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A PHYSICAL BASIS FOR KREIN'S PREDICTION FORMULA*

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A prediction problem of the following variety is considered. A stationary random process $w(\cdot)$ of known spectrum is observed over $|t| \leq a$. Using these observed values, $w(b)$ is to be predicted for some b with $|b| > a$. We present a physical interpretation of a solution to this problem due to Krein, which used the theory of inverse Sturm–Liouville problems. Our physical model involves a nonuniform lossless transmission line excited at one end by white noise. The signal at the other end is the process $w(t)$, and the prediction is found by calculating as intermediate quantities the voltage and current stored on the line at $t = 0$. These quantities are spatially uncorrelated and provide a spatial representation at $t = 0$ of the innovations of $w(t)$ over $|t| \leq a$.

Stationary processes   prediction
random process modelling

1. Introduction

The subject of this paper is largely another paper [6] of Krein. This other paper considers what is now viewed as a fairly standard problem: given a partial time record over an interval of a stationary process, predict (with minimum mean square error) the value that the process will take at some time outside the interval for which the record is available. Krein's solution of this problem is noteworthy for two reasons. First, the solution is of a form allowing ready adjustment of the end-points of the record interval as well as of the time argument of the estimate. Second, the solution uses ideas that are largely foreign to work in random processes, in particular Sturm–Liouville equations, their eigenfunctions and an associated inverse problem. Even the form of the solution looks quite unlike any other solution of the same problem, and is mildly surprising in that the estimate is usually displayed via two separate linear operators acting on the data, rather than one.

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Krein gives some physical significance to his formulas in terms of a nonuniform string. However, the physical characterization never goes as far as associating a random process with the string, and so can only be considered partial.

The main aim of this paper is to provide a physical basis for Krein's formula in terms of a nonuniform lossless transmission line, with the physical basis including a random process. A string could doubtless also have been used. So also could an acoustic line; the acoustic line model actually makes contact with a now large body of literature dealing with the technologically important area of speech recognition and synthesis, see e.g. [8] for a survey of many of the ideas, including ideas related to inverse problems, and filtering and prediction of random processes.

In the broadest of outline, we verify that the Krein formulas correspond to doing certain calculations on the quantities in the transmission line model. (Thus an independent proof of the Krein formula is not being offered; rather, an interpretation is being given.) Then we give an argument independent of the Krein formula as to why the calculations on the transmission line model quantities provide the desired prediction. If desired, this can be regarded as an independent proof of the Krein formula.

Our purpose is to explain, rather than to prove under the weakest set of smoothness or other assumptions. Since many of the prospective audience will be deficient in knowledge in one or more of the relevant background areas, e.g. transmission lines, Sturm–Liouville theory, partial differential equations, the paper has a more tutorial flavour than is common.

The paper is structured in the following way. In Section 2, Krein's solution is stated, and in Section 3, the physical interpretation of Krein's solution is presented. Sections 2 and 3 are essentially summary in nature. Section 4 is more tutorial: it provides various descriptions, including the Sturm–Liouville description, of a nonuniform transmission line. Given the summary of Sections 2 and 3 and the background of Section 4, Sections 5 and 6 verify the physical interpretation stated in Section 3. Section 7 argues the prediction formula of Krein from first principles, while Section 8 contains concluding remarks.

2. Krein's prediction formula

Imagine a real stationary random process $w(t)$ with known spectrum, and suppose that the process is known over $|t| \leq a$. An estimate of $w(b)$ is required in terms of the known values of the process for some b with $|b| > a$. Krein [6] offers a formula for this prediction.

The statement of Krein's prediction formula embodies three ideas:

(a) Construction of a Sturm–Liouville equation with spectrum (in the differential equation sense) equal to the prescribed spectrum (which is the spectrum of the random process).

(b) Construction of eigenfunctions of the Sturm–Liouville equation.

(c) The stating of a formula for the prediction in a space isomorphic to the Hilbert space spanned by the random process; the formula involves the eigenfunctions of the Sturm–Liouville equation.

We shall now review these ideas in detail, without at this stage, explaining why the Krein result is true.

2.1. The inverse Sturm–Liouville problem and its eigenfunctions

Consider the differential equation for $x \in [0, \infty)$

$$d^2y/dx^2 + [\lambda^2 - q(x)]y = 0 \tag{2.1}$$

with boundary conditions

$$y(0) = 1, \qquad y'(0) = h. \tag{2.2}$$

The function $q(x)$ is assumed to be continuous. Let $\phi(x, \lambda)$ be the solution. By analogy with the case when $q(x) = 0$ for all x, $h = 0$, so that $\phi(x, \lambda)$ is $\cos \lambda x$, we define for arbitrary $f \in L^2[0, \infty)$ a generalized transform

$$\mathbf{E}(\lambda) = \int_0^\infty f(x)\phi(x, \lambda)\, dx. \tag{2.3}$$

(Some care needs to be taken in defining the infinite integral, naturally.) One can show (and it is standard result [3]) that there exists a unique monotonic function $\rho(\lambda)$, bounded on each finite interval, such that $\rho(\lambda)$ provides a Parseval theorem:

$$\int_0^\infty f^2(x)\, dx = \int_{-\infty}^{+\infty} \mathbf{E}^2(\lambda)\, d\rho(\lambda). \tag{2.4}$$

(Again, some care is needed in defining the infinite integrals.) The function $\phi(x, \lambda)$ is termed an eigenfunction of (2.1) and $\rho(\lambda)$ is termed the spectrum of (2.1). The inverse Sturm–Liouville problem is: given $\rho(\lambda)$ (with suitable constraints) find $q(x)$ and h. The solution to this problem is now considered standard. For a discussion, see e.g. [3]. It is clear that, given $\rho(\lambda)$ and ability to solve the inverse problem, the eigenfunctions $\phi(x, \lambda)$ are in principle known.

Two other points can be made. First, the eigenfunctions satisfy an orthogonality condition

$$\int_{-\infty}^\infty \phi(y, \lambda)\phi(z, \lambda)\, d\rho(\lambda) = \delta(y - z). \tag{2.5}$$

This can be obtained formally from the Parseval theorem in the following way. First, by writing the Parseval theorem for $f_1(x) + f_2(x)$ and $f_1(x) - f_2(x)$, one derives

$$\int_0^\infty f_1(x) f_2(x)\, dx = \int_0^\infty E_1(\lambda) E_2(\lambda)\, d\rho(\lambda).$$

Then take $f_1(x) = \delta(x - y)$, $f_2(x) = \delta(x - z)$, and use (2.3).

Second, we shall have occasion to use functions $\psi(x, \lambda)$, defining them by

$$\psi(x, \lambda) = \frac{1}{\lambda}\phi(x, 0)\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\phi(x, \lambda)}{\phi(x, 0)}\right]. \tag{2.6}$$

It turns out that these functions satisfy a different, but related, Sturm–Liouville equation, with

$$\psi(0, \lambda) = 0, \qquad \psi'(0, \lambda) = -\lambda. \tag{2.7}$$

We will clarify the connection between (2.1) and the equation satisfied by the $\psi(x, \lambda)$ in a later section.

2.2. A standard isomorphism

Let $w(t)$, $-\infty < t < \infty$ be a real stationary process. We define a Hilbert space $\mathcal{H}$ via a more or less standard procedure: $\mathcal{H}$ consists of finite sums of the form

$$\sum_{i=1}^{n} a_i \int_{t_i}^{s_i} w(t)\,\mathrm{d}t$$

with $-\infty < t_i, s_i < \infty$, a_i real constants, together with limits of such random variables. The limit is defined using the norm

$$\|z\|_{\mathcal{H}}^2 = \mathbf{E}(z^2) \tag{2.8}$$

which induces the inner product

$$\langle z_1, z_2\rangle_{\mathcal{H}} = \mathbf{E}(z_1 z_2). \tag{2.9}$$

Suppose that

$$\mathbf{E}[w(t)w(t+\tau)] = \delta(\tau) + K(\tau) \tag{2.10}$$

where $K(\tau)$ is a smooth kernel, and let $\rho(\lambda)$ be the associated spectrum. Thus the standard relation

$$\delta(\tau) + K(\tau) = \lim_{N\to\infty}\int_{-N}^{N} \mathrm{e}^{\mathrm{j}\lambda\tau}\,\mathrm{d}\rho(\lambda) = \lim_{N\to\infty}\int_{-N}^{N} \cos\lambda\tau\,\mathrm{d}\rho(\lambda) \tag{2.11}$$

holds.

Now consider the mapping

$$w(t) \leftrightarrow \mathrm{e}^{\mathrm{j}\lambda t}, \quad \mathrm{j} = \sqrt{-1} \tag{2.12}$$

so that

$$\sum_{i=1}^{n} a_i \int_{t_i}^{s_i} w(t)\,\mathrm{d}t \leftrightarrow \sum_{i=1}^{n} a_i \frac{\mathrm{e}^{\mathrm{j}\lambda s_i} - \mathrm{e}^{\mathrm{j}\lambda t_i}}{\mathrm{j}\lambda}. \tag{2.13}$$

On the left-hand side in (2.12) t is a fixed argument, while on the right-hand side, λ is a running variable. Thus random variables are being mapped into analytic functions. If under this mapping, the random variables z_1, z_2 map into $f_1(\lambda)$, $f_2(\lambda)$, then one can verify that

$$\mathbf{E}[z_1 z_2] = \int_{-\infty}^{+\infty} f_1(\lambda) f_2^*(\lambda)\,\mathrm{d}\rho(\lambda) \tag{2.14}$$

and so an isomorphism is set up between the Hilbert space $\mathscr{H}$ and the Hilbert space $\tilde{\mathscr{H}}$ of functions $f(\lambda)$, satisfying

$$\|f\|^2_{\tilde{\mathscr{H}}} = \int_{-\infty}^{+\infty} |f(\lambda)|^2\,\mathrm{d}\rho(\lambda) < \infty. \tag{2.15}$$

Of course,

$$\langle f_1, f_2 \rangle_{\tilde{\mathscr{H}}} = \int_{-\infty}^{+\infty} f_1(\lambda) f_2^*(\lambda)\,\mathrm{d}\rho(\lambda). \tag{2.16}$$

2.3. *The projection problem*

The task of estimating $w(b)$ via $w(t)$, $|t| \leq a$, is the same as the task of projecting $w(b)$ onto the subspace of $\mathscr{H}$ spanned by $w(t)$, $|t| \leq a$. By virtue of the isomorphism set up above, this is identical with the task of projecting the function $\mathrm{e}^{\mathrm{j}\lambda b}$ onto the subspace of $\tilde{\mathscr{H}}$ spanned by the functions $\mathrm{e}^{\mathrm{j}\lambda t}$, $|t| \leq a$ (or equivalently, $\cos \lambda t$, $\sin \lambda t$ for $|t| \leq a$). Of course, the projection has to be a linear combination of the functions spanning the subspace, and the error has to be orthogonal to the subspace, i.e. the inner product of the error with any one of the functions which generate the subspace must be zero.

2.4. *The Krein solution*

With $w(t)$ and $K(t)$ as in Section 2.2, the spectrum $\rho(\lambda)$ is obtained and from it, the associated Sturm–Liouville equation, and thence the eigenfunctions $\phi(x, \lambda)$ and associated quantities $\psi(x, \lambda)$. Then the formula yielding the projection of $w(b)$ on $w(t)$, $|t| \leq a$, is presented in the isomorphic Hilbert space $\tilde{\mathscr{H}}$ as a formula for projecting $\mathrm{e}^{\mathrm{j}\lambda b}$ on $\mathrm{e}^{\mathrm{j}\lambda t}$, $|t| \leq a$. It is

$$P_a(\mathrm{e}^{\mathrm{j}\lambda b}) = \int_0^a \left[\int_{-\infty}^{+\infty} \cos \sigma b \phi(\sigma, z)\,\mathrm{d}\rho(\sigma)\right] \phi(\lambda, z)\,\mathrm{d}z + \mathrm{j}\int_0^a \left[\int_{-\infty}^{+\infty} \sin \sigma b \psi(\sigma, z)\,\mathrm{d}\rho(\sigma)\right] \psi(\lambda, z)\,\mathrm{d}z \tag{2.17}$$

Remark. It is not obvious that the right-hand side is contained in the subspace generated by $\mathrm{e}^{\mathrm{j}\lambda t}$, $|t| \leq a$. That this is so follows from the nonobvious fact that $\phi(\lambda, z)$ and $\psi(\lambda, z)$ as functions of λ are in the subspace generated by $\mathrm{e}^{\mathrm{j}\lambda t}$, $|t| \leq a$ (see [3]).

3. Physical description of the Krein formula

The purpose of this section is to indicate, in summary form, a physical basis for the Krein formula. Detailed derivation is left to a later section. We note that the relation of nonuniform lossless transmission lines with the inverse Sturm–Liouville problem is discussed in [11] and with an impedance modelling problem in [4].

3.1. Obtaining equal random process and differential equation spectra in the one physical object

An early point of puzzlement in the Krein approach is that two different concepts of spectra are in one sense married. This can be done physically in the following way. A nonuniform lossless transmission line has associated with it a Sturm–Liouville equation, in fact two such equations; if all excitations are sinusoidal in time, the variation of voltage and current along the line are described by related Sturm Liouville equations. Of course, there is a spectrum associated with each equation; we shall focus on the voltage equation, and its spectrum.

Next, suppose the line is finite, of electrical length L greater than b. Leave its left-hand end, corresponding to $x=0$, open-circuit, and terminate its right-hand end, corresponding to $x=L$, in a resistor, with a series white noise voltage source (see Fig. 1). (Any R ohm resistor at temperature T°A has a zero mean noise voltage

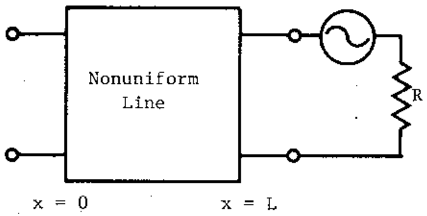

Fig. 1.

which is usually modelled as gaussian, with correlation function $2kTR\delta(t)$, k being Boltzmann's constant[9].) This noise source generates signals in the line, and a voltage at $x=0$. This voltage is a stationary random process, in general not white. With suitable smoothness properties in the line and matching of the resistor to the local characteristic impedance of the line at $x=L$, this voltage will have a covariance of the form $\delta(t)+K(t)$ (to within a scaling factor) and an associated spectrum.

It turns out that the two spectra (differential equation and random process) differ by a scaling constant only.

The above remarks are restricted to finite length lines, but it is possible to let the line length L approach infinity. In fact, the analysis of Section 5 is done for the infinite line case when the claim of this subsection is established. Notice that the infinite line case subsumes the finite line case, because an R ohm resistor can be replaced by an infinite uniform line of characteristic impedance R ohms [11].

3.2. Calculations for nonuniform transmission lines

Using a standard normalization which we describe in more detail later, it becomes possible to assume that wavefronts travel at unit velocity in a nonuniform line, i.e., if a line is initially unexcited, and an input is applied beginning at time $t=0$ at the left-hand end, corresponding to $x=0$, no response will be observed at the point $x=c$ until c seconds have elapsed. Under this sort of assumption, the equations for normalized voltage $\bar{v}$ and current $\bar{\imath}$ along the line take the form, as we clarify in the next section,

$$\bar{v}_{xx}-\bar{v}_{tt}=q\bar{v}, \qquad \bar{\imath}_{xx}-\bar{\imath}_{tt}=\hat{q}\bar{\imath}. \tag{3.1}$$

The point is that hyperbolic partial differential equations are involved. This means that a wide body of existing theory, see e.g. [7], can be used to conclude what is in principle knowable, or computable, for the line.

Particular conclusions of the theory are:

(a) Knowing $\bar{v}(0,t)$ for $-c\leqslant t\leqslant c$ and that $\bar{\imath}(0,t)=0$ for $-c\leqslant t\leqslant c$ (due to the open-circuit conditions), one can deduce $\bar{v}(x,t)$ and $\bar{\imath}(x,t)$ for all x,t inside the triangle of Fig. 2; in particular, one can deduce $\bar{v}(x,0)$ and $\bar{\imath}(x,0)$ for $0\leqslant x\leqslant c$, i.e., the quantities stored on the line at time $t=0$.

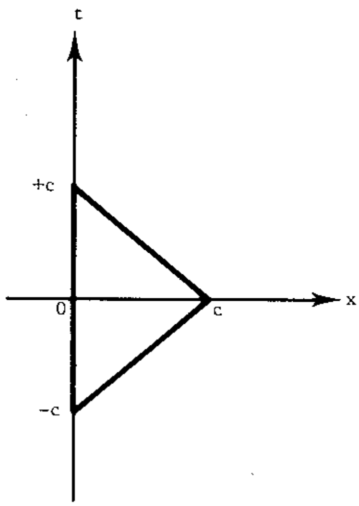

Fig. 2.

(b) Knowing $\bar{v}(x,0)$ and $\bar{\imath}(x,0)$ for $0\leqslant x\leqslant c$ and knowing that $\bar{\imath}(0,t)=0$ for $-c\leqslant t\leqslant c$, one can deduce $\bar{v}(x,t)$ and $\bar{\imath}(x,t)$ for all x, t inside the triangle of Fig. 2; in particular, one can deduce $\bar{v}(0,t)$, $-c\leqslant t\leqslant c$.

The first of these conclusions is a consequence of the fact that 'Cauchy data' are given, and that the characteristics passing through the end-points of the Cauchy data interval are the sloping sides of the triangle. The second is a little less obvious. Imagine the line is extended into the region $[-c, 0]$ as a mirror image of the given part over $[0, c]$, i.e., the inductance and capacitance per unit length at $x = d$ are the same as those at $x = -d$, $d \in [0, c]$. Adopt initial conditions $\bar{v}(-x, 0) = \bar{v}(x, 0)$, $\bar{\imath}(-x, 0) = -\bar{\imath}(x, 0)$. This Cauchy problem yields a solution in a region which includes the triangle of Fig. 2 and yields $\bar{\imath}(0, t) = 0$.

Calculations of the type referred to will be done in Section 6.

3.3. *Estimating $w(b)$ in physical terms*

We assume that the random process $w(t)$ is generated by the mechanism depicted in Fig. 1 and appears perhaps with scaling at the left-hand end of the line. Measurements of $w(t)$ are available over $|t| \leq a$. Since the normalization is known, this is equivalent to knowing $\bar{v}(0, t)$ for $-a \leq t \leq a$, while also $\bar{\imath}(0, t) \equiv 0$. Henceforth, we shall identify $w(t)$ with $\bar{v}(0, t)$.

From $\bar{v}(0, t)$, $|t| \leq a$, the quantities $\bar{v}(x, 0)$, $\bar{\imath}(x, 0)$ for $0 \leq x \leq a$ are found. These are (normalized) voltages and currents stored on the line at $t = 0$.

Next, we define functions $\hat{\bar{v}}(x, 0)$, $\hat{\bar{\imath}}(x, 0)$ for $0 \leq x \leq b$ by

$$\hat{\bar{v}}(x, 0) = \begin{cases} \bar{v}(x, 0), & 0 \leq x \leq a, \\ 0, & a < x \leq b, \end{cases} \tag{3.2}$$

and similarly for $\hat{\bar{\imath}}$. The superscript hat should be thought of as denoting an estimate; we know what the stored values of $\bar{v}(x, 0)$ and $\bar{\imath}(x, 0)$ are for $0 \leq x \leq a$, and the estimate is naturally set equal to these known values. However, the true stored values of $\bar{v}(x, 0)$ and $\bar{\imath}(x, 0)$ for $x > a$ are unknown. We estimate them as zero (and later we shall argue why this is an appropriate estimate).

We now have stored values on the line for $0 \leq x \leq b$, albeit in part estimated values. We can accordingly compute the resulting normalized voltage at $x = 0$, $\hat{\bar{v}}(0, t)$, for $-b \leq t \leq b$. For $-a \leq t \leq a$ this agrees with the known values $\bar{v}(0, t)$. For other values of t, and in particular for $t = b$, it turns out that it is the projection of the unknown $\bar{v}(0, b)$ onto $\bar{v}(0, t)$, $|t| \leq a$.

That this calculation parallels the calculation implied by the Krein formula is shown in Section 6. That the calculation does yield the projection is argued independently of any appeal to the Krein formula in Section 7.

4. Transmission line calculations

In this section we obtain various descriptions of nonuniform lossless transmission lines.

4.1. *Obtaining Sturm–Liouville equations from the telegraph equations*

For a nonuniform lossless line, in a region where no sources are present, the standard telegraph equations are, see e.g. [7, 11],

$$\frac{\partial v(y,t)}{\partial y}=-\frac{\partial}{\partial t}[l(y)i(y,t)], \qquad \frac{\partial i(y,t)}{\partial y}=-\frac{\partial}{\partial t}[c(y)v(y,t)]. \tag{4.1}$$

Here, y is a distance variable, $l(y)$ and $c(y)$ are the inductance and capacitance per unit length, assumed positive for all y; voltage and current are denoted by v and i.

We shall describe two (standard) normalizations of these equations and obtain coupled first-order partial differential equations for the normalized variables, uncoupled second-order partial differential equations for the normalized variables, and uncoupled Sturm–Liouville equations associated with the normalized variables when sinusoidal time variation is assumed.

The first normalization is to deform (in general nonlinearly) the distance variable so as to obtain a situation in which wavefronts travel at unit velocity. The independent variable y is replaced by x where

$$x(y)=\int_0^y \sqrt{l(\alpha)c(\alpha)}\,\mathrm{d}\alpha. \tag{4.2}$$

The variable x measures electrical length. The voltage and current now become functions of x and t, and (4.1) is replaced by

$$\frac{\partial v(x,t)}{\partial x}=-\frac{\partial}{\partial t}[z_0(x)i(x,t)], \qquad \frac{\partial i(x,t)}{\partial x}=-\frac{\partial}{\partial t}[z_0^{-1}(x)v(x,t)]. \tag{4.3}$$

Here

$$z_0(x)=\sqrt{l(x)/c(x)} \tag{4.4}$$

is termed the characteristic impedance of the line.

The second normalization represents a position-dependent scaling of the voltage and current. Define

$$\tilde{v}(x,t)=\frac{v(x,t)}{\sqrt{z_0(x)}}, \qquad \tilde{\imath}(x,t)=\sqrt{z_0(x)}\,i(x,t) \tag{4.5}$$

and define also

$$s(x)=\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\ln z_0(x). \tag{4.6}$$

We assume that $l(x)$, $c(x)$ have sufficient smoothness properties as to ensure that $s(x)$ is itself continuously differentiable. With these definitions, (4.3) yield the normalized telegraph equations

$$\frac{\partial \bar{v}(x,t)}{\partial x} = -\frac{\partial \bar{\imath}(x,t)}{\partial t} - s(x)\bar{v}(x,t),$$
$$\frac{\partial \bar{\imath}(x,t)}{\partial x} = -\frac{\partial \bar{v}(x,t)}{\partial t} + s(x)\bar{\imath}(x,t) \tag{4.7}$$

and in turn, (4.7) yield

$$\bar{v}_{xx} = \bar{v}_{tt} + (s^2 - s_x)\bar{v}, \tag{4.8a}$$

$$\bar{\imath}_{xx} = \bar{\imath}_{tt} + (s^2 + s_x)\bar{\imath}. \tag{4.8b}$$

The derivation of the hyperbolic equations (4.8) was foreshadowed in the last section.

The last step in obtaining a Sturm–Liouville equation is to assume that

$$\bar{v}(x,t) = V(x,\lambda)\,\mathrm{e}^{\mathrm{j}\lambda t}, \qquad \bar{\imath}(x,t) = I(x,\lambda)\,\mathrm{e}^{\mathrm{j}\lambda t} \tag{4.9}$$

for some complex $V(x,\lambda), I(x,\lambda)$, i.e., that signals of only one frequency are present. Then (4.8) yield

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} + [\lambda^2 - q(x)]V = 0, \tag{4.10a}$$

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} + [\lambda^2 - \hat{q}(x)]I = 0 \tag{4.10b}$$

where

$$q(x) = s^2(x) - \frac{\mathrm{d}s(x)}{\mathrm{d}x}, \qquad \hat{q}(x) = s^2(x) + \frac{\mathrm{d}s(x)}{\mathrm{d}x}.$$

4.2. *Boundary conditions for the Sturm–Liouville equation*

Suppose there is an open circuit at $x=0$. Set $x=0$ in (4.7). We see that $\bar{v}_x(0,t) = -s(0)\bar{v}(0,t)$, and so

$$V'(0,\lambda) = -s(0)V(0,\lambda). \tag{4.11}$$

Eigenfunctions of (4.10a) will be used with the defining initial conditions

$$V(0,\lambda) = \phi(0,\lambda) = 1, \qquad V'(0,\lambda) = \phi'(0,\lambda) = -s(0). \tag{4.12}$$

For the current equation we have of course $\bar{\imath}(0,t) = 0$, while $\bar{\imath}_x(0,t) = -\bar{v}_t(x,t)$. So parallel with (4.11) we have

$$I(0,\lambda) = 0, \qquad I'(0,\lambda) = -\mathrm{j}\lambda V(0,\lambda) = -\mathrm{j}\lambda. \tag{4.13}$$

4.3. The eigenfunction $\psi(x, \lambda)$

In this section we shall exhibit the importance of the function $\psi(x, \lambda)$, defined in (2.6), under the open-circuit boundary conditions just discussed.

With $\phi(x, \lambda)$ the solution of

$$\frac{d^2V(x)}{dx^2}+\left[\lambda^2-s^2(x)+\frac{ds(x)}{dx}\right]V(x)=0, \tag{4.14}$$

subject to $\phi(0, \lambda)=1$, $\phi'(0, \lambda)=-s(0)$, we recall the definition of the function $\psi(x, \lambda)$:

$$\psi(x, \lambda)=\frac{1}{\lambda}\phi(x, 0)\frac{d}{dx}\left[\frac{\phi(x, \lambda)}{\phi(x, 0)}\right]. \tag{4.15}$$

We claim:

(a) $$s(x)=\frac{\phi'(x, 0)}{\phi(x, 0)}; \tag{4.16}$$

(b) $$\phi'(x, \lambda)=\lambda\psi(x, \lambda)-s(x)\phi(x, \lambda), \tag{4.17a}$$

$$\psi'(x, \lambda)=-\lambda\phi(x, \lambda)+s(x)\psi(x, \lambda) \tag{4.17b}$$

and

$$\psi(0, \lambda)=0, \qquad \psi'(0, \lambda)=-\lambda; \tag{4.18}$$

(c) The solution of

$$\frac{d^2I(x)}{dx^2}+\left[\lambda^2-s^2(x)-\frac{ds(x)}{dx}\right]I(x)=0, \tag{4.19}$$

subject to $I(0, \lambda)=0$, $I'(0, \lambda)=-j\lambda V(0, \lambda)=-j\lambda$, is

$$I(x, \lambda)=j\psi(x, \lambda); \tag{4.20}$$

(d) $$\int_{-\infty}^{+\infty}\psi(x, \lambda)\psi(y, \lambda)\, d\rho(\lambda)=\delta(x-y). \tag{4.21}$$

Proofs involve straightforward manipulation and are omitted. Several comments are in order. The most important equations are (4.20) and (4.21); (4.20) says that *with an open-circuit at* $x=0$, $V(x)=\phi(x, \lambda)$ *is accompanied by* $I(x)=j\psi(x, \lambda)$, *with* ψ *readily computable from* ϕ, while (4.21) is an orthogonality result paralleling that for $\phi(x, \lambda)$. Next, though equations (4.17) and (4.18) are mainly of use to prove (4.20), equations (4.17) are linked to a frequency domain statement of the first-order coupled equations (4.7) linking $\bar{v}_x$, $\bar{\imath}_x$, $\bar{v}_t$ and $\bar{\imath}_t$.

The frequency domain version of (4.7) under the boundary conditions (4.12) and (4.13), with $I(x, \lambda)$ identified as $\mathrm{j}\psi(x, \lambda)$, becomes

$$\phi'(x, \lambda) = -\mathrm{j}\lambda[j\psi(x, \lambda)] - s(x)\phi(x, \lambda),$$

$$\mathrm{j}\psi'(x, \lambda) = -\mathrm{j}\lambda\phi(x, \lambda) + \mathrm{j}s(x)\psi(x, \lambda);$$

i.e., we recover (4.17).

5. Equality of the spectra

In this section, we shall show (as claimed in Section 3.1) that the spectrum of the Sturm–Liouville equation for the transmission line normalized voltage (under conditions of sinusoidal time-variation and open-circuit at $x = 0$) and the spectrum of the random process generated according to the earlier described arrangement and depicted in Fig. 1 differ by a constant multiple. Both spectra are linked to the voltage at $x = 0$ which results from an impulsive current input at $x = 0$.

5.1. Transmission line equations with impulsive current input at $x = 0$, $t = 0$

Suppose that a distributed current input $i_\varepsilon(x, t)$ (the units are amps/meter) is applied to the line, being confined to $0 \leqslant x \leqslant \varepsilon$. The telegraph equations have the v equation unaltered, while the i equation is

$$\frac{\partial i(x, t)}{\partial x} = -\frac{\partial}{\partial t}[z_0^{-1}(x)v(x, t)] + i_\varepsilon(x, t).$$

Letting $\varepsilon \to 0$ with $\int_0^\varepsilon i_\varepsilon(x, t)\,\mathrm{d}x = I(t)$, i.e., concentrating the current input to a single point, yields

$$\frac{\partial i(x, t)}{\partial x} = -\frac{\partial}{\partial t}[z_0^{-1}(x)v(x, t)] + \delta(x)I(t).$$

When we introduce the normalization, there results after straightforward calculations

$$\begin{aligned} \bar{v}_x(x, t) &= -\bar{\imath}_t(x, t) - s(x)\bar{v}(x, t), \\ \bar{\imath}_x(x, t) &= -\bar{v}_t(x, t) + s(x)\bar{\imath}(x, t) + \sqrt{z_0(0)}\delta(x)\delta(t), \end{aligned} \tag{5.1}$$

when $I(t)$ is an impulse. Also, we have

$$\bar{v}_{xx} = \bar{v}_{tt} + (s^2 - s_x)\bar{v} - \sqrt{z_0(0)}\delta(x)\delta^{(1)}(t). \tag{5.2}$$

5.2. Input impedance of transmission line and its relation to a Sturm–Liouville spectrum

The voltage at $x = 0$ in response to a current impulse is the input impedance (in the time domain) of the line. For an infinite length line, or for $0 \leqslant t \leqslant 2L$, this impulse response, call it $z(t)$, is computable as

$$z(t) = \sqrt{z_0(0)}\bar{v}(0, t) \tag{5.3}$$

where $\bar{v}(x, t)$ is found as the solution of (5.2) with boundary conditions

$$\bar{v}_x(0, t) = -s(0)\bar{v}(0, t), \qquad \bar{v}(x, 0) = 0 \quad (x \geqslant 0), \qquad \bar{v}_t(x, 0) = 0 \quad (x \geqslant 0). \tag{5.4}$$

Equivalently, $\bar{v}(x, t)$ is the solution of the homogeneous equation associated with (5.2), viz.

$$\bar{v}_{xx} = \bar{v}_{tt} + (s^2 - s_x)\bar{v} \tag{5.5}$$

with boundary conditions

$$\bar{v}_x(0, t) = -s(0)\bar{v}(0, t), \qquad \bar{v}(x, 0) = \sqrt{z_0(0)}\delta(x), \qquad \bar{v}_t(x, 0) = 0, \quad x \geqslant 0. \tag{5.6}$$

We claim that the solution of (5.5) *with boundary conditions* (5.6) *is*

$$\bar{v}(x, t) = \sqrt{z_0(0)} \int_{-\infty}^{+\infty} \phi(x, \lambda) \cos \lambda t \, \mathrm{d}\rho(\lambda) \tag{5.7}$$

where $\phi(x, \lambda)$ is the solution of (4.10a), *subject to $\phi(0, \lambda) = 1$, $\phi'(0, \lambda) = -s(0)$, and $\rho(\lambda)$ is the associated spectrum of* (4.10a). It is trivial to verify that (5.5) is satisfied, that $\bar{v}_x(0, t) = -s(0)\bar{v}(0, t)$ and that $\bar{v}_t(x, 0) = 0$ for $x \geqslant 0$ on using the evenness with respect to λ of $\phi(x, \lambda)$ and oddness of $\rho(\lambda)$. Lastly, observe that

$$\begin{aligned}\bar{v}(x, 0) &= \sqrt{z_0(0)} \int_{-\infty}^{+\infty} \phi(x, \lambda) \, \mathrm{d}\rho(\lambda) = \sqrt{z_0(0)} \int_{-\infty}^{+\infty} \phi(x, \lambda)\phi(0, \lambda) \, \mathrm{d}\rho(\lambda) \\ &= \sqrt{z_0(0)}\delta(x)\end{aligned}$$

on using the orthogonality property of the eigenfunctions described in Section 2.

By taking $x = 0$ in (4.17), we obtain

$$z(t) = z_0(0) \int_{-\infty}^{+\infty} \cos \lambda t \, \mathrm{d}\rho(\lambda). \tag{5.8}$$

Remark. This calculation could have been done for a finite line; the calculations must include consideration of boundary conditions at $x = L$.

5.3. Input impedance of a line and its relation to a random process spectrum

Consider the set up of Fig. 1. It is a well-known result (Nyquist's theorem) [9], that with the voltage generator a white noise source, the autocorrelation function of the voltage observed at $x=0$ is, to within a scaling constant, given by $z(|t|)$, where $z(\cdot)$ is as defined above. If $\psi(\lambda)$ is the associated spectrum, we have

$$z(|t|)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos\lambda t\,\mathrm{d}\psi(\lambda). \tag{5.9}$$

Comparing (5.8) and (5.9), we observe equality of the spectra (differential equation and random process) to within a scaling constant.

6. Transmission line calculations and the prediction formula

Suppose that a random process $w(t)$ is observed as shown in Fig. 1. If $w(t)$ has known spectrum, the line is constructible, this being an inverse problem of the sort discussed at the end of the last section and in [3, 4, 11]. (In fact, if $w(t)$ has a covariance $\delta(t)+K(t)$ known only for $|t|\leq c$, we can construct a line of length $\frac{1}{2}c$ modelling this covariance, since as shown in [4], if a line input impedance is specified over $[0, c]$ in the time domain this determines the line over $[0, \frac{1}{2}c]$). We now consider the effect of carrying out the calculations outlined in Section 3, in which from $v(t)$, $|t|\leq a$, the (normalized) voltage and current $\bar{v}(x, 0)$, $\bar{\imath}(x, 0)$ are computed for $x\leq a$ and then by 'extending' these quantities for $x>a$ with assumed values of zero, the terminal voltage is again computed. We shall relate the calculation to the Krein prediction formula.

6.1. Calculations of stored quantities

Suppose that $\bar{v}(0, t)=\mathrm{e}^{\mathrm{j}\lambda t}$ is observed for $|t|\leq a$. We claim that

$$\bar{v}(x, 0)=\phi(x, \lambda), \quad 0\leq x\leq a, \tag{6.1a}$$

$$\bar{\imath}(x, 0)=\mathrm{j}\psi(x, \lambda), \quad 0\leq x\leq a \tag{6.1b}$$

where $\psi(x, \lambda)$ is as defined in (2.6).

To verify this claim, recall (see Fig. 2 with $c=a$) that no matter what values $\bar{v}(0, t)$ and $\bar{\imath}(0, t)$ take for $|t|>a$, $\bar{v}(x, 0)$ and $\bar{\imath}(x, 0)$ for $0\leq x\leq a$ are uniquely determined by knowledge of $\bar{v}(0, t)$ and $\bar{\imath}(0, t)$ for $|t|\leq a$. Therefore, we may *assume* values for $\bar{v}(0, t)$, $\bar{\imath}(0, t)$ in $|t|>a$ which are helpful in doing the calculations. The values which turn out to be helpful are $\bar{v}(0, t)=\mathrm{e}^{\mathrm{j}\lambda t}$ and $\bar{\imath}(0, t)=0$.

Then, in view of the discussion of Section 4, we see that

$$\bar{v}(x, t)=\phi(x, \lambda)\,\mathrm{e}^{\mathrm{j}\lambda t}, \tag{6.2a}$$

$$\bar{\imath}(x, t)=\mathrm{j}\psi(x, \lambda)\,\mathrm{e}^{\mathrm{j}\lambda t} \tag{6.2b}$$

where ϕ and ψ are as defined earlier, and (6.2) are correct values in the triangle depicted in Fig. 2, with $c=a$. Eq. (6.1) is immediate.

6.2. *Calculation of terminal quantities, given stored quantities*

We now consider the task of computing $\bar{v}(0,t)$, $|t|\leqslant b$ given that

$$\begin{aligned}\bar{v}(x,0)&=\begin{cases}\phi(x,\lambda), & 0\leqslant x\leqslant a,\\ 0, & a<x\leqslant b,\end{cases}\\ \bar{\imath}(x,0)&=\begin{cases}\mathrm{j}\psi(x,\lambda), & 0\leqslant x\leqslant a,\\ 0, & a<x<b,\end{cases}\end{aligned}\tag{6.3}$$

and also given that $\bar{\imath}(0,t)=0$, $|t|\leqslant b$. We claim that *the pair*

$$\begin{aligned}\bar{v}(x,t)=&\int_{-\infty}^{+\infty}\phi(x,\sigma)\cos\sigma t\left\{\int_0^a\phi(y,\sigma)\phi(y,\lambda)\,\mathrm{d}y\right\}\mathrm{d}\rho(\sigma)\\ &+\mathrm{j}\int_{-\infty}^{+\infty}\phi(x,\sigma)\sin\sigma t\left\{\int_0^a\psi(y,\sigma)\psi(y,\lambda)\,\mathrm{d}y\right\}\mathrm{d}\rho(\sigma)\end{aligned}\tag{6.4}$$

and

$$\begin{aligned}\bar{\imath}(x,t)=&-\int_{-\infty}^{+\infty}\psi(x,\sigma)\sin\sigma t\left\{\int_0^a\phi(y,\sigma)\phi(y,\lambda)\,\mathrm{d}y\right\}\mathrm{d}\rho(\sigma)\\ &+\mathrm{j}\int_{+\infty}^{+\infty}\psi(x,\sigma)\cos\sigma t\left\{\int_0^a\psi(y,\sigma)\psi(y,\lambda)\,\mathrm{d}y\right\}\mathrm{d}\rho(\sigma)\end{aligned}\tag{6.5}$$

satisfy the transmission line equations (4.7), *and give rise to* $\bar{v}(x,0)$, $\bar{\imath}(x,0)$ *as in* (6.3) *and to* $\bar{\imath}(0,t)=0$.

Assuming for the moment the truth of this claim, it follows, considering Fig. 2 with $c=b$, that $\bar{v}(x,t)$ and $\bar{\imath}(x,t)$ are the unique solution within the triangle consistent with the prescribed values of $\bar{v}(x,0)$, $\bar{\imath}(x,0)$ and $\bar{\imath}(0,t)$ and therefore $\bar{v}(0,t)$ for $|t|=b$ is

$$\begin{aligned}\bar{v}(0,b)=&\int_{-\infty}^{+\infty}\cos\sigma b\left\{\int_0^a\phi(y,\sigma)\phi(y,\lambda)\,\mathrm{d}y\right\}\mathrm{d}\rho(\sigma)\\ &+\mathrm{j}\int_{-\infty}^{+\infty}\sin\sigma b\left\{\int_0^a\psi(y,\sigma)\psi(y,\lambda)\,\mathrm{d}y\right\}\mathrm{d}\rho(\sigma)\end{aligned}\tag{6.6}$$

This is the Krein formula of Section 2. Note that (6.4) and (6.5) may not represent the correct values of $\bar{v}(x,t)$ and $\bar{\imath}(x,t)$ for all x and t. But if they represent correct values for $0\leqslant x\leqslant b$ and $t=0$ and if they represent $i(0,t)=0$ for $|t|\leqslant b$, they represent correct values in the triangle of Fig. 2 with $c=b$, and this is all we need.

To verify the claim, we consider first the satisfaction of (4.7). Form $\bar{v}_x(x,t)+s(x)\bar{v}(x,t)$. Recalling (4.17a), this quantity evaluates as

$$\sigma\int_{-\infty}^{+\infty}\psi(x,\sigma)\cos\sigma t\left\{\int_0^a\phi(y,\sigma)\phi(y,\lambda)\,\mathrm{d}y\right\}\mathrm{d}\rho(\sigma)+$$

$$+\mathrm{j}\sigma\int_{-\infty}^{+\infty}\psi(x,\sigma)\sin\sigma t\left\{\int_0^a\psi(y,\sigma)\psi(y,\lambda)\,\mathrm{d}y\right\}\mathrm{d}\rho(\sigma)$$

which on inspection of (6.5) is easily seen to be $-\bar{\imath}_t(x,t)$. So (4.7a) is satisfied. The checking of (4.7b) proceeds the same way.

As for the boundary conditions, that $\bar{\imath}(0,t)=0$ follows by recalling that $\psi(0,\lambda)=0$ for all λ. Next, (6.4) yields

$$\bar{v}(x,0)=\int_{-\infty}^{+\infty}\phi(x,\sigma)\int_0^a\phi(y,\sigma)\phi(y,\lambda)\,\mathrm{d}y\;\mathrm{d}\rho(\sigma)$$

$$=\int_0^a\phi(y,\lambda)\,\mathrm{d}y\int_{-\infty}^{+\infty}\phi(x,\sigma)\phi(y,\sigma)\,\mathrm{d}\rho(\sigma)$$

$$=\int_0^a\phi(y,\lambda)\delta(x-y)\,\mathrm{d}y=\begin{cases}\phi(x,\lambda), & x\leq a,\\ 0, & x>a.\end{cases}$$

(We use the orthogonality property of (2.5), viz. that $\int_{-\infty}^{+\infty}\phi(x,\sigma)\phi(y,\sigma)\,\mathrm{d}\rho(\sigma)=\delta(x-y)$.) In a similar way, using this time the orthogonality property for the $\psi(x,\sigma)$ of (4.21), we verify the boundary condition $\bar{\imath}(x,0)$.

6.3. *Linkage with the prediction problem*

What the theory of partial differential equations tells us is that there exists a linear transformation

$$\mathcal{K}:\quad \bar{v}(0,t),\ |t|\leq a\ \rightarrow\int_{-a}^{+a}k(b,t)\bar{v}(0,t)\,\mathrm{d}t$$

computable as the cascade $\mathcal{K}_2\mathcal{K}_1$ of two transformations

$$\mathcal{K}_1:\quad \bar{v}(0,t),\ |t|\leq a\ \rightarrow\begin{cases}\bar{v}(x,0), & 0\leq x\leq a,\\ \bar{\imath}(x,0), & 0\leq x\leq a\end{cases}$$

(computed with $\bar{\imath}(0,t)=0$) and

$$\mathcal{K}_2:\quad \begin{cases}\bar{v}(x,0), & 0\leq x\leq a\\ \bar{\imath}(x,0), & 0\leq x\leq a\end{cases}\rightarrow\ \bar{v}(0,b),\ b>a,$$

(computed by assuming $\bar{v}(x,0)=0$, $x>a$ and $\bar{\imath}(x,0)=0$, $x>a$).

What Sections 6.1 and 6.2 have done is to compute the result of this transformation when $\bar{v}(0,t)=\mathrm{e}^{\mathrm{j}\lambda t}$, $|t|\leq a$. Comparison with Krein's formula shows that this transformation gives the same result as Krein's formula.

The mapping $\mathcal{K}_1$ constructs from $e^{j\lambda t}$, $|t|<a$ the pair $\phi(x,\lambda)$, $\psi(x,\lambda)$ for $x\leq a$. The mapping $\mathcal{K}_2$ constructs the estimate as in (6.6) from this pair. Thus in (6.6) the kernel associated with $\mathcal{K}_2$ is explicitly displayed, while that associated with $\mathcal{K}_1$ has not been displayed here. It can, however, be found elsewhere, e.g., in [3].

In view of the isomorphism between the value at time t of a stationary random process $w(\cdot)$ and the function $e^{j\lambda t}$ (t fixed, λ variable) described in Section 2, and the fact that Krein has established that his formula defines the projection of $e^{j\lambda b}$ onto $e^{j\lambda t}$, $|t|\leq a$, it follows that the prediction of $\bar{v}(0,b)$ in terms of $\bar{v}(0,t)$, $|t|\leq a$ when $\bar{v}(0,t)$ is a sample function of a random process of spectrum $\rho(\lambda)$, is achieved by the transformation $\mathcal{K}=\mathcal{K}_2\mathcal{K}_1$.

7. Prediction without Krein or Sturm–Liouville

The preceding sections have taken the viewpoint that a prediction formula is known, and have sought to give a physical interpretation of it. In this section, we argue the physical interpretation of the prediction formula from first principles, making no appeal to Krein's ideas.

7.1. *Prediction given spatial whiteness*

We recall the set up of Fig. 1, with R assumed to equal $z_0(L)$. In the next sections, we shall argue that for $x\neq y$, $[\bar{v}(x,t)\ \ \bar{\imath}(x,t)]$ is uncorrelated with $[\bar{v}(y,t)\ \ \bar{\imath}(y,t)]$ for any $y\neq x$, i.e. *at any fixed instant of time* $[\bar{v}\ \ \bar{\imath}]$ *is spatially a white process.* Based on this fact, which at present is an assumption, we shall now explain why the prediction procedure works, the procedure being to construct from $\bar{v}(0,t)$, $|t|\leq a$ the quantities $\bar{v}(x,0)$, $\bar{\imath}(x,0)$, $0\leq x\leq a$ and then to compute $\hat{\bar{v}}(0,b)$ by taking $\hat{\bar{v}}(x,0)=0$, $\hat{\bar{\imath}}(x,0)=0$ for $b\geq x>a$. Reserving $\hat{\bar{v}}(0,b)$ for the quantity obtained by this procedure, let $z(t)$ denote $\mathbf{E}[\bar{v}(0,t)|\bar{v}(0,\tau),\ \tau\leq a]$.

The defining property for the prediction is

$$\mathbf{E}\{[\bar{v}(0,t)-z(t)]\bar{v}(0,\tau)\}=0,\quad |\tau|\leq a. \tag{7.1}$$

Since when $\bar{\imath}(0,\tau)=0$, $\bar{v}(0,\tau)$, $\tau\leq a$ is obtainable from $\bar{v}(x,0)$, $\bar{\imath}(x,0)$ for $0\leq x\leq a$ by linear operations, and vice versa, an equivalent defining property is

$$\mathbf{E}\left\{[\bar{v}(0,t)-z(t)]\begin{bmatrix}\bar{v}(x,0)\\ \bar{\imath}(x,0)\end{bmatrix}^{\mathrm{T}}\right\}=0,\quad 0\leq x\leq a. \tag{7.2}$$

Suppose $\bar{v}(0,t)$ for $|t|\leq b$ were known, while also $\bar{\imath}(0,t)=0$. Then we could compute $\bar{v}(x,0)$, $\bar{\imath}(x,0)$ for $0\leq x\leq b$ (agreeing with the values computed over $0\leq x\leq a$) by linear operations, and conversely. Suppose these linear operations are used on $z(t)$, $|t|\leq b$ to compute quantities which we shall denote by $\bar{v}_z(x,0)$,

$\bar{\imath}_z(x, 0)$ for $0 \leqslant x \leqslant b$. Eq. (7.2) may therefore be replaced by the equivalent equation

$$\mathbf{E}\left\{\left(\begin{bmatrix}\bar{v}(y,0)\\ \bar{\imath}(y,0)\end{bmatrix}-\begin{bmatrix}\bar{v}_z(y,0)\\ \bar{\imath}_z(y,0)\end{bmatrix}\right)\begin{bmatrix}\bar{v}(x,0)\\ \bar{\imath}(x,0)\end{bmatrix}^{\mathrm{T}}\right\}=0,\quad 0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b.$$

Clearly $z(t)$ must agree on $|t| \leqslant a$ with $\bar{v}(0, t)$ and consequently $\bar{v}_z(x, 0)$, $\bar{\imath}_z(x, 0)$ must agree with $\bar{v}(x, 0)$, $\bar{\imath}(x, 0)$ on $|t| \leqslant a$. So (7.2) is equivalent to

$$\mathbf{E}\left\{\left(\begin{bmatrix}\bar{v}(y,0)\\ \bar{\imath}(y,0)\end{bmatrix}-\begin{bmatrix}\bar{v}_z(y,0)\\ \bar{\imath}_z(y,0)\end{bmatrix}\right)\begin{bmatrix}\bar{v}(x,0)\\ \bar{\imath}(x,0)\end{bmatrix}^{\mathrm{T}}\right\}=0,\quad 0\leqslant x\leqslant a,\ a< y\leqslant b.$$

Applying the assumption of spatial whiteness, yet to be proved,

$$\mathbf{E}\left\{\begin{bmatrix}\bar{v}_z(y,0)\\ \bar{\imath}_z(y,0)\end{bmatrix}[\bar{v}(x,0)\quad \bar{\imath}(x,0)]\right\}=0 \tag{7.3}$$

for $0 \leqslant x \leqslant a$, $a < y \leqslant b$.

Because $z(t)$, $|t| \leqslant b$, has to lie in the space spanned by $\bar{v}(0, t)$, $|t| \leqslant a$ or equivalently $\bar{v}(x, 0)$, $\bar{\imath}(x, 0)$, $0 \leqslant x \leqslant a$, and because $\bar{v}_z(y, 0)$, $\bar{\imath}_z(y, 0)$ are obtained from $z(t)$ by linear operations, it follows that $\bar{v}_z(y, 0)$, $\bar{\imath}_z(y, 0)$ for $a < y \leqslant b$ are obtainable from $\bar{v}(x, 0)$, $\bar{\imath}(x, 0)$, $0 \leqslant x \leqslant a$ by linear operations. In light of (7.3), this means that

$$\begin{bmatrix}\bar{v}_z(y,0)\\ \bar{\imath}_z(y,0)\end{bmatrix}=0,\quad a<y\leqslant b. \tag{7.4}$$

Turning this round, we conclude that the estimate $z(t)$ for $a < |t| \leqslant b$ can be obtained by setting $\hat{\bar{v}}(x, 0) = 0$, $\hat{\bar{\imath}}(x, 0) = 0$ for $a < x \leqslant b$ and computing $\hat{\bar{v}}(0, t)$.

It is evident that the prediction procedure is not far different in concept from more standard procedures [5, 10]. The constructions of the stored voltage and current at $t = 0$ corresponds to the construction of innovations data from the measurement, and then the construction of the prediction corresponds to prediction using the known part of the innovations. It is interesting to observe that a (scalar) record of the innovations of time length 2a is provided by a (2-vector) record of spatial length a, with spatial whiteness corresponding to time whiteness. We shall indicate elsewhere the precise relations between the innovations and this spatial record.

7.2. Spatial whiteness in the finite-dimensional case

To get a feel for the spatial whiteness result, consider the arrangement of Fig. 3. A transmission line can be thought of as a limiting form of the finite network depicted. Take as the state variable

$$x = [\sqrt{C_0}v_{C_0}\ \sqrt{L_1}i_{L_1}\ \sqrt{C_1}\,v_{C_1}\ \cdots\ \sqrt{C_n}\,v_{C_n}]. \tag{7.5}$$

With $u(t)$ an arbitrary voltage input, the state variable equation is

$$\dot{x} = Fx + Gu \tag{7.6}$$

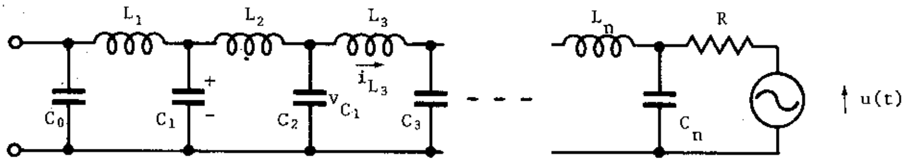

Fig. 3.

where

$$F=\begin{bmatrix} 0 & -\frac{1}{\sqrt{L_1C_0}} & 0 & & \\ \frac{1}{\sqrt{L_1C_0}} & & -\frac{1}{\sqrt{L_1C_1}} & & \\ & \frac{1}{\sqrt{L_1C_1}} & 0 & & \\ & & \ddots & & -\frac{1}{\sqrt{L_nC_n}} \\ & & & \frac{1}{\sqrt{L_nC_n}} & -\frac{1}{RC_n} \end{bmatrix},$$

$$G=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{R\sqrt{C_n}} \end{bmatrix}. \tag{7.7}$$

If u is white noise with $\mathbf{E}[u(t)u(s)]=R\delta(t-s)$, then the steady state covariance matrix $\mathbf{E}[x(t)x^{\mathrm{T}}(t)]=P$ is given by

$$PF^{\mathrm{T}}+FP=-GG^{\mathrm{T}}. \tag{7.8}$$

It is easily checked that

$$P=\tfrac{1}{2}I. \tag{7.9}$$

(If $\mathbf{E}[u(t)u(s)]=2kTR\delta(t-s)$, corresponding to Johnson resistor noise, then (7.9) with scaling says that the average stored energy of each reactive component is $\frac{1}{2}kT$.

This is the equipartition principle.) The significance of (7.9) for our purposes is that it shows $\mathbf{E}[x_i(t)x_j(t)]=0$ for $i \neq j$, corresponding to spatial whiteness.

7.3. *Spatial whiteness in piecewise constant lines*

In [8] signals in lossless nonuniform acoustic lines are considered. (Abstractly, these behave like lossless nonuniform electrical transmission lines.) For simplicity the acoustic lines are assumed to comprise juxtaposed sections of constant characteristic impedance and identical length. An extensive theory is described, covering such topics as the inverse problem, and prediction. A spatial whiteness property is also established. To the extent that an arbitrary line can be regarded as a limiting form of the type of line considered in [8], the spatial whiteness claim appears reasonable.

7.4. *Spatial whiteness on the transmission line*

In this section we formulate an infinite-dimensional version of the result of the previous section. Consider the set up of Fig. 1 but where the generator is deterministic, generating an impulse $\delta(t)$. Let the resulted normalized voltage and current at x, t be denoted by $h_v(x,t)$ and $h_i(x,t)$. When the generator is white noise with covariance $\delta(t)$, and a steady state has been reached, one obtains

$$\mathbf{E}\left\{\begin{bmatrix}\bar{v}(x,t)\\ \bar{\imath}(x,t)\end{bmatrix}[\bar{v}(y,t)\quad \bar{\imath}(y,t)]\right\}=\int_0^\infty \begin{bmatrix}h_v(x,t)\\ h_i(x,t)\end{bmatrix}[h_v(y,t)\quad h_i(y,t)]\,\mathrm{d}t. \tag{7.10}$$

A rather lengthy calculation using the partial differential equations satisfied by h_v and h_i shows that this quantity is zero for $x \neq y$.

7.5. *Another approach*

The approach of Gelfand and Levitan [3] to the inverse Sturm–Liouville problem involves the construction of various kernels satisfying covariance factorization types of identity. The various kernels we have had occasion to use can also be linked into these identities, and the identities themselves can be viewed with physical insight via the transmission line model. In this way one can recover the spatial whiteness and link the stored voltage and current at $t=0$ to the innovations of the observed random process. We shall set out these calculations elsewhere.

8. Conclusions

The main conclusion is that a number of ideas for future research are opened up. We list the following avenues along which we are currently working:

(1) Prediction when covariances are not of the form $\delta(t)+K(t)$ with smooth $K(t)$. The case when the delta function is absent can perhaps be handled by modelling the covariance using a line and lumped reactive elements. The case when the delta function is replaced by a sum of delta functions with different arguments can perhaps be handled by working with lines with discontinuities in characteristic impedance.

(2) Use of a line as a whitening filter [5].

(3) Use of a line as a Kalman filter, and even as a smoother [5].

(4) Eliminating many of the smoothness assumptions (this is linked with point 1).

(5) Clarifying the meaning of the calculations using covariance factorization ideas, and particularly, identifying stored voltage and current as a spatial representation of the innovations process.

(6) The vector process case. There are several relevant issues here. First, if the covariance is not symmetric (as opposed to self adjoint), the line will have to be a multiport structure with some sort of nonreciprocity. Second, if a multiport line is given, it is not clear how a distance scaling operation could be set up to achieve unit velocity of propagation on all channels. On the other hand, given a covariance, it seems reasonable to assert that from it, a multiport line could be found with unit velocity of propagation on all channels.

(7) Time-varying lines and their connections with non-stationary processes. The work of [1] may be helpful in this regard.

(8) Discrete versions. Two discretizations are possible. If we break the line up into a set of pieces of the same electrical length and discretize in time, we can draw some parallels with Levinson filtering as applied in speech processing [8] or seismic data processing [2]. The other form of discretization is to replace the line by a lumped inductor–capacitor ladder network.

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