Realization of Digital Transfer Functions Using Cascaded Lattice and Ladder Block Structures

by P. ANANTHAKRISHNA

Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106, U.S.A.

B. D. O. ANDERSON

Department of Systems Engineering, Australian National University, Canberra, ACT, Australia

and S. K. MITRA

Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106, U.S.A.

ABSTRACT: *Several techniques are available for implementing a scalar digital filter using lattice and ladder structures which are known to yield low noise realizations. Of these, the lattice and ladder structures proposed by Gray and Markel have the advantage of being internally scaled. Block implementation of a scalar digital filter offers a number of advantages. In this paper, new algorithms to realize block digital filters in the form of cascaded lattice or ladder two-pairs are proposed. Block lattice and ladder structures similar to Gray and Markel's scalar structure are obtained by using Levinson recursion. Also, normalized block lattice and ladder structures are developed.*

I. Introduction

Block implementation of a digital transfer function has some advantages such as fewer computations, the possibility of using fast convolution techniques for intermediate computations, efficient implementation by parallel processors, and reduced roundoff noise. A scalar digital filter can be implemented in block form in many ways. Block state structures have been studied by Mitra and Gnanasekaran (1), Barnes and Shinnaka (2), Ananthakrishna and Mitra (3). In this paper, some techniques for realizing a block digital filter in block lattice and ladder form are presented.

Lattice and ladder realizations of scalar digital filters have been proposed by Mitra, Kamat, and Huey (4), and Gray and Markel (5). Gray and Markel's structure has the advantage of being internally scaled. Gnanasekaran has proposed a technique for realizing the block lattice and ladder realizations of a digital filter (6). However, in his realization, the design equations become very complicated when the order of the filter is large.

In this paper new algorithms to realize block digital filters in the form of cascaded lattice or ladder block two-pairs are proposed. Block lattice and ladder structures are obtained by using the Levinson recursion. These block structures are internally scaled and, as a result, may find application in block linear prediction. From this structure, normalized block lattice and ladder structures are derived.

The general block implementation for an Nth order scalar digital transfer function

$$h(z) = \left(\sum_{i=0}^{N} b_i z^{-i}\right) \Big/ \left(\sum_{i=0}^{N} a_i z^{-i}\right); \quad a_0 \neq 0$$

is given by the block difference equation (3)

$$A_{p,0} Y_k + A_{p,1} Y_{k-1} + \cdots + A_{p,p} Y_{k-p} = P_{p,0} U_k + P_{p,1} U_{k-1} + \cdots + P_{p,p} U_{k-p} \quad (1.1)$$

where p, the order of the difference equation, is the smallest integer such that $pL \geqslant N$, Y_k and U_k are the kth output and input block of length L,

$$Y_k = [y_{kL} \; y_{kL+1} \cdots y_{kL+L-1}]^t; \quad U_k = [u_{kL} \; u_{kL+1} \cdots u_{kL+L-1}]^t,$$

$A_{p,k}$ and $P_{p,k}$ are the $L \times L$ Toeplitz matrices given by

$$A_{p,k} = \begin{bmatrix} a_{kL} & \cdots & a_{kL-L+1} \\ \vdots & & \vdots \\ a_{kL+L-1} & \cdots & a_{kL} \end{bmatrix}; \quad P_{p,k} = \begin{bmatrix} b_{kL} & \cdots & b_{kL-L+1} \\ \vdots & & \vdots \\ b_{kL+L-1} & \cdots & b_{kL} \end{bmatrix}$$

where $a_i = b_i = 0$ for $i < 0$ or $i > N$.

The basic building block in the realization scheme is the block digital two-pair with $L \times 1$ input vectors $U^{(2)}$, $Y^{(1)}$, and $L \times 1$ output vectors $Y^{(2)}$ and $U^{(1)}$ as shown in Fig. 1. The inputs and outputs of the block digital two-pair are related by the block-chain matrix

$$\begin{bmatrix} U^{(2)} \\ Y^{(2)} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U^{(1)} \\ Y^{(1)} \end{bmatrix} \quad (1.2)$$

or by the block-transfer matrix

$$\begin{bmatrix} Y^{(2)} \\ U^{(1)} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} U^{(2)} \\ Y^{(1)} \end{bmatrix} \quad (1.3)$$

where the transfer parameter matrices are related to the chain parameter matrices by the relations

$$T_{11} = CA^{-1}, \; T_{12} = (D - CA^{-1}B), \; T_{21} = A^{-1}, \text{ and } T_{22} = -A^{-1}B.$$

The block digital filter is realized by cascading p such blocks.

In Section II, two general methods of synthesizing a block digital filter in lattice and ladder forms are given. In Section III, block lattice and ladder structures are developed using a modified Levinson recursion algorithm. In Section IV, the normalized block lattice and ladder structures are derived, and finally, in Section V, a numerical example is included to illustrate the procedures.

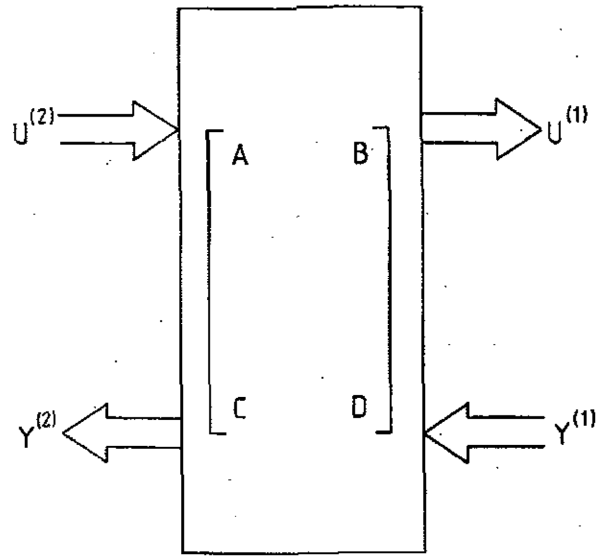


FIG. 1. Block two-pair.

II. Cascaded Block Lattice and Ladder Structures

Two methods of developing cascaded block lattice and ladder structures are described in this section. The first method leads to a structure which is essentially a block version of the scalar structure proposed by Mitra, Kamat, and Huey (4), and the second one is a block version of the Gray and Markel's structure (5). Note that neither of the block structures is automatically scaled in the synthesis procedure; as it turns out, there are difficulties with each method.

A. First method

The main idea in this realization scheme is to realize the pth order block transfer matrix $H_p(Z)$ given by

$$H_p(Z) = P_p(Z)A_p^{-1}(Z) \tag{2.1}$$

as a constrained block two-pair with a constraining block transfer matrix $H_{p-1}(Z)$ of lower block order than $H_p(Z)$. In (2.1), Z^{-1} is the block delay variable and is related to the sample delay z^{-1} by $Z^{-1} = z^{-L}$. Next $H_{p-1}(Z)$ is realized in the same manner. This process is continued until the constraining transfer matrix reduces to a constant matrix.

The ith stage of the cascaded block lattice structure is given in Fig. 2, where $U^{(i)}$, $Y^{(i-1)}$ are the input vectors, $Y^{(i)}$, $U^{(i-1)}$ are the output vectors and A_i, B_i, C_i, and D_i are the block-chain matrix parameters. The transfer matrix $H_i(Z)$ is given by

$$H_i(Z) = [C_i + D_i H_{i-1}][A_i + B_i H_{i-1}]^{-1} \tag{2.2}$$

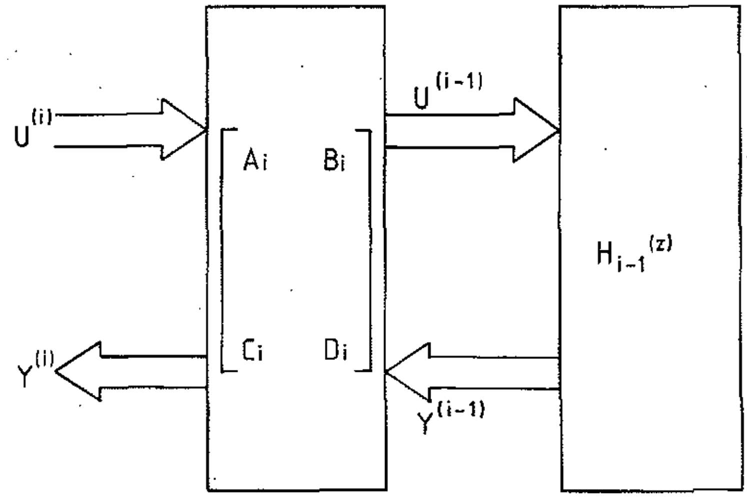


FIG. 2. ith constrained two-pair.

from which $H_{i-1}(z)$ can be expressed as

$$H_{i-1}(Z) = [D_i - H_i B_i]^{-1}[H_i A_i - C_i]. \tag{2.3}$$

The input to the pth stage, $U^{(p)}$ is the same as the input U.

Observe from (1.1) that

$$H_p(Z) = [A_{p,0} + A_{p,1}Z^{-1} + \cdots + A_{p,p}Z^{-p}]^{-1}[P_{p,0} + \cdots + P_{p,p}Z^{-p}].$$

Because the numerator and denominator of this matrix function commute (6), and because $A_{p,0}$ is triangular with $a_0 \neq 0$ down the main diagonal and so is nonsingular,

$$H_p(Z) = [P_{p,0}A_{p,0}^{-1} + \cdots + P_{p,p}A_{p,0}^{-1}Z^{-p}]\,[I + A_{p,1}A_{p,0}^{-1}Z^{-1} + \cdots + A_{p,p}A_{p,0}^{-1}Z^{-p}]^{-1}$$

where I is the identity matrix. By redefinition of symbols then, we can regard the task as one of synthesizing a pth order block transfer function

$$H_p(Z) = P_p(Z)A_p(Z)^{-1} = \left(\sum_{i=0}^{p} P_{p,1}Z^{-i}\right)\left(\sum_{i=0}^{p} A_{p,i}Z^{-1}\right)^{-1} \tag{2.4}$$

where, without loss of generality, $A_{p,0}$ can be assumed to be an identity matrix. The procedure is as follows. The chain parameters for the pth stage are chosen to be

$$A_p = I,\quad B_p = A_{p,p}Z^{-1},\quad C_p = P_{p,0},\quad D_p = P_{p,p}Z^{-1}. \tag{2.5}$$

The lower order constraining transfer function matrix H_{p-1} is of the form

$$H_{p-1}(Z) = P_{p-1}(Z)A_{p-1}(Z)^{-1} = \left(\sum_{i=0}^{p-1} P_{p-1,i}Z^{-i}\right)\left(\sum_{i=0}^{p-1} A_{p-1,i}Z^{-i}\right)^{-1} \tag{2.6}$$

where $A_{p-1,0}$ is assumed again to be an identity matrix. $H_{p-1}(Z)$ is found as follows.

From (2.2), (2.5), and (2.6), we have

$$H_p = (C_p + D_p H_{p-1})(A_p + B_p H_{p-1})^{-1}$$
$$= (P_{p,0} + P_{p,p} Z^{-1} H_{p-1})(I + A_{p,p} Z^{-1} H_{p-1})^{-1}. \quad (2.7)$$

Using the expressions for H_p and H_{p-1} from (2.4) and (2.6), respectively, it can be shown that

$$P_{p,0} A_{p-1,k} + P_{p,p} P_{p-1,k-1} = P_{p,k}; \quad A_{p-1,k} + A_{p,p} P_{p-1,k-1} = A_{p,k} \quad (2.8)$$

for $k = 1, \ldots, (p-1)$, and

$$P_{p-1,p-1} = I. \quad (2.9)$$

Equation (2.8) can be solved for $P_{p-1,k-1}$ and $A_{p-1,k}$, for $k = 1, \ldots, (p-1)$ yielding

$$\begin{aligned} P_{p-1,k-1} &= (P_{p,p} - P_{p,0} A_{p,p})^{-1} (P_{p,k} - P_{p,0} A_{p,k}), \\ A_{p-1,k} &= A_{p,k} - A_{p,p} P_{p-1,k-1} \end{aligned} \quad (2.10)$$

and this defines $H_{p-1}(Z)$ in terms of $H_p(Z)$ provided the inverse in (2.10) exists.† No adjustment to the procedure is available even in the scalar case, should the inverse fail to exist. This process is repeated (assuming at each stage the inverse exists) until $H_0(Z)$, a constant, (in fact the identity), matrix is obtained. The ith stage will then be given by

$$\begin{bmatrix} U^{(i)} \\ Y^{(i)} \end{bmatrix} = \begin{bmatrix} I & \hat{B}_i \\ C_i & \hat{D}_i \end{bmatrix} \begin{bmatrix} U^{(i-1)} \\ Z^{-1} Y^{(i-1)} \end{bmatrix} \quad (2.11)$$

where $\hat{B}_i = A_{i,i}$ and $\hat{D}_i = P_{i,i}$. Note from (2.10) that $\hat{D}_i = I$ for all i except for $i = p$. Hence only the left most stage will have three block multipliers and the other stages will have only two block multipliers. The resulting lattice structure is given in Figs. 3(a) and (b). The ladder structure can be obtained using the transfer parameters and the ith stage of the ladder structure is given in Fig. 4.

B. Second method

The block transfer matrix of Eq. (2.4) can also be implemented using block lattice and ladder structures as a tapped cascaded block lattice or ladder structure. The procedure for obtaining this structure is outlined below. The block lattice structure will have $(2p+1)$ matrix parameters, which separate into p block multiplier parameter matrices K_m and $(p+1)$ tap parameter matrices v_m. These are obtained recursively from the block transfer matrix of (2.4) as follows (7). As before, $A_{p,0}$ is assumed to be an identity matrix. For $m = p, (p-1), \ldots, 1$ we define recursively

$$B_m(Z)^{-m} = Z^{-m} A_m(Z^{-1}) = \sum_{i=0}^{m} B_{m,m-i} Z^{-i}, \quad (2.12)$$

$$K_{m-1} = A_{m,m}, \quad (2.13)$$

† The inverse will not exist if in (1.1) $pL > N$. If $pL = N$, the inverse fails to exist if and only if $H(0) - H(\infty)$ is finite and singular.

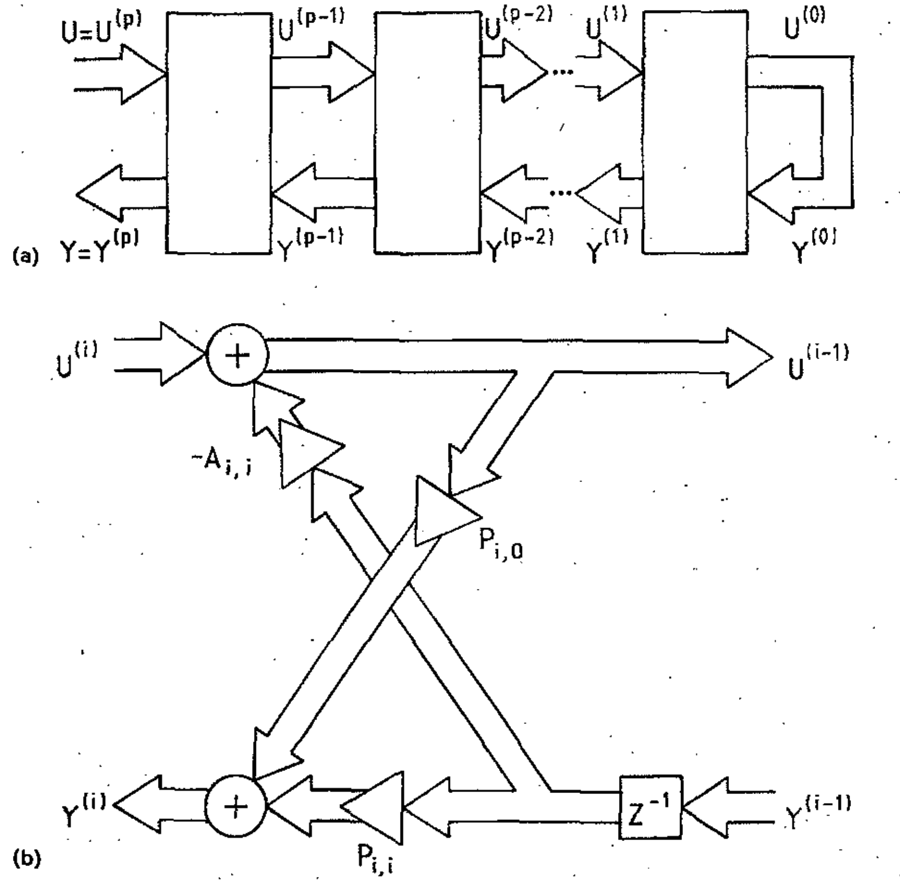


FIG. 3. (a) Cascaded block lattice structures. (b) Lattice structure for the *i*th stage.

$$A_{m-1}(Z) = (I-K_{m-1}^2)^{-1}[A_m(Z)-K_{m-1}B_m(Z)], \tag{2.14}$$

$$v_m = P_{m,m}, \tag{2.15}$$

$$P_{m-1}(Z) = P_m(Z)-v_mB_m(Z), \tag{2.16}$$

with $v_0 = P_{0,0}$, provided $(I-K_{m-1}^2)^{-1}$ exists at each step. A necessary and sufficient condition for the inverses to exist at each step is that the magnitudes of the eigenvalues of K_m, for all m, be different from unity. In the scalar case it has been proved (8) that $K_m < 1$ for a stable filter H_p, and if H_p is scalar then the procedure is indeed the Levinson recursion algorithm. But in the block case this is not so, and it does not seem the case that, in general, the stability of $H_p(Z)$ would guarantee that $|\lambda_i(K_m)| < 1$. In the next section we outline the block version of the Levinson algorithm; again an inverse appears, but with $H_p(Z)$ stable it is guaranteed to exist. In a sense, then, the method above is an incorrect generalization of the associated scalar method, whereas the method of the next section is the correct generalization.

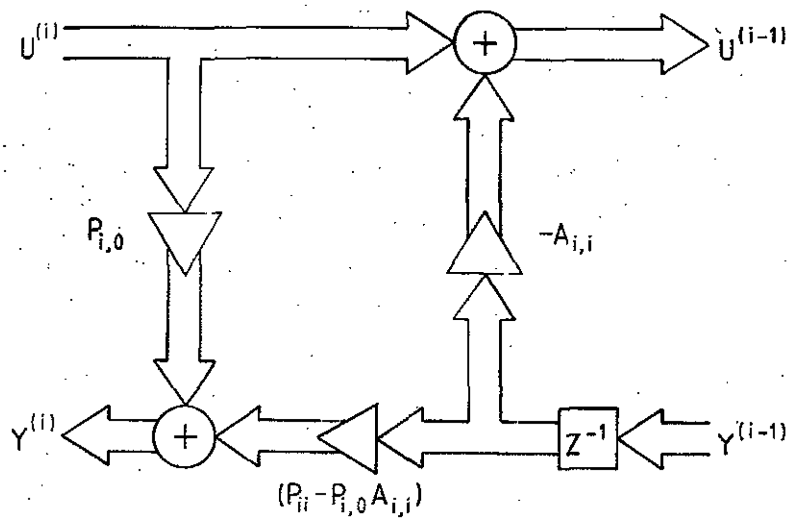


FIG. 4. Ladder structure for the *i*th stage.

From relations (2.12)–(2.14), it can be shown that $A_{m-1,0} = I$ for all m and

$$\begin{bmatrix} A_{m+1} \\ B_{m+1} \end{bmatrix} = \begin{bmatrix} I & K_m \\ K_m & I \end{bmatrix} \begin{bmatrix} A_m \\ Z^{-1}B_m \end{bmatrix}. \tag{2.17}$$

Also, $A_0(Z) = B_0(Z) = A_{0,0} = B_{0,0} = I$. It is seen that (2.17) can be implemented in block lattice form except that matrix polynomials rather than transfer functions are involved. We shall now explain how to remedy this situation. Using (2.16) recursively we can rewrite $P_p(Z)$ as

$$P_p = \sum_{m=0}^{p} v_m B_m. \tag{2.18}$$

Hence the output is given by

$$Y = H_p U = \sum_{m=0}^{p} v_m B_m A_p^{-1} U \tag{2.19}$$

and this suggests the rewriting of (2.17) as

$$\begin{bmatrix} A_{m+1}A_p^{-1}U \\ B_{m+1}A_p^{-1}U \end{bmatrix} = \begin{bmatrix} I & K_m \\ K_m & I \end{bmatrix} \begin{bmatrix} A_m A_p^{-1}U \\ Z^{-1}B_m A_p^{-1}U \end{bmatrix}. \tag{2.20}$$

From (2.19) and (2.20), it is seen that the overall block filter can be implemented using a tapped cascaded block lattice or block ladder structure. The individual stages are shown in Fig. 5.

We have described two methods of implementing a block digital filter in a block lattice and block ladder structure. Unfortunately, these structures are not automatically scaled, and the methods may break down due to the non-existence of certain matrix inverses. However, both of these structures do reduce to the corresponding scalar structures when the block length is unity.

III. Block Levinson Filters

We wish to realize the block transfer matrix of (2.4) as a tapped cascaded block lattice or block ladder structure in such a way that we can readily relate an input bound to a bound on internal variables. To do this we would like to use a Levinson-like approach. We begin by describing vector Levinson filters, and then indicate a

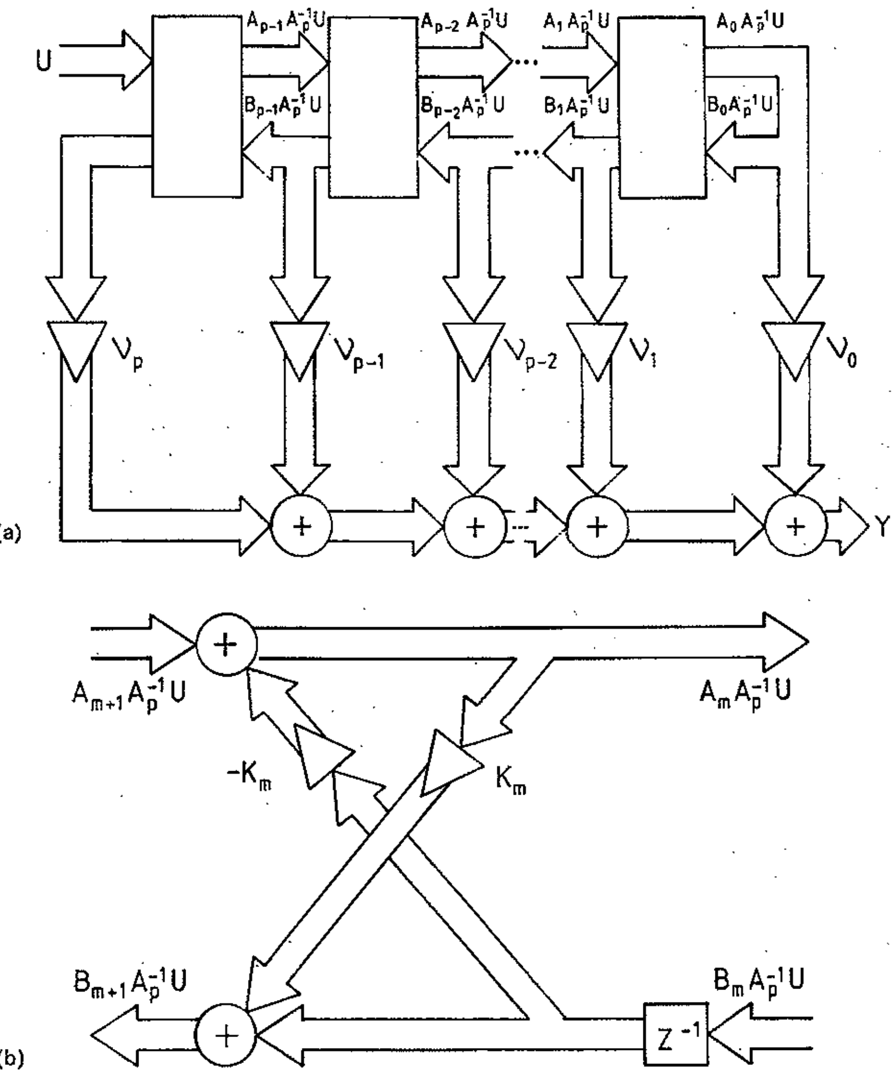
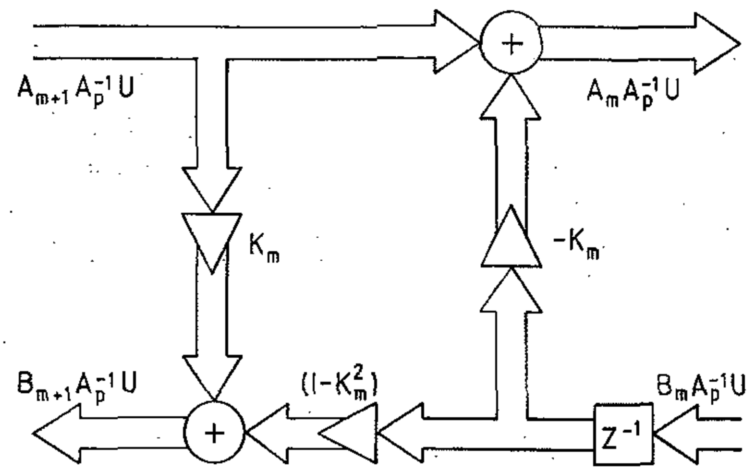


FIG. 5.



(c)

FIG. 5. (a) Tapped cascaded lattice or ladder structure. (b) Lattice structure for the $(m+1)$th stage of Fig. 5(a). (c) Ladder structure for the $(m+1)$th stage of Fig. 5(a).

simplification of the results for block Levinson filters which will give us the desired block lattice or block-ladder structure. Levinson filters are virtually restricted in application to realizing stable transfer functions, and so henceforth we shall assume that the stability property is satisfied.

A. Vector Levinson filters

Gray and Markel have used the scalar version of Levinson filters to obtain the lattice and ladder structures (**5**, **8**, **9**). The vector version of Levinson filters is discussed elsewhere (**10–12**).

In order to implement the transfer function of Eq. (2.4)

$$H_p(Z) = P_p(Z)A_p^{-1}(Z),$$

where $A_{p,0} = I$ using Levinson filters, we need to find $(3p+1)$ matrix parameters, viz., p K_i^α-parameter matrices, $K_0^\alpha, K_1^\alpha, \ldots, K_{p-1}^\alpha$; p K_i^β-parameter matrices, $K_0^\beta, K_1^\beta, \ldots, K_{p-1}^\beta$, and $(p+1)$ tap parameter matrices $v_0, v_1, \ldots, v_p$. These parameter matrices will be found recursively starting from $P_p(Z)$ and $A_p(Z)$. The lattice structure will then be as shown in Fig. 6, where A_i and B_i are matrix polynomials the definitions of which will be given below. The lattice gains matrices K_i^α, K_i^β and the matrix polynomials A_i and B_i depend on $A_p(Z)$ only.

First we define a sequence X from $A_p(Z)$ as

$$X = A_p^{-1}U \tag{3.1}$$

where U is the input. Let the power spectrum of U be given by $\Phi_{UU}(Z) = I$. The power spectrum of X is hence given by

$$\Phi_{XX}(Z) = R(Z) = \sum_{i=-\infty}^{\infty} R_iZ^{-i} = A_p^{-1}(Z)\Phi_{UU}(Z)A_p^{-*}(Z) = A_p^{-1}(Z)A_p^{-*}(Z) \tag{3.2}$$

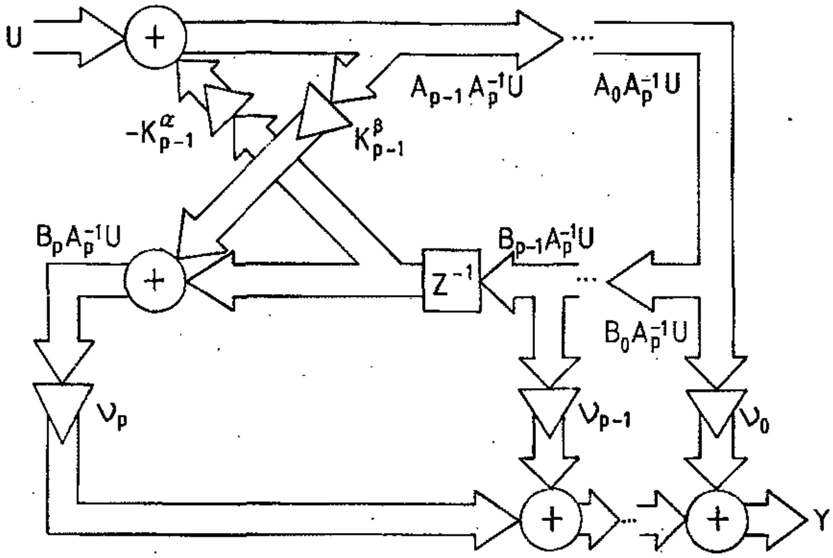


FIG. 6. Lattice structure for Levinson filters.

where

$$A^*(Z) = A^t(Z^{-1}), \quad A^{-*}(Z) = [A^*(Z)]^{-1}$$

and

$$R_i = E\{X_k X_{k-i}^t\}.$$

A_m and B_m are then obtained as polynomials that define mth order optimal forward and backward predictors, respectively, of the sequence X with power spectrum

$$\sum_{i=-\infty}^{\infty} R_i Z^{-i}.$$

The prediction errors are given by

$$\varepsilon_m(k) = \sum_{i=0}^{m} A_{m,i} X_{k-i}; \quad r_m(k) = \sum_{i=0}^{m} B_{m,i} X_{k-m+i}$$

with $A_{m,0} = I$ and $B_{m,0} = I$. The corresponding Z-transforms are given by

$$\varepsilon_m = A_m(Z)X; \quad r_m = B_m(Z)X \tag{3.3}$$

where $A_m(Z)$ and $B_m(Z)$ and defined in obvious manner. The power spectra of ε_m and r_m are given by

$$\Phi_{\varepsilon_m\varepsilon_m}(Z) = A_m \Phi_{XX}(Z) A_m^* = \sum_{i=-\infty}^{\infty} E\{\varepsilon_m(k)\varepsilon_m^t(k-i)\} Z^{-i} \tag{3.4}$$

and

$$\Phi_{r_m r_m}(Z) = B_m \Phi_{XX}(Z) B_m^* = \sum_{i=-\infty}^{\infty} E\{r_m(k) r_m^t(k-i)\} Z^{-i}. \tag{3.5}$$

The optimal predictors of order m satisfy the equation (see e.g. (10–12))

$$\begin{bmatrix} I & A_{m,1} & \cdot & \cdot & \cdot & A_{m,m} \\ B_{m,m} & \cdot & \cdot & \cdot & B_{m,1} & I \end{bmatrix} \mathscr{R}_m = \begin{bmatrix} \pi_m & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \Gamma_m \end{bmatrix} \tag{3.6}$$

where

$$\mathscr{R}_m = \begin{bmatrix} R_0 & \cdot & \cdot & R_m \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ R_{-m} & \cdot & \cdot & R_0 \end{bmatrix} \tag{3.7}$$

and, π_m and Γ_m are the forward and backward prediction error covariances, respectively, given by

$$\pi_m = E\{\varepsilon_m(k)\varepsilon_m^t(k)\}; \quad \Gamma_m = E\{r_m(k)r_m^t(k)\}. \tag{3.8}$$

Several observations can be made here (10–12). First, if $A_p(Z)$ is given, $B_p(Z)$ can be found as follows: use Eq. (3.2) and $\Phi_{UU}(Z) = I$ to derive $R_{-p}, \ldots, R_0, \ldots, R_p$ and then use (3.6) to define $B_p(Z)$. Second, the optimal forward predictor of order p is $A_p(Z)$. Third (and this observation underpins the lattice structure), given the two mth order predictors, $(m+1)$th order predictors can be obtained from the equations (10–12)

$$\begin{bmatrix} A_{m+1} \\ B_{m+1} \end{bmatrix} = \begin{bmatrix} I & Z^{-1}K_m^\alpha \\ K_m^\beta & Z^{-1}I \end{bmatrix} \begin{bmatrix} A_m \\ B_m \end{bmatrix} \tag{3.9}$$

where K_m^α, K_m^β are obtained as follows:

$$\begin{bmatrix} I & A_{m,1} & \cdot & \cdot & A_{m,m} & 0 \\ 0 & B_{m,m} & \cdot & \cdot & B_{m,1} & I \end{bmatrix} \mathscr{R}_{m+1} = \begin{bmatrix} \pi_m & 0 & \cdot & \cdot & \alpha_m \\ \beta_m & \cdot & \cdot & 0 & \Gamma_m \end{bmatrix}, \tag{3.10}$$

$$K_m^\alpha = -\alpha_m \Gamma_m^{-1}; \quad K_m^\beta = -\beta_m \pi_m^{-1}. \tag{3.11}$$

Further,

$$\begin{aligned} \beta_m &= \alpha_m^t, \\ \pi_{m+1} &= \pi_m - \alpha_m \Gamma_m^{-1} \alpha_m^t = (I - K_m^\alpha K_m^\beta)\pi_m, \\ \Gamma_{m+1} &= \Gamma_m - \alpha_m^t \pi_m^{-1} \alpha_m = (I - K_m^\beta K_m^\alpha)\Gamma_m. \end{aligned} \tag{3.12}$$

From (3.9) the structure of Fig. 6 becomes evident. Combining (3.1) and (3.3) it is easily seen that

$$\varepsilon_m = A_m A_p^{-1} U; \quad r_m = B_m A_p^{-1} U; \quad m = 0, 1, \ldots, p.$$

From Fig. 6 it is seen that the input in the forward path to the mth stage is $\varepsilon_m = A_m A_p^{-1} U$ and that in the backward path is $r_{m-1} = B_{m-1} A_p^{-1} U$. The outputs of the mth stage are $\varepsilon_{m-1} = A_{m-1} A_p^{-1} U$ and $r_m = B_m A_p^{-1} U$ in the forward and backward paths, respectively. The mean-square forward and backward prediction errors are given by

$$\bar{\varepsilon}_m = E\{\varepsilon_m^t(k)\varepsilon_m(k)\} = \text{tr}\ \pi_m,$$
$$\bar{r}_m = E\{r_m^t(k)r_m(k)\} = \text{tr}\ \Gamma_m. \tag{3.13}$$

We now outline a procedure† for obtaining the matrix polynomials A_m, B_m, the block multipliers K_m^α, K_m^β, and block tap parameters v_m from the transfer matrix. First B_p is obtained from A_p by solving Eq. (3.6) for $m = p$. Next K_m^α, K_m^β, and v_m are obtained recursively (with m decreasing in the recursion) from the following relations (obtained by rewriting (3.9) and (3.11))

$$K_m^\alpha = A_{m+1,m+1}; \quad K_m^\beta = B_{m+1,m+1}, \tag{3.14}$$

$$A_m = (I - K_m^\alpha K_m^\beta)^{-1}(A_{m+1} - K_m^\alpha B_{m+1}),$$
$$B_m = Z(I - K_m^\beta K_m^\alpha)^{-1}(-K_m^\beta A_{m+1} + B_{m+1}), \tag{3.15}$$

for $m = p-1, \ldots, 0$ and

$$v_m = P_{m,m},$$
$$P_{m-1} = P_m - v_m B_m \tag{3.16}$$

for $m = p, \ldots, 1$ and with $v_0 = P_{0,0}$. As a result of the stability of $H_p(Z)$, it can be shown (11) that the inverses in (3.15) all exist. Using (3.16) recursively it can be shown that

$$P_p(Z) = \sum_{i=0}^{p} v_i B_i. \tag{3.17}$$

Hence the output of the filter will be given by

$$Y = H_p U = \sum_{i=0}^{p} v_i B_i A_p^{-1} U. \tag{3.18}$$

Now we shall discuss the bounding of internal variables in terms of the bounds on input U. We could consider two cases, corresponding to a stationary random input and a deterministic input signal. However, here we shall focus on only the stochastic case, and following convention, derive bounds when U is a white noise process, i.e. $\Phi_{UU}(Z) = I$. Let X denote any internal block variable with $\Phi_{XX}(Z)$ as its power spectrum. X is assumed to be random. Note that with U stationary, X must be stationary. We define the norm of X by

$$\|X\|^2 = E\{X_k^t X_k\} = \text{tr}\ E\{X_k X_k^t\} = \frac{1}{2\pi j}\,\text{tr} \oint \Phi_{XX}(Z) Z^{-1}\, dz \tag{3.19}$$

† While this procedure is not necessarily the most attractive, it serves as the basis for the procedure to be used when the transfer function matrix is obtained by blocking a scalar transfer function.

where the integral is counterclockwise around the boundary of the unit circle. Now it follows that

$$\|A_m A_p^{-1} U\|^2 = \|\varepsilon_m\|^2 = E\{\varepsilon_m^t(k)\varepsilon_m(k)\} = \frac{1}{2\pi j}\,\mathrm{tr}\oint \Phi_{\varepsilon_m\varepsilon_m}(Z)Z^{-1}\,\mathrm{d}Z = \mathrm{tr}\,\pi_m \tag{3.20}$$

where

$$\Phi_{\varepsilon_m\varepsilon_m}(Z) = A_m A_p^{-1}\Phi_{UU}(Z)A_p^{-*}A_m^* = A_m A_p^{-1}A_p^{-*}A_m^*. \tag{3.21}$$

Similarly it can be shown that

$$\|B_m A_p^{-1} U\|^2 = \|r_m\|^2 = \mathrm{tr}\,\Gamma_m. \tag{3.22}$$

Now by virtue of (3.12), we have

$$\mathrm{tr}\,\pi_{m+1} \leqslant \mathrm{tr}\,\pi_m \quad \text{and} \quad \mathrm{tr}\,\Gamma_{m+1} \leqslant \mathrm{tr}\,\Gamma_m. \tag{3.23}$$

Hence the norms of the internal variables at any stage are less than or equal to those of the variables at the stage one to the right. The variable with the largest norm is the right-most one, $A_0 A_p^{-1} U = B_0 A_p^{-1} U$, whose norm is given by $\mathrm{tr}\,\pi_0 = \mathrm{tr}\,\Gamma_0 = \mathrm{tr}\,R_0$.

In this part we have described vector Levinson filters. In part B we simplify these results for the case of block filters derived from an underlying scalar process.

B. *Levinson filters for blocked scalar processes*

In scalar Levinson filters (Gray and Markel's lattice structure), it is found that the internal variables in the backward path have the same norms as the corresponding variables in the forward path. In this section we show the carryover of this result to block Levinson filters derived by blocking an underlying scalar process. Suppose that

$$H_p(Z) = P_p(Z)D_p^{-1}(Z) = \left(\sum_{i=0}^{p} P_{p,i}Z^{-i}\right)\left(\sum_{i=0}^{p} D_{p,i}Z^{-i}\right)^{-1} \tag{3.24}$$

where $H_p(Z)$ is the block version of the scalar transfer function

$$h_N(z) = \left(\sum_{i=0}^{N} b_i z^{-i}\right)\Big/\left(\sum_{i=0}^{N} a_i z^{-i}\right).$$

(Of course, $H_p(Z)$ is stable precisely when $h_N(z)$ is stable.) Then $P_{p,i}$ and $D_{p,i}$ are given by

$$P_{p,i} = \begin{bmatrix} b_{iL} & \cdots & b_{iL-L+1} \\ \vdots & & \vdots \\ b_{iL+L-1} & \cdots & b_{iL} \end{bmatrix}, \quad D_{p,i} = \begin{bmatrix} a_{iL} & \cdots & a_{iL-L+1} \\ \vdots & & \vdots \\ a_{iL+L-1} & \cdots & a_{iL} \end{bmatrix} \tag{3.25}$$

and p is the smallest integer such that $pL \geqslant N$. Further suppose that, without real loss of generality, $a_0 \neq 0$. Then we may assume $a_0 > 0$, again without loss of generality, this ensures that $D_{p,0}$ is lower triangular, with positive diagonal elements, Now if V and Y are the input and output, respectively, of the filter then

$$Y = P_p D_p^{-1} V = P_p A_p^{-1} U \tag{3.26}$$

where

$$A_p \triangleq D_{p,0}^{-1} D_p, \tag{3.27}$$

$$U \triangleq D_{p,0}^{-1} V. \tag{3.28}$$

With these definitions $A_{p,0} = I$. Now let us define X as

$$X \triangleq D_p^{-1} V = A_p^{-1} U. \tag{3.29}$$

Now to understand the simplification to the Levinson algorithm that results from the block structure, it is helpful to think in terms of random processes. Accordingly, we suppose that V is a sequence of independent, zero mean unit variance, Gaussian random variables, i.e.

$$\Phi_{VV}(Z) = I$$

so that $\Phi_{XX}(Z)$, the power spectrum of X is given by

$$\Phi_{XX}(Z) \triangleq \bar{R}(Z) = D_p^{-1} D_p^{-*} = \sum_{k=-\infty}^{\infty} \bar{R}_k Z^{-k}. \tag{3.30}$$

Notice that $\bar{R}(Z)$ is not the same as $R(Z)$ in (3.3). Now we prove the following lemma.

Lemma 3.1

With the $s \times s$ matrix J_s defined by

$$J_s = \begin{bmatrix} 0 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 0 \end{bmatrix}, \tag{3.31}$$

$$J_L \bar{R}_k J_L = \bar{R}_{-k} \tag{3.32}$$

and

$$J_L \bar{R}(Z) J_L = \bar{R}(Z^{-1}). \tag{3.33}$$

Proof. $\bar{R}_k$ in (3.30) is given by

$$\bar{R}_k = E\{X_n X_{n-k}^t\}.$$

With X defined by (3.29), X is the block version of the scalar x where

$$x(z) = v(z) \bigg/ \left(\sum_{i=0}^{N} a_i z^{-i} \right)$$

and V is the block version of v, which will itself be a sequence of independent, zero mean, unit variance random variables. We shall call such an input sequence a *standard input* sequence. Hence

$$X_k = [x_{kL} \;\; x_{kL+1} \cdots x_{kL+L-1}]^t.$$

This leads to the following expressions

$$(\bar{R}_k)_{ij} = E\{x_{nL+i-1} x_{nL-kL+j-1}\},$$

$$(\bar{R}_{-k})_{L-i+1,L-j+1} = E\{x_{nL+L-i} x_{nL+kL+L-j}\}.$$

It can then be seen that for a stationary scalar process x, $(\bar{R}_k)_{ij}$ and $(\bar{R}_{-k})_{L-i+1,L-j+1}$ will be equal, whence (3.32) and (3.33) are immediate.

It is possible to give a direct algebraic proof of this lemma which makes no appeal to the random process ideas. For this proof, see Appendix A.

Define

$$\bar{\mathcal{R}}_m \triangleq \begin{bmatrix} \bar{R}_0 & \cdot & \cdot & \bar{R}_m \\ \bar{R}_{-m} & \cdot & \cdot & \bar{R}_0 \end{bmatrix}. \tag{3.34}$$

Then using (3.32) it can be shown that

$$\bar{\mathcal{R}}_m = J_{(m+1)L}\bar{\mathcal{R}}_m J_{(m+1)L} \tag{3.35}$$

for all m.

Next, define for $m < p$ the optimal m-step forward prediction of the X sequence by

$$\hat{X}_k = -\sum_{i=1}^{m} A_{m,i}X_{k-i} \tag{3.36}$$

(where $\Phi_{VV}(Z) = I$ remains in force) and

$$\sum_{i=0}^{m} A_{m,i}Z^{-i} = A_m(Z).$$

Though the symbol $A_p(Z)$ has been used to define a 'normalized' version of $D_p(Z)$, as the following lemma shows, $A_p(Z)$ can also be interpreted as the optimal pth order forward predictor, so that (3.36) applies also to $m = p$.

Lemma 3.2.

With $\Phi_{VV}(Z) = I$, the optimal pth order forward predictor of the X sequence is

$$\hat{X}_k = -\sum_{i=1}^{p} D_{p,0}^{-1}D_{p,i}X_{k-i} = -\sum_{i=1}^{p} A_{p,i}X_{k-i}. \tag{3.37}$$

Proof. For the above predictor the error sequence ε_p is given by

$$\varepsilon_p(k) = \sum_{i=0}^{p} A_{p,i}X_{k-i} \tag{3.38}$$

where $A_{p,0} = I$. Taking Z-transforms, this equation can be rewritten as

$$X = A_p^{-1}\varepsilon_p. \tag{3.39}$$

Comparing this with (3.29) which defines the sequence X it is found that $\varepsilon_p = U$. With U defined by (3.28) as $D_{p,0}^{-1}V$, its power spectrum is given by $\Phi_{UU}(Z) = D_{p,0}^{-1}D_{p,0}^{-t}$. Hence the power spectrum of ε_p is given by

$$\Phi_{\varepsilon_p\varepsilon_p}(Z) = D_{p,0}^{-1}D_{p,0}^{-t}. \tag{3.40}$$

From (3.39) it can be seen that X_k is given by

$$X_k = \sum_{i=0}^{\infty} \mathcal{H}_i\varepsilon_p(k-i) \tag{3.41}$$

where

$$A_p^{-1} = \sum_{i=0}^{\infty} \mathscr{H}_i Z^{-i}.$$

We now define the inner product of two vectors X and Y as

$$\langle X, Y \rangle = E\{Y^t X\} \tag{3.42}$$

where X and Y are random. Two vectors X and Y are said to be orthogonal when their inner product is zero and we denote it by $X \perp Y$. The pth order forward predictor of (3.37) is optimal if and only if

$$\varepsilon_p(k) \perp X_{k-i}; \quad i = 1, 2, \ldots, p. \tag{3.43}$$

From (3.40) and (3.41) it can be seen that the above condition is satisfied. Hence the pth order forward predictor of the X sequence given in (3.37) is optimal.

The orthogonality relations of (3.43) lead to the following equation for $A_p(Z)$:

$$[I \quad A_{p,1} \quad \cdots \quad A_{p,p}]\mathscr{R}_p = [\pi_p \quad 0 \quad \cdots \quad 0] \tag{3.44}$$

which can be rewritten as

$$[J_L A_{p,p} J_L \quad \cdots \quad J_L A_{p,1} J_L \quad I] \quad \mathscr{R}_p = [0 \quad \cdots \quad 0 \quad J_L \pi_p J_L] \tag{3.45}$$

by pre- and post-multiplying by J_{pL+L}. Comparing (3.45) with (3.6), we see that (3.45) identifies $B_p(Z)$, which defines the pth order backward predictor of $X = D_p^{-1} V$, as

$$B_p(Z) = \sum_{i=0}^{p} B_{p,p-i} Z^{-i}$$

where

$$B_{p,i} = J_L A_{p,i} J_L. \tag{3.46}$$

This immediate calculation of the backward predictor from knowledge of the forward predictor is a key advance over the situation in normal matrix Levinson filters. From A_p, B_p, and P_p the lower order polynomials A_m, B_m, and P_m, the block multipliers K_m^α, K_m^β, and the block tap parameters ν_m can be obtained using (3.14)–(3.16). The mth order predictor matrix polynomials A_m and B_m satisfy the equations

$$\begin{bmatrix} I & A_{m,1} & \cdot & \cdot & A_{m,m} & 0 \\ 0 & B_{m,m} & \cdot & \cdot & B_{m,1} & I \end{bmatrix} \mathscr{R}_{m+1} = \begin{bmatrix} \pi_m & 0 & \cdot & \cdot & \alpha_m \\ \beta_m & \cdot & \cdot & 0 & \Gamma_m \end{bmatrix}. \tag{3.47}$$

Since $\mathscr{R}_m = J_{mL+L} \mathscr{R}_m J_{mL+L}$, the matrix polynomial

$$B_m(Z) = \sum_{i=0}^{m} B_{m,m-i} Z^{-i} \tag{3.48}$$

can be obtained from $A_m(Z)$ by

$$B_{m,i} = J_L A_{m,i} J_L \tag{3.49}$$

and further

$$\Gamma_m = J_L \pi_m J_L, \tag{3.50}$$

$$\alpha_m^t = \beta_m = J_L \alpha_m J_L. \tag{3.51}$$

Using the above relations for Γ_m and β_m in (3.11) results in K_m^β given by

$$K_m^\beta = J_L K_m^\alpha J_L. \tag{3.52}$$

Hence the $(m+1)$th stage of the cascaded block lattice structure will be as shown in Fig. 7 and the overall tapped cascaded block lattice structure will be as shown in Fig. 8. The block ladder structure for the $(m+1)$th stage is shown in Fig. 9.

The internal block variables $A_m D_p^{-1} V$ and $B_m D_p^{-1} V$ are precisely ε_m and r_m, respectively. The norms of the internal variables are now given by the following equations for standard inputs:

$$\|V\| = L^{1/2}, \tag{3.53}$$

$$\|U\| = [\mathrm{tr}\,(D_{p,0}^{-1} D_{p,0}^{-t})]^{1/2}, \tag{3.54}$$

$$\|A_m D_p^{-1} V\|^2 = \|\varepsilon_m\|^2 = \mathrm{tr}\ \pi_m. \tag{3.55}$$

Similarly we have

$$\|B_m D_p^{-1} V\|^2 = \|r_m\|^2 = \mathrm{tr}\ \Gamma_m = \mathrm{tr}\ J_L \pi_m J_L = \mathrm{tr}\ \pi_m. \tag{3.56}$$

In part A of Section III it was shown that $\mathrm{tr}\ \pi_{m+1} \leqslant \mathrm{tr}\ \pi_m$. Hence for block Levinson filters, we have

$$\|A_{m+1} D_p^{-1} V\| \leqslant \|A_m D_p^{-1} V\| = (\mathrm{tr}\ \pi_m)^{1/2}, \quad m = 0,\ldots,p-1 \tag{3.57}$$

and

$$\|B_m D_p^{-1} V\| = \|A_m D_p^{-1} V\|, \quad m = 0,\ldots,p. \tag{3.58}$$

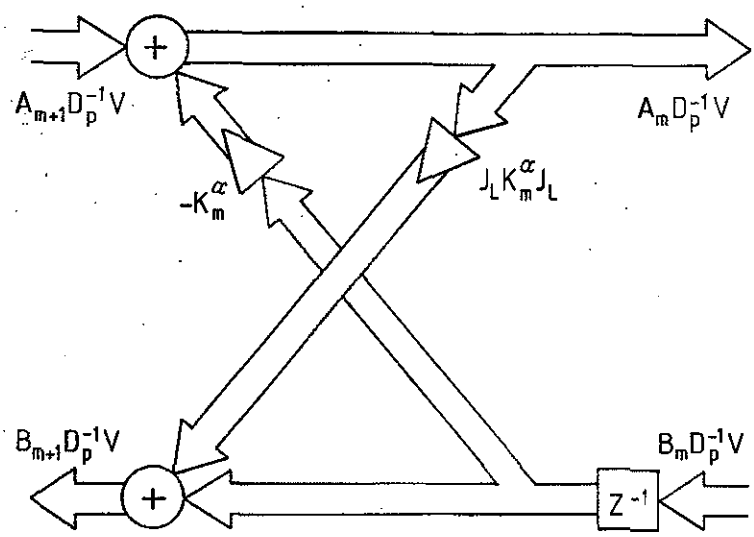


FIG. 7. Lattice structure for the $(m+1)$th stage of Fig. 8.

Vol. 316, No. 1, pp. 93–124, July 1983
Printed in Great Britain

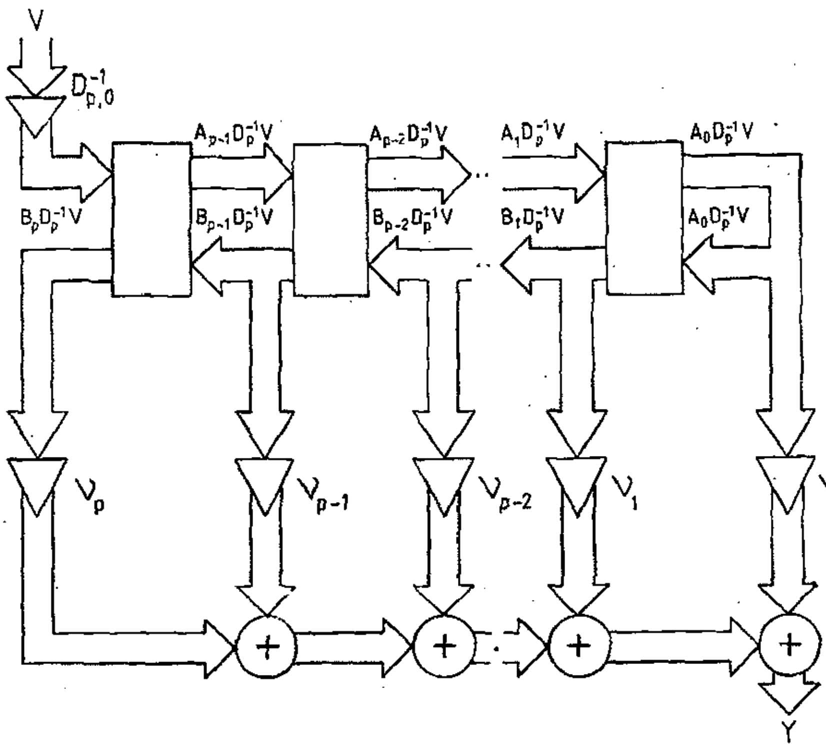


FIG. 8. Tapped cascaded block lattice or block ladder structure.

From part A we also know that the last stage has the largest norm and if the input is bounded all the internal variables will be bounded.

In this section we have presented a tapped cascaded block lattice and a tapped block ladder structure where the structure is characterized by $2p+1$ parameters: p K_i^α-parameters and $(p+1)$ ν_i-parameters. These are found recursively from $P_p(Z)$ and

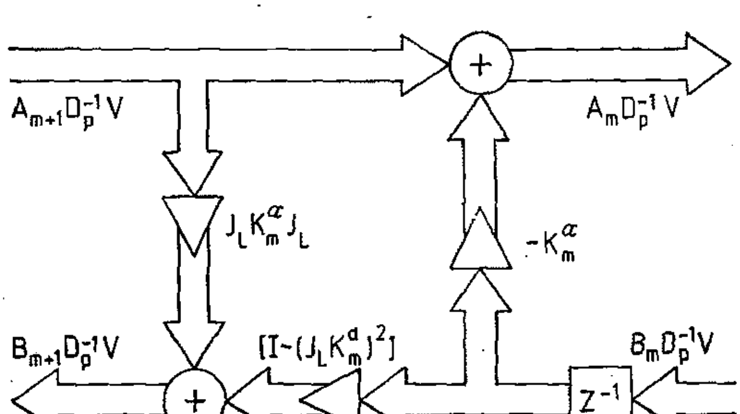


FIG. 9. Ladder structure for the $(m+1)$th stage of Fig. 8.

$D_p(Z)$ using the (3.59)–(3.64) derived by specializing equations given earlier in this section. [These equations are much simpler to use as B_m is obtained easily from A_m; also as multiplication by J_L is nothing but reversing the order of the elements of the vector, we need to store only one matrix multiplier, K_m^α, per stage.]

$$A_p(Z) = D_{p,0}^{-1}D_p, \tag{3.59}$$

$$B_{m+1}(Z) = Z^{-m-1}J_L A_{m+1}(Z^{-1})J_L, \tag{3.60}$$

$$K_m^\alpha = A_{m+1,m+1}, \tag{3.61}$$

$$A_m = [I-(K_m^\alpha J_L)^2]^{-1}[A_{m+1}-K_m^\alpha B_{m+1}], \tag{3.62}$$

for $m = p-1,\ldots,0$, and

$$v_m = P_{m,m}, \tag{3.63}$$

$$P_{m-1} = P_m - v_m - v_m B_m \tag{3.64}$$

for $m = p,\ldots,1$ with $v_0 = P_{0,0}$. (Existence of the inverses in (3.62) is guaranteed when the underlying scalar transfer function is stable.) Also the covariances of the internal variables $A_m D_p^{-1}V$ and $A_{m+1}D_p^{-1}V$, π_m and π_{m+1}, respectively are related by

$$\pi_m = [I-(K_m^\alpha J_L)^2]^{-1}\pi_{m+1} \tag{3.65}$$

for $m = p-1,\ldots,0$ with

$$\pi_p = D_{p,0}^{-1}D_{p,0}^{-t}. \tag{3.66}$$

The lattice structure for the $(m+1)$th stage becomes evident by the following equations for A_m and B_{m+1}:

$$A_m = A_{m+1} - K_m^\alpha Z^{-1}B_m; \quad B_{m+1} = (J_L K_m^\alpha J_L)A_m + Z^{-1}B_m. \tag{3.67}$$

This lattice and ladder structure reduces to scalar Gray and Markel's structure with two multipliers (5), when the block length L is 1. However, the lattice structure discussed in this section is not scaled. Scaling of this structure to obtain a normalized lattice structure is discussed in the next section.

Also, interpretation of the algebraic relations between the elements of α_m, β_m, Γ_m, and π_m, which arise because of the origin of the transfer function matrix $P_p A_p^{-1}$ is given in Appendix B.

IV. Normalized Block Lattice Structure

We wish to scale the lattice and ladder structure of the previous section in such a way that the internal variables at all the nodes have the same norm. Then if the input is bounded the internal variables will also be bounded by the same bound. Normalized vector Levinson filters are discussed in (11) but of course these filters are not for processes derived by blocking a scalar process.

We retain the same notation as in Part B of the previous section. Then in order to obtain a normalized structure we define new polynomials $\hat{A}_m$, $\hat{B}_m$ as

$$\hat{A}_m \triangleq \gamma_m A_m; \quad \hat{B}_m \triangleq \delta_m B_m \tag{4.1}$$

where γ_m and δ_m are multiplier matrices that will be chosen later to yield a normalized structure. With this definition it is found that under standard inputs

$$\|\hat{A}_m D_p^{-1} V\|^2 = \operatorname{tr}(\gamma_m \pi_m \gamma_m^t) \tag{4.2}$$

and

$$\|\hat{B}_m D_p^{-1} V\|^2 = \operatorname{tr}(\delta_m \Gamma_m \delta_m^t). \tag{4.3}$$

By using $\hat{A}_m$ and $\hat{B}_m$ as defined by (4.1) in (3.67) for the lattice structure of A_m and B_m, we obtain

$$\begin{aligned} \hat{A}_m &= \gamma_m \gamma_{m+1}^{-1} \hat{A}_{m+1} - \gamma_m K_m^\alpha \delta_m^{-1} Z^{-1} \hat{B}_m, \\ \hat{B}_{m+1} &= \delta_{m+1}(J_L K_m^\alpha J_L)\gamma_{m+1}^{-1} \hat{A}_{m+1} + \delta_{m+1}[I - J_L K_m^\alpha J_L K_m^\alpha]\delta_m^{-1} Z^{-1} \hat{B}_m \end{aligned} \tag{4.4}$$

which is rewritten as

$$\begin{aligned} \hat{A}_m &= M_{(m+1)1} \hat{A}_{m+1} - M_{(m+1)2} Z^{-1} \hat{B}_m, \\ \hat{B}_{m+1} &= M_{(m+1)3} \hat{A}_{m+1} + M_{(m+1)4} Z^{-1} \hat{B}_m \end{aligned} \tag{4.5}$$

where $M_{(m+1)i}$, the multiplier matrices of the $(m+1)$th stage are defined as

$$\begin{aligned} M_{(m+1)1} &\triangleq \gamma_m \gamma_{m+1}^{-1} \\ M_{(m+1)2} &\triangleq \gamma_m K_m^\alpha \delta_m^{-1}, \\ M_{(m+1)3} &\triangleq \delta_{m+1}(J_L K_m^\alpha J_L)\gamma_{m+1}^{-1}, \\ M_{(m+1)4} &\triangleq \delta_{m+1}[I - J_L K_m^\alpha J_L K_m^\alpha]\delta_m^{-1}. \end{aligned} \tag{4.6}$$

We now define the square root of a positive definite matrix A as a lower triangular matrix $A^{1/2}$ with positive diagonal elements satisfying $A = A^{1/2} A^{t/2}$, where $A^{t/2} = (A^{1/2})^t$. Suppose we choose γ_m and δ_m as

$$\begin{aligned} \gamma_m &= \pi_m^{-1/2}, \\ \delta_m &= J_L \gamma_m J_L \end{aligned} \tag{4.7}$$

then we have

$$\begin{aligned} \|\hat{A}_m D_p^{-1} V\|^2 &= \operatorname{tr}(\gamma_m \pi_m \gamma_m^t) = \operatorname{tr}(\pi_m^{-1/2} \pi_m \pi_m^{-t/2}) = L, \\ \|\hat{B}_m D_p^{-1} V\|^2 &= \operatorname{tr}(\delta_m \Gamma_m \delta_m^t) = \operatorname{tr}(J_L \gamma_m \pi_m \gamma_m^t J_L) = \operatorname{tr}(J_L^2) = L \end{aligned}$$

and

$$\hat{B}_m = Z^{-m} J_L \hat{A}_m(Z^{-1}) J_L.$$

From (3.66) we have $\pi_p = D_{p,0}^{-1} D_{p,0}^{-t}$ where $D_{p,0}$ is lower triangular with positive diagonal elements. Therefore $\gamma_p = D_{p,0}$ and $\hat{A}_p = D_p$. It was shown in the previous section that for standard inputs, the norm of the input is given by

$$\|V\| = L^{1/2}.$$

Hence with γ_m and δ_m chosen as in (4.7), we have

$$\|V\| = \|\hat{A}_m \hat{A}_p^{-1} V\| = \|\hat{B}_m \hat{A}_p^{-1} V\| = L^{1/2}, \quad m = 0, 1, \ldots, p. \tag{4.8}$$

The $(m+1)$th stage of the normalized lattice structure is as shown in Fig. 10(a) and the tapped cascaded structure is as shown in Fig. 10(b).

Motivated by results available in the scalar case (9), we now proceed to obtain alternative expressions for the multipliers of the $(m+1)$th stage, see (4.18)–(4.21)

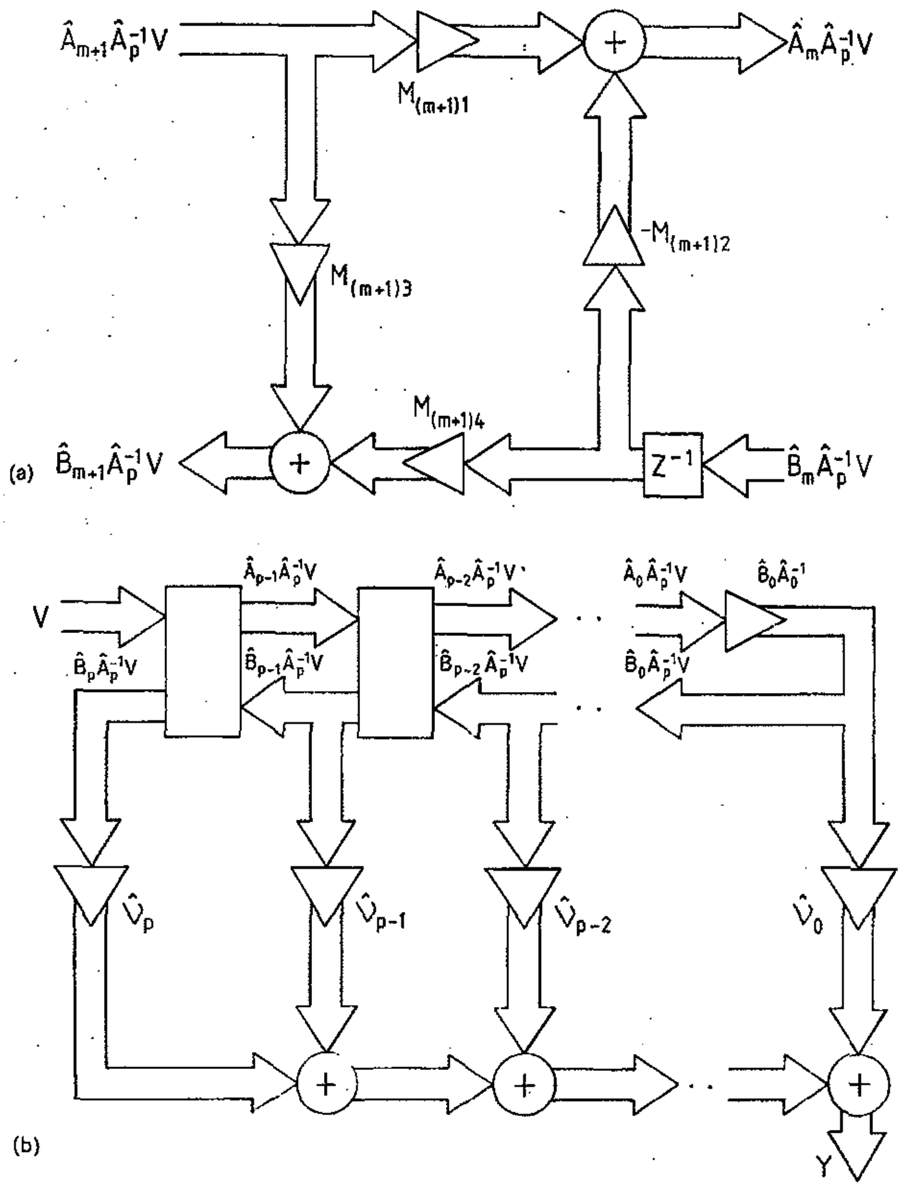


FIG. 10. (a) Ladder structure for the $(m+1)$th stage of the normalized block filter. (b) Normalized block structure I.

below. From (3.12), we have

$$\pi_{m+1} = \pi_m - \alpha_m \Gamma_m^{-1} \alpha_m^t$$

which can be rewritten as

$$\pi_{m+1} = \pi_m^{1/2}[I - \pi_m^{-1/2}\alpha_m J_L \pi_m^{-t/2} J_L J_L \pi_m^{-1/2} J_L \alpha_m^t \pi_m^{-t/2}]\pi_m^{t/2}.$$

Now defining a matrix $\hat{K}_m^{\alpha}$ as

$$\hat{K}_m^{\alpha} \triangleq -\pi_m^{-1/2}\alpha_m J_L \pi_m^{-t/2} J_L \tag{4.9}$$

we have

$$\begin{aligned} (\hat{K}_m^{\alpha})^t &= J_L \hat{K}_m^{\alpha} J_L, \\ \pi_{m+1} &= \pi_m^{1/2}[I - \hat{K}_m^{\alpha}(\hat{K}_m^{\alpha})^t]\pi_m^{t/2} \end{aligned} \tag{4.10}$$

as $\alpha_m^t = J_L \alpha_m J_L$ from (3.51). Hence, we have

$$\pi_{m+1}^{1/2} = \pi_m^{1/2} S_{m+1}^{1/2} \tag{4.11}$$

where

$$S_{m+1} \triangleq [I - \hat{K}_m^{\alpha}(\hat{K}_m^{\alpha})^t] = [I - (\hat{K}_m^{\alpha} J_L)^2] \tag{4.12}$$

and this leads to the relations

$$\gamma_{m+1} = \pi_{m+1}^{-1/2} = S_{m+1}^{-1/2}\gamma_m, \tag{4.13}$$

$$M_{(m+1)1} = \gamma_m \gamma_{m+1}^{-1} = S_{m+1}^{1/2}. \tag{4.14}$$

Now using the definition of K_m^{α} as in (3.11), we have

$$M_{(m+1)2} = \gamma_m K_m^{\alpha} \delta_m^{-1} = -\pi_m^{-1/2}\alpha_m J_L \pi_m^{-1}\pi_m^{1/2} J_L = \hat{K}_m^{\alpha}. \tag{4.15}$$

From the definition of $M_{(m+1)3}$ and the expressions obtained above for $M_{(m+1)1}$ and $M_{(m+1)2}$ we have

$$\begin{aligned} M_{(m+1)3} &= J_L \gamma_{m+1} K_m^{\alpha} J_L \gamma_{m+1}^{-1} = J_L \gamma_{m+1} \gamma_m^{-1} \hat{K}_m^{\alpha} J_L \gamma_m \gamma_{m+1}^{-1} \\ &= J_L S_{m+1}^{-1/2} \hat{K}_m^{\alpha} J_L S_{m+1}^{1/2}. \end{aligned} \tag{4.16}$$

From the definition of γ_m we have

$$\gamma_m^{-1}\gamma_m^{-t} = \pi_m$$

and π_{m+1} is given from (3.65) as

$$\pi_{m+1} = [I - (K_m^{\alpha} J_L)^2]\pi_m$$

Hence we have

$$\gamma_{m+1}^{-1}\gamma_{m+1}^{-t} = [I - (K_m^{\alpha} J_L)^2]\gamma_m^{-1}\gamma_m^{-t}.$$

Using this relation in the definition of $M_{(m+1)4}$, we have

$$\begin{aligned} M_{(m+1)4} &= J_L \gamma_{m+1}[I - (K_m^{\alpha} J_L)^2]\gamma_m^{-1} J_L = J_L \gamma_{m+1}^{-t} \gamma_m^t J_L \\ &= J_L (M_{(m+1)1})^t J_L = J_L S_{m+1}^{t/2} J_L. \end{aligned} \tag{4.17}$$

Now summarizing the results, the multiplier matrices of the $(m+1)$th stage of the normalized lattice structure are given by

$$M_{(m+1)1} = S_{m+1}^{1/2},$$
$$M_{(m+1)2} = \hat{K}_m^\alpha,$$
$$M_{(m+1)3} = J_L S_{m+1}^{-1/2} \hat{K}_m^\alpha J_L S_{m+1}^{1/2},$$
$$M_{(m+1)4} = J_L S_{m+1}^{t/2} J_L$$

where $S_{m+1}^{1/2}$ is defined via (4.11), $\hat{K}_m^\alpha$ is defined in any one of a number of ways, including the following, obtainable from (4.15):

$$\hat{K}_m^\alpha = \pi_m^{-1/2} K_m^\alpha J_L \pi_m^{1/2} J_L. \tag{4.18}$$

The output of the filter is obtained as

$$Y = \sum_{m=0}^{p} v_m B_m \hat{A}_p^{-1} V = \sum_{m=0}^{p} \hat{v}_m \hat{B}_m \hat{A}_p^{-1} V$$

where

$$\hat{v}_m = v_m \delta_m^{-1}.$$

In this section we have obtained a normalized block filter where all the internal variables and the input have the same bound, by using normalized Levinson recursion algorithm. This structure will reduce to the normalized filter structure of Gray and Markel (9) when the block length is unity. Also in this structure the covariances of all the internal variables are identity matrices.

V. A Numerical Example

The example considered here is a fourth-order digital transfer function given by

$$h(z) = \frac{2-(1/6)z^{-1}+(1/5)z^{-2}+(1/8)z^{-3}-(1/3)z^{-4}}{1-(7/12)z^{-1}-(1/6)z^{-2}+(7/48)z^{-3}-(1/48)z^{-4}}.$$

The block version of this transfer function for a block length of 2 is given as

$$H(Z) = \left\{\begin{bmatrix} 2 & 0 \\ -1/6 & 2 \end{bmatrix} + \begin{bmatrix} 1/5 & -1/6 \\ 1/8 & 1/5 \end{bmatrix} Z^{-1} + \begin{bmatrix} -1/3 & 1/8 \\ 0 & -1/3 \end{bmatrix} Z^{-2}\right\}$$
$$\left\{\begin{bmatrix} 1 & 0 \\ -7/12 & 1 \end{bmatrix} + \begin{bmatrix} -1/6 & -7/12 \\ 7/48 & -1/6 \end{bmatrix} Z^{-1} + \begin{bmatrix} -1/48 & 7/48 \\ 0 & -1/48 \end{bmatrix} Z^{-2}\right\}^{-1}.$$

This can be rewritten as

$$H(Z) = P_2 A_2^{-1}$$

where

$$P_2 = \sum_{i=0}^{2} P_{2,i} Z^{-i}, \quad A_2 = \sum_{i=0}^{2} A_{2,i} Z^{-i}$$

and the matrices $P_{2,i}$ and $A_{2,i}$ are

$$P_{2,2}=\begin{bmatrix}-0.2604 & 0.1250\\ -0.1944 & -0.3333\end{bmatrix},\quad P_{2,1}=\begin{bmatrix}0.1028 & -0.1667\\ 0.2417 & 0.2\end{bmatrix},$$

$$P_{2,0}=\begin{bmatrix}2 & 0\\ 1 & 2\end{bmatrix},\quad A_{2,2}=\begin{bmatrix}0.0642 & 0.1458\\ -0.0122 & -0.0208\end{bmatrix},$$

$$A_{2,1}=\begin{bmatrix}-0.5069 & -0.5833\\ 0.0486 & -0.1667\end{bmatrix},\quad A_{2,0}=I.$$

First method

From A_2 and P_2 lower order polynomials A_1, P_1, A_0, and P_0 are obtained using the relations

$$P_{m-1,m-1}=A_{m-1,0}=I,$$
$$P_{m-1,i-1}=(P_{m,m}-P_{m,0}A_{m,m})^{-1}(P_{m,i}-P_{m,0}A_{m,i}),$$
$$A_{m-1,i}=A_{m,i}-A_{m,m}P_{m-1,i-1}$$

for $m=2,1$; $i=1,\ldots,m$ and are found to be

$$P_{1,1}=I,$$

$$P_{1,0}=\begin{bmatrix}-2.8989 & -1.9179\\ 0.0641 & -1.5249\end{bmatrix},\quad A_{1,1}=\begin{bmatrix}-0.3301 & -0.2377\\ 0.0147 & -0.2217\end{bmatrix},$$

$$A_{1,0}=I,\quad P_{0,0}=I,\quad A_{0,0}=I.$$

The lattice and ladder structure is now obtained from Figs. 3 (a) and (b) and Fig. 4.

Second method

From P_2 and A_2, the multiplier matrices K_0, K_1, and the tap parameter matrices v_0, v_1, and v_2 are obtained recursively using the relations

$$K_{m-1}=A_{m,m};\quad A_{m-1}=(I-K_{m-1}^2)^{-1}[A_m-K_{m-1}Z^{-m}A_m(z^{-1})],$$
$$v_m=P_{m,m};\quad P_{m-1}=P_m-v_mB_m$$

for $m=2,1$ with $v_0=P_{0,0}$. These are found to be

$$K_1=\begin{bmatrix}0.0642 & 0.1458\\ -0.0122 & -0.0208\end{bmatrix},\quad A_{1,1}=\begin{bmatrix}-0.4823 & -0.5239\\ 0.0436 & -0.1767\end{bmatrix},\quad A_{1,0}=I,$$

$$K_0=\begin{bmatrix}-0.4823 & -0.5239\\ 0.0436 & -0.1767\end{bmatrix},\quad A_{0,0}=I,\quad v_2=\begin{bmatrix}-0.2604 & 0.1250\\ -0.1944 & -0.3333\end{bmatrix},$$

$$P_{1,1}=\begin{bmatrix}-0.0353 & -0.2977\\ 0.1593 & 0.0310\end{bmatrix},\quad P_{1,0}=\begin{bmatrix}2.0182 & 0.0406\\ 1.0084 & 2.0214\end{bmatrix},$$

$$v_1=\begin{bmatrix}-0.0353 & -0.2977\\ 0.1593 & 0.0310\end{bmatrix},$$

$$P_{0,0} = \begin{bmatrix} 2.0142 & -0.0305 \\ 1.0839 & 2.1104 \end{bmatrix}, \qquad v_0 = \begin{bmatrix} 2.0142 & -0.0305 \\ 1.0839 & 2.1104 \end{bmatrix}.$$

Now the block lattice and ladder structures are obtained from Figs. 5 (a), (b) and (c).

Third method

Block lattice and ladder structures are obtained using block Levinson filters. The block transfer matrix is now written as

$$H(Z) = P_2 D_2^{-1}$$

where P_2 and D_2 are second-order block polynomials given by

$$P_{2,2} = \begin{bmatrix} 2 & 0 \\ -1/6 & 2 \end{bmatrix}, \quad P_{2,1} = \begin{bmatrix} 1/5 & -1/6 \\ 1/8 & 1/5 \end{bmatrix}, \quad P_{2,0} = \begin{bmatrix} -1/3 & 1/8 \\ 0 & -1/3 \end{bmatrix},$$

$$D_{2,2} = \begin{bmatrix} 1 & 0 \\ -7/12 & 1 \end{bmatrix}, \quad D_{2,1} = \begin{bmatrix} -1/6 & -7/12 \\ -7/12 & 1 \end{bmatrix}, \quad D_{2,1} = \begin{bmatrix} -1/6 & -7/12 \\ 7/48 & -1/6 \end{bmatrix},$$

$$D_{2,0} = \begin{bmatrix} -1/48 & 7/48 \\ 0 & -1/48 \end{bmatrix}.$$

From D_2 a polynomial A_2 is obtained as

$$A_2 = D_{2,0}^{-1} D_2$$

and is found to be

$$A_{2,2} = \begin{bmatrix} -0.0208 & 0.1458 \\ -0.0122 & 0.0642 \end{bmatrix}, \quad A_{2,1} = \begin{bmatrix} -0.1667 & -0.5833 \\ 0.0486 & -0.5069 \end{bmatrix}, \quad A_{2,0} = I.$$

Now the multiplier matrices K_1^α, K_0^α and the tap parameter matrices v_2, v_1, and v_0 are obtained recursively from the relations

$$B_m = Z^{-m} J_2 A_m(Z^{-1}) J_2; \quad K_{m-1}^\alpha = A_{m,m},$$

$$A_{m-1} = [I - (K_{m-1}^\alpha J_2)^2]^{-1} [A_m - K_{m-1}^\alpha B_m],$$

$$v_m = P_{m,m}; \quad P_{m-1} = P_m - v_m B_m$$

for $m = 2, 1$ with $v_0 = P_{0,0}$. These are found to be

$$K_1^\alpha = \begin{bmatrix} -0.0208 & 0.1458 \\ -0.0122 & 0.0642 \end{bmatrix},$$

$$A_{1,1} = \begin{bmatrix} -0.0942 & -0.5679 \\ 0.0790 & -0.4999 \end{bmatrix}, \quad A_{1,0} = I,$$

$$K_0^\alpha = \begin{bmatrix} -0.0942 & -0.5679 \\ 0.0790 & -0.4999 \end{bmatrix},$$

$$A_{0,0} = I,$$

$$v_2 = \begin{bmatrix} -0.3333 & 0.125 \\ 0 & -0.3333 \end{bmatrix}$$

$$P_{1,1} = \begin{bmatrix} 0.1039 & -0.1296 \\ -0.0694 & 0.1444 \end{bmatrix}, \quad P_{1,0} = \begin{bmatrix} 2.0031 & -0.0014 \\ -0.1180 & 1.9930 \end{bmatrix},$$

$$v_1 = \begin{bmatrix} 0.1039 & -0.1296 \\ -0.0694 & 0.1444 \end{bmatrix},$$

$$P_{0,0} = \begin{bmatrix} 1.9815 & -0.0219 \\ -0.0707 & 2.0122 \end{bmatrix},$$

$$v_0 = \begin{bmatrix} 1.9815 & -0.0219 \\ -0.0707 & 2.0122 \end{bmatrix}.$$

The block lattice and ladder structure is now obtained from Figs. 7–9.

The covariances of the internal variables in the forward and backward paths are given by π_m and $J_2\pi_m J_2$, respectively. These are found to be

$$\pi_2 = \begin{bmatrix} 1 & 0.5833 \\ 0.5833 & 1.3403 \end{bmatrix},$$

$$\pi_1 = \begin{bmatrix} 1.0186 & 0.5914 \\ 0.5914 & 1.3438 \end{bmatrix}, \quad \pi_0 = \begin{bmatrix} 1.6937 & 1.0620 \\ 1.0620 & 1.6937 \end{bmatrix}.$$

The norms of the internal variables are the same in the forward and backward paths and these are found to be $\|A_2 D_2^{-1} V\| = \|B_2 D_2^{-1} V\| = 1.5298$, $\|A_1 D_2^{-1} V\| = \|B_1 D_2^{-1} V\| = 1.5370$, $\|A_0 D_2^{-1} V\| = \|B_0 D_2^{-1} V\| = 1.8405$.

Fourth method

The normalized form of the lattice structure obtained by the above method is derived here. New polynomials $\hat{A}_m$ and $\hat{B}_m$ are obtained as

$$\hat{A}_m = \gamma_m A_m, \quad \hat{B}_m = \delta_m B_m$$

where $\gamma_m = \pi_m^{-1/2}$ and $\delta_m = J_2 \gamma_m J_2$. The polynomial $\hat{A}_m$, the multipliers $M_{m,i}$ and the tap parameters $\hat{v}_m$ are found to be

$$\hat{A}_{2,2} = \begin{bmatrix} -0.0208 & 0.1458 \\ 0 & -0.0208 \end{bmatrix}, \quad \hat{A}_{2,1} = \begin{bmatrix} -0.1667 & -0.5833 \\ 0.1458 & -0.1667 \end{bmatrix},$$

$$\hat{A}_{2,0} = \begin{bmatrix} 1 & 0 \\ -0.5833 & 1 \end{bmatrix}, \quad \hat{A}_{1,1} = \begin{bmatrix} -0.0934 & -0.5627 \\ 0.1337 & -0.1702 \end{bmatrix},$$

$$\hat{A}_{1,0} = \begin{bmatrix} 0.9908 & 0 \\ -0.5804 & 0.9998 \end{bmatrix}, \quad \hat{A}_{0,0} = \begin{bmatrix} 0.7684 & 0 \\ -0.6185 & 0.9864 \end{bmatrix},$$

$$M_{21} = \begin{bmatrix} 0.9908 & 0 \\ 0.0028 & 0.9998 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0.0206 & -0.13373 \\ 0.00005 & 0.0206 \end{bmatrix},$$

$$M_{23} = \begin{bmatrix} 0.0208 & 0 \\ -0.13368 & 0.0208 \end{bmatrix}, \quad M_{24} = \begin{bmatrix} 0.9998 & 0 \\ 0.0028 & 0.9908 \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} 0.7755 & 0 \\ -0.0463 & 0.9866 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0.0734 & 0.6270 \\ -0.1381 & 0.0734 \end{bmatrix},$$

$$M_{13}=\begin{bmatrix}0.0934 & -0.13373\\ 0.6227 & 0.0934\end{bmatrix},\quad M_{14}=\begin{bmatrix}0.9866 & 0\\ -0.0463 & 0.7755\end{bmatrix},$$

$$\hat{v}_2=\begin{bmatrix}-0.3333 & -0.0694\\ 0 & -0.3333\end{bmatrix},\quad \hat{v}_1=\begin{bmatrix}0.1040 & -0.0699\\ -0.0694 & 0.1050\end{bmatrix},$$

$$\hat{v}_0=\begin{bmatrix}2.0089 & 1.5886\\ -0.0717 & 2.5610\end{bmatrix}.$$

The normalized structure is now obtained from Figs. 10 (a) and (b). The multiplier $\hat{B}_0\hat{A}_0^{-1}$ is found to be

$$\hat{B}_0\hat{A}_0^{-1}=\begin{bmatrix}0.7790 & -0.6270\\ 0.6270 & 0.7790\end{bmatrix}.$$

From this example it is clear that the internal variables in the structure obtained by using a vector Levinson filter for the blocked transfer function matrix, and also the associated normalized structure are *not* blocked versions of the corresponding scalar variables. Nevertheless, the fact that the block filter so obtained is ultimately associated with a scalar transfer function means that the structure can be found with computational saving.

VI. Conclusion

Some new algorithms to realize block digital filters in lattice and ladder form are presented. Levinson recursion algorithms have been used to obtain a cascaded block lattice and ladder structure where the internal node variables are bounded if the input is bounded. This structure is then normalized to yield a structure where all the internal node variables and the input have the same bounds. All these structures reduce to the corresponding scalar structures when the block length is unity. Finally a numerical example is given to illustrate the procedure.

Acknowledgement

This work was supported by the National Science Foundation under Grant ECS79-18028, and the Australian Research Grants Committee.

References

(1) S. K. Mitra and R. Gnanasekaran, "Block implementation of recursive digital filters—New structures and properties", *IEEE Trans. Circuits Syst.*, Vol. CAS-25, pp. 200–207, 1978.

(2) C. W. Barnes and S. Shinnaka, "Finite word effects in block-state realizations of fixed-point digital filters", *IEEE Trans. Circuits Syst.*, Vol. CAS-27, pp. 345–349, 1980.

(3) P. Ananthakrishna and S. K. Mitra, "Block-state recursive digital filters with minimum round-off noise", "Proc. IEEE Int. Conf. Acoust. Speech, and Signal Processing", Vol. 1, pp. 81–84, April 1980.

(4) S. K. Mitra, P. S. Kamat and D. C. Huey, "Cascaded lattice realization of digital filters", *Circuit Theory Appl.*, Vol. 5, pp. 3–11, 1977.

(5) A. H. Gray, Jr. and J. D. Markel, "Digital lattice and ladder filter synthesis", *IEEE Trans. Audio Electroacoust.*, Vol. AU-21, pp. 491–500, 1973.
(6) R. Gnanasekaran, "Block implementation of one-dimensional recursive digital filters", Ph.D. Dis., University of California, Santa Barbara, May 1978.
(7) P. Ananthakrishna, S. K. Mitra and B. D. O. Anderson, "Digital lattice and ladder block structures", Proc. Asilomar Conf. on Circuits, Systems and Computers, pp. 234–238, 1980.
(8) J. D. Markel and A. H. Gray, Jr., "On autocorrelation equations as applied to speech analysis", *IEEE Trans. Audio Electroacoust.*, Vol. AU-21, No. 2, 1973.
(9) A. H. Gray, Jr. and J. D. Markel, "A normalized digital filter structure", *IEEE Trans. Acoust. Speech and Signal Processing*, Vol. ASSP-23, No. 3, 1975.
(10) T. Kailath, "A view of three decades of linear filtering theory", *IEEE Trans. Inf. Theory*, Vol. IT-20, No. 2, 1974.
(11) M. Morf, A. Vieira, D. T. L. Lee and T. Kailath, "Recursive multichannel maximum entropy spectral estimation", *IEEE Trans. GeoScience Electronics*, Vol. GE-16, No. 2, April 1978.
(12) B. D. O. Anderson and J. B. Moore, "Optimal Filtering", Prentice-Hall, Englewood Cliffs, N.J., 1979.

Appendix A

Here we shall give a direct algebraic proof of the identity $\bar{R}_{-k} = J_L \bar{R}_k J_L$ where $\bar{R}$ is defined as

$$\bar{R}(Z) = \sum_{k=-\infty}^{\infty} \bar{R}_k Z^{-k} \triangleq D_p^{-1} D_p^{-*} \tag{A.1}$$

where $D_p(Z)$ is the block version of the polynomial

$$d(Z) = \sum_{i=0}^{pL} d_i Z^{-i}$$

and is given by

$$D_p(Z) = \sum_{i=0}^{p} D_{p,i} Z^{-1} \tag{A.2}$$

where

$$D_{p,i} = \begin{bmatrix} d_{iL} & \cdot & \cdot & d_{iL-L+1} \\ d_{iL+L-1} & \cdot & \cdot & d_{iL} \end{bmatrix}. \tag{A.3}$$

Now $J_L \bar{R}(Z) J_L = \bar{R}(Z^{-1})$ is a necessary and sufficient condition for $\bar{R}_{-k} = J_L \bar{R}_k J_L$ where J_L is a $L \times L$ matrix given by

$$J_L = \begin{bmatrix} 0 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 0 \end{bmatrix}.$$

This is verified easily by comparing the coefficients of like powers of Z in the Laurent expansion of both sides,

$$J_L\bar{R}(Z)J_L = \sum_{k=-\infty}^{\infty} J_L\bar{R}_kJ_LZ^{-k},$$

$$\bar{R}(Z^{-1}) = \sum_{k=-\infty}^{\infty} \bar{R}_{-k}Z^{-k}.$$

Equivalently, we need to prove that $J_L\bar{R}^{-1}(Z)J_L = \bar{R}^{-1}(Z^{-1})$.

Now from (A.1), we have

$$J_L\bar{R}^{-1}(Z)J_L = J_LD_p^*D_pJ_L. \tag{A.4}$$

It is easily seen from (A.2) and (A.3) that

$$J_LD_pJ_L = D_p^t \tag{A.5}$$

and this reduces (A.4) to

$$J_L\bar{R}^{-1}(Z)J_L = D_p(Z^{-1})D_p^t(Z). \tag{A.6}$$

Now we can write $D_p(Z^{-1})$ as

$$D_p(Z^{-1}) = \sum_{i=0}^{p} D_{p,i}Z^i = Z^p \sum_{i=0}^{p} D_{p,p-i}Z^{-i} \triangleq Z^pC_p^t(Z) \tag{A.7}$$

where $C_p(Z)$ is the block version of the polynomial,

$$c(z) = \sum_{i=0}^{pL} c_iz^{-i}$$

and

$$c_i = \mathrm{d}_{pL-i}.$$

This reduces (A.6) to

$$J_L\bar{R}^{-1}(Z)J_L = Z^pC_p^t(Z)D_p^t(Z). \tag{A.8}$$

Now from (A.1) and (A.7), we have

$$\bar{R}^{-1}(Z^{-1}) = D_p^*(Z^{-1})D_p(Z^{-1}) = Z^pD_p^t(Z)C_p^t(Z). \tag{A.9}$$

Hence from (A.8) and (A.9), $\bar{R}^{-1}(Z^{-1})$ and $J_L\bar{R}^{-1}(Z)J_L$ will be equal if and only if $C_p^t(Z)$ and $D_p^t(Z)$ commute, that is iff $C_p(Z)$ and $D_p(Z)$ commute. We will show by the following two lemmas that C_p and D_p commute.

Lemma A.1

If L_1 and L_2 are two lower triangular Toeplitz matrices of the same order, then L_1 and L_2 commute, and their product L_3 is also lower triangular and Toeplitz, more precisely, if

$$L_1 = \begin{bmatrix} x_1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_N & \cdot & \cdot & x_1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} y_1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_N & \cdot & \cdot & y_1 \end{bmatrix},$$

then

$$L_3 = \begin{bmatrix} z_1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_N & \cdot & \cdot & z_1 \end{bmatrix} = L_1 L_2 = L_2 L_1$$

with

$$z_i = \sum_{k=1}^{i} x_k y_{i-k+1} = \sum_{k=1}^{i} x_{i-k+1} y_k.$$

This can be proved by direct calculation.

Lemma A.2

If $C(Z)$ and $D(Z)$ are the block versions of two polynomials $c(z)$ and $d(z)$ (in z^{-1}), then $C_p(Z)$ and $D_p(Z)$ commute.

Consider the square matrices L_1 and L_2 of order $(K_1+K_2+1)L$, where K_1 and K_2 are the maximum powers of Z^{-1} in C and D, respectively, given by

$$L_1 = \begin{bmatrix} C_0 & \cdot & \cdot & 0 \\ C_1 & C_0 & \cdot & \cdot \\ C_{K_1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & C_{K_1} & \cdot & C_0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} D_0 & \cdot & \cdot & 0 \\ D_1 & D_0 & \cdot & \cdot \\ D_{K_2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & D_{K_2} & \cdot & D_0 \end{bmatrix}.$$

Now it can be observed by considering the C_i and D_i matrices in detail that L_1 and L_2 are lower triangular and Toeplitz (as well as being block Toeplitz matrices). Hence by Lemma A.1 they commute, and so

$$\begin{aligned} &C_0 D_0 = D_0 C_0, \\ &C_1 D_0 + C_0 D_1 = D_1 C_0 + D_0 C_1, \\ &C_{K_1} D_0 + \cdots + C_0 D_{K_1} = D_0 C_{K_1} + \cdots + D_{K_1} C_0, \\ &\quad \vdots \\ &C_{K_1} D_{K_2} = D_{K_2} C_{K_1}. \end{aligned}$$

However, these equations represent a term by term equating of the coefficients of the powers of Z^{-1} in $C(Z)D(Z)$ and $D(Z)C(Z)$. Hence $C(Z)$ and $D(Z)$ commute.

From the above two lemmas we see that C_p and D_p commute and hence $J_L \bar{R}^{-1}(Z) J_L = \bar{R}^{-1}(Z^{-1})$ and this completes the proof of the identity $\bar{R}_{-k} = J_L \bar{R}_k J_L$.

Appendix B

In Section III it was shown that the matrices Γ_m and β_m could be obtained from π_m and α_m, respectively, by a type of rotation of the matrices (pre- and post-multiplication by J_L). Here we give an interpretation of the elements of ε_m, r_m, π_m, Γ_m, α_m and β_m leading to the same relations.

Consider the signal X defined in (3.29) as

$$X \triangleq D_p^{-1} V \tag{B.1}$$

We define the mth order optimal forward predictor of the X sequence as

$$E[X_k/X_{k-1},\ldots,X_{k-m}] = \hat{X}_k = -\sum_{i=1}^{m} A_{m,i}X_{k-i} \tag{B.2}$$

where $\hat{X}_k$ is the predicted value of X_k given $X_{k-1},\ldots,X_{k-m}$ and the mth order backward predictor as

$$E[X_{k-m}/X_k,\ldots,X_{k-m+1}] = \tilde{X}_{k-m} = -\sum_{i=1}^{m} B_{m,i}X_{k-m+i} \tag{B.3}$$

where $\tilde{X}_{k-m}$ is the predicted value of X_{k-m} given $X_{k-m+1},\ldots,X_k$. The forward and backward prediction errors $\varepsilon_m(k)$ and $r_m(k)$, respectively at kth instant, are then defined as

$$\varepsilon_m(k) = X_k - E[X_k/X_{k-1},\ldots,X_{k-m}], \tag{B.4}$$

$$r_m(k) = X_{k-m} - E[X_{k-m}/X_k,\ldots,X_{k-m+1}]. \tag{B.5}$$

With X defined by (B.1), X is the block version of the scalar sequence x (see part B, Section III) and can therefore be written as

$$X_k = [x_{kL} \quad x_{kL+1} \cdots x_{kL+L-1}]^t. \tag{B.6}$$

Using this, the forward and backward prediction errors can be rewritten as

$$\varepsilon_m(k) = \begin{bmatrix} x_{kL} - E[x_{kL}/x_{kL-1},\ldots,x_{kL-mL}] \\ x_{kL+1} - E[x_{kL+1}/x_{kL-1},\ldots,x_{kL-mL}] \\ \vdots \\ x_{kL+L-1} - E[x_{kL+L-1}/x_{kL-1},\ldots,x_{kL-mL}] \end{bmatrix}, \tag{B.7}$$

$$r_m(k) = \begin{bmatrix} x_{kL-mL} - E[x_{kL-mL}/x_{kL+L-1},\ldots,x_{kL-mL+L}] \\ x_{kL-mL+L} - E[x_{kL-mL+1}/x_{kL+L-1},\ldots,x_{kL-mL+L}] \\ \vdots \\ x_{kL-mL+L-1} - E[x_{kL-mL+L-1}/x_{kL+L-1},\ldots,x_{kL-mL+L}] \end{bmatrix}. \tag{B.8}$$

Hence the ith elements of $\varepsilon_m(k)$ and $r_m(k)$ are given by

$$\begin{aligned} [\varepsilon_m(k)]_i &= x_{kL+i-1} - E[x_{kL+i-1}/x_{kL-1},\ldots,x_{kL-mL}], \\ [r_m(k)]_i &= x_{kL-mL+i-1} - E[x_{kL-mL+i-1}/x_{kL+L-1},\ldots,x_{kL-mL+L}]. \end{aligned} \tag{B.9}$$

Now we define a scalar j-step, qth order forward predictor of x sequence as

$$E[x_n/x_{n-j},\ldots,x_{n-j-q+1}] = -\sum_{i=0}^{q-1} a_{q,j,i}x_{n-j-i}$$

and a scalar j-step, qth order backward predictor of x sequence as

$$E[x_n/x_{n+j},\ldots,x_{n+j+q-1}] = -\sum_{i=0}^{q-1} b_{q,j,i}x_{n+j+i}.$$

With these definitions $[\varepsilon_m(k)]_i$ is the forward prediction error in predicting x_{kL+i-1} using an i-step, mLth order forward predictor and $[r_m(k)]_i$ is the backward prediction error in predicting $x_{kL-mL+i-1}$ using an $(L-i+1)$-step, mLth order backward predictor. Hence $[r_m(k)]_{L-i+1}$ is the backward prediction error in predicting $x_{kL-mL+L-i}$ using an i-step mLth order backward predictor. The forward and backward error variances are the same for the scalar problem.

Hence the mean square values of $[r_m(k)]_{L-i+1}$ and $[\varepsilon_m(k)]_i$ are the same. The error covariances π_m and Γ_m are given by

$$[\pi_m]_{ij} = E\{[\varepsilon_m(k)]_i[\varepsilon_m(k)]_j\}, \tag{B.10}$$

$$[\Gamma_m]_{L-i+1,L-j+1} = E\{[r_m(k)]_{L-i+1}[r_m(k)]_{L-j+1}\} \tag{B.11}$$

and these two are equal. That is

$$[\pi_m]_{ij} = [\Gamma_m]_{L-i+1,L-j+1}$$

and hence

$$\pi_m = J_L\Gamma_m J_L. \tag{B.12}$$

Now we turn to the matrices α_m and β_m. It can be shown that these are given by (12)

$$\alpha_m = \beta_m^t = E\{[X_{k+1} - E[X_{k+1}/X_k,\ldots,X_{k-m+1}]]X_{k-m}^t\}, \tag{B.13}$$

$$\beta_m = \alpha_m^t = E\{[X_{k-m} - E[X_{k-m}/X_k,\ldots,X_{k-m+1}]]X_{k+1}^t\}. \tag{B.14}$$

Hence the ijth elements of α_m and β_m are given by

$$[\alpha_m]_{ij} = E\{[x_{kL+L+i-1} - E[x_{kL+L+i-1}/x_{kL+L-1},\ldots,x_{kL+L-mL}]]x_{kL-mL+j-1}\}, \tag{B.15}$$

$$[\beta_m]_{ij} = E\{[x_{kL-mL+i-1} - E[x_{kL-mL+i-1}/x_{kL+L-1},\ldots,x_{kL+L-mL}]]x_{kL+L+j-1}\}. \tag{B.16}$$

Therefore $[\alpha_m]_{ij}$ is the expected value of the product of the error in predicting $x_{kL+L+i-1}$ using an i-step, mLth order forward predictor and $x_{kL-mL+j-1}$. Now (B.16) can be rewritten as

$$[\beta_m]_{L-i+1,L-j+1} = E\{[x_{kL-mL+L-i} - E[x_{kL-mL+L-i}/x_{kL+L-1},\ldots,x_{kL+L-mL}]]x_{kL+2L-j}\}.$$

That is $[\beta_m]_{L-i+1,L-j+1}$ is the expected value of the product of the error in predicting $x_{kL-mL+L-1}$ using an i-step, mLth order backward predictor and $x_{kL+2L-j}$. It can be noticed that

$$(kL+L+i-1)-(kL-mL+j-1) = (kL+2L-j)-(kL-mL+L-i) = mL+L+i-j.$$

Hence $[\alpha_m]_{ij}$ and $[\beta_m]_{L-i+1,L-j+1}$ are equal leading to the relation

$$\alpha_m = J_L\beta_m J_L = \beta_m^t. \tag{B.17}$$