*$$

and (5.8) is immediate.

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