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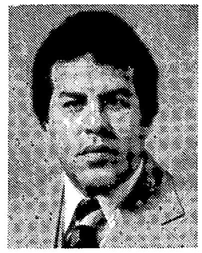

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Transactions Briefs

Simplified Stability Tests for Delay-Differential Systems

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Abstract —**The task of checking that the zeros of a prescribed real exponential polynomial all have negative real part arises in examining the stability of delay-differential systems. We describe new tests which are akin to the reduced order Hermite, reduced Markov, and Liénard–Chipart stability tests for ordinary real polynomials.**

I. Introduction

Systems which include the usual lumped linear elements as well as delays have characteristic equations of the form

$$g(z)=\sum_{i=0}^{n}\sum_{k=0}^{m} g_{ik}z^{i}e^{-\omega_k z}=0,\qquad g_{ik}\ \text{real},\quad 0=\omega_0<\omega_1<\cdots<\omega_m. \tag{1.1}$$

This paper discusses stability criteria for such systems which are analogous to the reduced or half-size type criteria applicable when $g(z)$ is polynomial, such as the Liénard–Chipart, reduced Hermite, and reduced Markov criteria, [1], [2].

Let us write

$$f(z)=e^{\omega_m z}g(z)=\sum_{k=0}^{m} p_k(z)e^{\sigma_k z} \tag{1.2}$$

where $\sigma_k=\omega_m-\omega_{m-k}$ and p_k are polynomials; notice that $f(\cdot)$ and $g(\cdot)$ have the same zeros. It is known that the collection of pairs $(\sigma_i, \deg p_i)$, $i=0,,\cdots,m$ allows easy classification of (1.1) as being of advanced, neutral, or retarded type [3]. For retarded systems, exponential asymptotic stability is equivalent to $f(z)$ possessing all zeros in $\mathrm{Re}[z]<0$. For neutral systems, if one can show that all zeros of $f(z)$ lie in $\mathrm{Re}[z]<-\delta<0$ for some δ, exponential stability follows (but the converse is not true). Advanced systems are never stable.

For these reasons, we shall be concerned with the problem of zero location for the so-called exponential polynomial, typified by (1.2).

For an introduction to such equations, see [3]. For a more sophisticated treatment, based on properties of entire functions, see [4], [5]. We note also that some early results involving extensions to exponential polynomials [6], [7] of the Hermite and Hurwitz criteria are observed in [5] to contain errors.

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II. Review of Krein's Results

With $f(z)$ as in (1.2), define

$$\phi(z)=\frac{f(jz)}{j}. \tag{2.1}$$

Our task is to make statements about the zeros of $f(z)$. It is trivial to check if $f(z)$ has a zero at $z=0$. Since such a zero would rule out the exponential stability property, we shall only consider $f(z)$ with $f(0)\neq 0$, and thus $\phi(0)\neq 0$. Also, while for the most part, we shall be interested in real $f(z)$, we shall in the early part of this section make only the inessential restriction that $f(0)$ is real. Thus with

$$\phi(z)=g(z)-jh(z) \tag{2.2}$$

we have

$$h(0)\neq 0 \qquad g(0)=0. \tag{2.3}$$

(Some of the first results continue to be valid for $g(0)\neq 0$.) The restriction allows us to define, at least in a neighborhood of the origin,

$$\frac{g(z)}{h(z)}=s_0+s_1z+s_2z^2+\cdots \tag{2.4}$$

and an associated quadratic form

$$S(g,h;\xi_0,\xi_1,\cdots)=\sum_{i,k=0}^{\infty}s_{i+k+1}\xi_i\xi_k. \tag{2.5}$$

Now $\phi(z)$ is certainly of exponential type because $f(z)$ is, and it also has positive defect, see [5, p. 323], provided it is not an ordinary polynomial. (We assume this to be so, in order to avoid known situations.) If $f(z)$ has all zeros in $\mathrm{Re}[z]<0$, then $\phi(z)$ has all zeros in $\mathrm{Im}[z]>0$. Accordingly, $\phi(z)$ is in the class P of [5, p. 319], or the class H_2 of [4, p. 249]. This by theorem 11 of [4] allows us to conclude:

$$f(z) \text{ has all zeros in } \mathrm{Re}[z]<0 \;\Rightarrow\; S \text{ is a positive form.} \tag{2.6}$$

The converse result is, see [4]

$$S \text{ is a positive form } \Rightarrow \quad \phi(z)=e(z)\phi_1(z), \quad e(z) \text{ real}, \quad \phi_1(z) \text{ possesses all zeros } \mathrm{Im}[z]>0 \tag{2.7}$$

$$\Rightarrow \quad f(z) \text{ possesses all zeros in } \mathrm{Re}[z]<0 \text{ except possibly for zeros on } \mathrm{Re}[z]=0, \text{ or located symmetrically with respect to } \mathrm{Re}[z]=0. \tag{2.8}$$

If S is merely a nonnegative form, i.e., has finite rank, we are in the classical situation of rational g/h [1]. We shall omit further consideration of this point.

This result is very similar to the Markov stability criterion for polynomial stability [1] save that the expansion (2.4) would be replaced by a power series in z^{-1}. The theorem provides a formal solution to the problem of checking whether zeros lie in $\mathrm{Re}[z]<0$: One forms S and checks for positivity. If this fails, the stability test fails. If S is positive, one then checks $f(z-\delta)$ for very small positive δ for stability, to eliminate the possibility (see (2.8)) of purely imaginary zeros of $f(z)$, or of zeros located symmetrically with respect to the imaginary axis. This "shifted" f approach is also relevant in studying the stability of neutral systems.

We shall term the above criterion (2.8) for zero positions the Krein criterion.

Bezoutian Approach

Define the Bezoutian

$$\frac{g(z)h(u)-g(u)h(z)}{z-u}=\sum_{p,q=0}^{\infty}b_{pq}z^pu^q \tag{2.9}$$

and the Bezoutian form

$$B(g,h;\eta_0,\eta_1,\cdots)=\sum_{p,q=0}^{\infty}b_{pq}\eta_p\eta_q. \tag{2.10}$$

Then, see [4, p. 253]

$$S(g,h;\xi_0,\xi_1,\cdots,\xi_n)=B(g,h;\eta_0,\eta_1,\cdots,\eta_n) \qquad \forall n \tag{2.11}$$

when

$$\eta_i=c_0\xi_i+c_1\xi_{i+1}+\cdots+c_{n-1}\xi_{n-1+i} \tag{2.12}$$

and

$$\frac{1}{h(z)}=c_0+c_1z+\cdots. \tag{2.13}$$

The coefficients in the quadratic form B are readily given in terms of coefficients in power series expansions of $g(\cdot)$ and $h(\cdot)$:

$$g(z)=a_0+a_1z+a_2z^2+\cdots \qquad h(z)=b_0+b_1z+b_2z^2+\cdots \tag{2.14}$$

$$b_{ij}=(i+1,j)+(i+2,j-1)+\cdots+(i+j+1,0) \tag{2.15}$$

where

$$(\alpha,\beta)=a_\alpha b_\beta-a_\beta b_\alpha. \tag{2.16}$$

Of course, the stability results involving the form S carry over to ones involving the form B.

Hurwitz Approach

Define for $n=0,1,2,\cdots$

$$B_n=\begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0,n-1}\\ \vdots & & & \vdots\\ b_{n-1,0} & & & b_{n-1,n-1}\end{bmatrix}$$

$$S_n=\begin{bmatrix} s_1 & s_2 & \cdots & s_n\\ s_2 & s_3 & & s_{n+1}\\ \vdots & & & \\ s_n & s_{n+1} & \cdots & s_{2n-1}\end{bmatrix} \tag{2.17}$$

$$H_{2n}=\begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{2n-1}\\ a_0 & a_1 & a_2 & \cdots & a_{2n-1}\\ 0 & b_0 & b_1 & \cdots & b_{2n-2}\\ 0 & a_0 & a_1 & \cdots & a_{2n-2}\\ & & & & \vdots \end{bmatrix}, \qquad n=0,1,2,\cdots. \tag{2.18}$$

Then

$$\det H_{2n}=c_0^{-2n}|S_n|=|B_n|. \tag{2.19}$$

The implications of this identity are obvious.

The above calculations have made no assumption of realness of $f(z)$. We shall now observe what happens in this case.

Bezoutians and Real f(z)

Let

$$f(z)=f_0+f_1z+f_2z^2+\cdots. \tag{2.20}$$

Then

$$g(z)=f_1z-f_3z^3+f_5z^5-\cdots \qquad h(z)=f_0-f_2z^2+f_4z^4\cdots. \tag{2.21}$$

Define

$$g_1(z)=f_1+f_3z+f_5z^2+\cdots \qquad h_1(z)=f_0+f_2z+\cdots \tag{2.22}$$

so that

$$g(z)=zg_1(-z^2) \qquad h(z)=h_1(-z^2). \tag{2.23}$$

Now by trivial calculation,

$$\frac{g(z)h(u)-g(u)h(z)}{z-u} = \frac{z^2g_1(-z^2)h_1(-u^2)-u^2g_1(-u^2)h_1(-z^2)}{z^2-u^2} + uz\frac{g_1(-z^2)h_1(-u^2)-g_1(-u^2)h_1(-z^2)}{z^2-u^2}.$$

Now if

$$\frac{-xg_1(x)h_1(y)+yg_1(y)h_1(x)}{-x+y}=\sum_{p,q=0}^{\infty}\gamma_{pq}x^py^q$$

and

$$\frac{[-g_1(x)]h_1(y)-[-g_1(y)]h_1(x)}{x-y}=\sum_{p,q=0}^{\infty}\delta_{pq}x^py^q$$

while as in (2.9),

$$\frac{g(z)h(u)-g(u)h(z)}{z-u}=\sum_{p,q=0}^{\infty}g_{pq}z^pu^q$$

we have

$$\sum_{p,q=0}^{\infty}g_{pq}z^pu^q=\sum_{p,q=0}^{\infty}(-1)^{p+q}\gamma_{pq}z^{2p}u^{2q} + \sum_{p,q=0}^{\infty}(-1)^{p+q}\delta_{pq}z^{2p+1}u^{2q+1}$$

and accordingly,

$$B(g,h;\xi_0,\xi_1,\cdots)=B(zg_1,h_1;\xi_0,-\xi_2,\xi_4,\cdots) + B(-g_1,h_1;\xi_1,-\xi_3,\xi_5,\cdots). \tag{2.24}$$

Accordingly, the form on the left is positive if and only if the two forms on the right are positive. This generalizes the idea known for real polynomials [2] that the Bezoutian matrix can be represented (with row and column reordering) as a direct sum of two Bezoutian matrices.

Krein and Hurwitz Criteria for real f(z)

Suppose that

$$\frac{g_1(z)}{h_1(z)}=\bar{s}_0+\bar{s}_1z+\bar{s}_2z^2+\cdots. \tag{2.25}$$

Then positivity of the Bezoutian of zg_1, h_1 requires (see the connection of (2.11)) that

$$\bar{S}_n=\begin{bmatrix} \bar{s}_0 & \bar{s}_1 & \cdots & \bar{s}_{n+1} \\ \bar{s}_1 & & & \vdots \\ \vdots & & & \\ \bar{s}_{n+1} & \bar{s}_{n+2} & \cdots & \bar{s}_{2n} \end{bmatrix} \tag{2.26}$$

have positive determinant for all n, while positivity of the Bezoutian of $-g_1, h_1$ requires that

$$\bar{T}_n=\begin{bmatrix} -\bar{s}_1 & -\bar{s}_2 & \cdots & -\bar{s}_n \\ -\bar{s}_2 & -\bar{s}_3 & \cdots & -\bar{s}_{n+1} \\ \vdots & \vdots & & \vdots \\ -\bar{s}_n & -\bar{s}_{n+1} & \cdots & -\bar{s}_{2n} \end{bmatrix} \tag{2.27}$$

have positive determinant for all n.

Finally, let us indicate the effect on the Hurwitz test. Requiring the pair zg_1, h_1 to have a positive Bezoutian form is equivalent to requiring all even order leading principal minors of the following matrix to be positive, provided that $f_0>0$:

$$H_1=\begin{bmatrix} f_0 & f_2 & f_4 & f_6 & \cdots \\ 0 & f_1 & f_3 & f_5 & \cdots \\ 0 & f_0 & f_2 & f_4 & \cdots \\ 0 & 0 & f_1 & f_3 & \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{2.28}$$

Requiring the pair $-g_1, h_1$ to have a positive Bezoutian form is equivalent to requiring all even order leading principal minors of the following matrix to be positive, provided that $f_0>0$:

$$H_2=\begin{bmatrix} f_0 & f_2 & f_4 & \cdots \\ -f_1 & -f_3 & -f_5 & \cdots \\ 0 & f_0 & f_2 & \cdots \\ 0 & -f_1 & -f_3 & \cdots \end{bmatrix}. \tag{2.29}$$

Together, these conditions amount to requiring positivity of all leading principal minors of the following matrix, provided that $f_0>0$:

$$H=\begin{bmatrix} f_1 & f_3 & f_5 & f_7 & \cdots \\ f_0 & f_2 & f_4 & f_6 & \cdots \\ 0 & f_1 & f_3 & f_5 & \cdots \\ 0 & f_0 & f_2 & f_4 & \cdots \end{bmatrix}. \tag{2.30}$$

There is of course no real loss of generality in demanding that $f_0>0$.

III. Reduced Criteria

We begin with the following preliminary observation.

Lemma: Let $f(z)$ be a real exponential polynomial with $f(0)>0$ and with all zeros in $\mathrm{Re}[z]<0$. Let

$$f(z)=\sum_{i=0}^{\infty}f_iz^i.$$

Then $f_i>0$ for all i.

Proof: Let $\phi(z)=f(jz)/j=(1/j)\Sigma p_k(jz)e^{j\sigma_kz}$. By [5, see p. 323, example 1] the defect is positive. (We restrict ourselves to the case when not all σ_k are zero). Also, $\phi(z)$ is certainly of exponential type, and has no roots in $\mathrm{Im}[z]<0$. Then [5, p. 319], $\phi(z)$ is of class P, with a Hadamard decomposition of the form

$$\phi(z)=z^me^{az+b}\prod_{k=1}^{\infty}\left(1-\frac{z}{\alpha_k}\right)e^{z/\alpha_k}$$

with

$$\mathrm{Im}[a]\geqslant 0 \quad\text{and}\quad \sum_{k=1}^{\infty}\left|\mathrm{Im}\left[\frac{1}{\alpha_k}\right]\right|<\infty.$$

Hence

$$f(z)=z^me^{\hat{a}z+\hat{b}}\prod_{k=1}^{\infty}\left(1-\frac{z}{\hat{\alpha}_k}\right)e^{z/\hat{\alpha}_k}$$

with

$$\mathrm{Re}[\hat{a}] \geqslant 0, \quad \sum_{k=1}^{\infty} \left| \mathrm{Re}\left[\frac{1}{\hat{\alpha}_k} \right] \right| < \infty$$

and with obvious connection between a and $\hat{a}$, etc. Hence we may write this as

$$f(z) = z^m e^{\hat{a}z+\hat{b}} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\hat{\alpha}_k}\right). \tag{3.1}$$

Since $f(\cdot)$ is real, with all zeros in $\mathrm{Re}[z] < 0$, it follows that $m = 0$, $\hat{a}$ is real and the $\hat{\alpha}_k$ are either real or occur in complex conjugate pairs. Also $f(0) = e^{\hat{b}}$ and so $\hat{b}$ is real.

Considering $\phi(z)$ again, we have

$$\phi(z) = \left(\frac{1}{j} e^{\hat{b}}\right) e^{j\hat{a}z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{(\hat{\alpha}_k / j)}\right) \tag{3.2}$$

with $\mathrm{Re}[\hat{\alpha}_k] < 0$ and $\hat{a}$ real.

Now

$$\frac{\phi(z)}{\bar{\phi}(z)} = e^{j(2\hat{a}z + \pi)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right)\left(1 - \frac{z}{\bar{\alpha}_k}\right)^{-1}$$

and, [5, pp. 319–320], $\hat{a}$ is the defect of $\phi(z)$, which is known to be positive. Hence we have

$$f(z) = f(0) e^{\hat{a}z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\hat{\alpha}_k}\right) \tag{3.3}$$

with $\hat{a} > 0$ and $\mathrm{Re}[\hat{\alpha}_k] < 0$, and complex $\hat{\alpha}_k$ occurring in complex conjugate pairs. It follows easily that the coefficients f_i are all positive. Q.E.D.

If $f(z)$ is an ordinary real polynomial, it is known [1], [2] that one can check for stability by examining the sign of the coefficients of $f(z)$ and by checking for positive definiteness a "half-size" Bezout matrix, or by checking whether alternative Hurwitz determinants are positive. (A simplified Markov stability test also exists.) We now seek an extension of this idea to exponential polynomials. The following main result should be read in conjunction with the identity (2.24), which expresses one Bezout form as the sum of two others.

Theorem: Let $f(z)$ be an exponential polynomial and suppose that in the power series expansion

$$f(z) = f_0 + f_1 z + f_2 z^2 + \cdots \tag{3.4}$$

all f_{2k} are positive (or all f_{2k+1} and f_0 are positive).

Let

$$g_1(z) = f_1 + f_3 z + f_5 z^2 + \cdots \qquad h_1(z) = f_0 + f_2 z + \cdots. \tag{3.5}$$

Then $B(-g_1, h_1; \xi_0, \xi_1, \cdots)$ is a positive Bezout form if and only if $B(zg_1, h_1, \xi_0, \xi_1, \cdots)$ is also positive.

Proof: We shall assume that $B(-g_1, h_1; \xi_0, \xi_1, \cdots)$ is positive. Notice first that with

$$\phi(z) \triangleq \frac{f(jz)}{j} \tag{3.6}$$

we have

$$f_1 - f_3 z^2 + f_5 z^4 \cdots = \frac{\phi(z) + \bar{\phi}(z)}{2z}$$

and

$$f_0 - f_2 z^2 + f_4 z^4 \cdots = \frac{j\phi(z) - j\bar{\phi}(z)}{2}$$

from which we can conclude that $g_1(z)$ and $h_1(z)$ are entire functions. Also, $h_1(0) \neq 0$. Using the positivity of $B(-g_1, h_1; \xi_0, \cdots)$ we have by [4, p.255, th. 11′] that

$$p(z) = -g_1(z) - jh_1(z) = E_1(z) H_1(z) \tag{3.7}$$

where $E_1(z)$ is a real function and $H_1(z)$ is an entire function of one of the classes H_p. By [5, p. 307] and the properties of such functions,

$$\left|\frac{p(z)}{\bar{p}(z)}\right| = \left|\frac{H_1(z)}{\bar{H}_1(z)}\right| < 1 \qquad \text{for } \mathrm{Im}[z] > 0. \tag{3.8}$$

The properties of the classes H_p imply that

$$H_1(z) = P_1(z) - jQ_1(z)$$

where $P_1(z)$ and $Q_1(z)$ are both real and have no common roots. Also, $H_1(z)$ possesses all zeros in $\mathrm{Im}\, z > 0$ [4].

Condition (3.8) implies by a trivial calculation, see [5, pp. 307, 308], that

$$\psi(z) \triangleq -\frac{Q_1}{P_1} = \frac{h_1}{g_1} \tag{3.9}$$

maps $\mathrm{Im}[z] > 0$ into $\mathrm{Im}[\psi] > 0$, and so [5, p. 308, th. 1] that

$$\psi(z) = c \frac{z - a_0}{z - b_0} \prod_{\substack{k \neq 0 \\ k = -\infty}}^{+\infty} \frac{1 - \frac{z}{a_k}}{1 - \frac{z}{b_k}} \tag{3.10}$$

where $b_k < a_k < b_{k+1}$ for all k, with $a_{-1} < 0 < b_1$ and $c > 0$.

Now $h_1(u) > 0$ from $f_{2k} > 0$ for all k. Therefore, the indexing on k in (3.10) can only run over negative values, and we have $\cdots b_{-k} < a_{-k} < b_{-k+1} \cdots < b_0 < a_0 < 0$. The same conditions are satisfied in the case of $f_0 > 0, f_{2k+1} > 0$ because of $g_1(u) > 0$ and $h_1(0) > 0$. Then

$$\phi(z) \triangleq -\frac{zP_1}{Q_1} = \left(\frac{b_0}{c}\right) \frac{z}{z - a_0} \prod \left(\frac{1 - \frac{z}{b_0}}{1 - \frac{z}{a_{-1}}} \right) \left(\frac{1 - \frac{z}{b_{-1}}}{1 - \frac{z}{a_{-2}}} \right) \cdots \tag{3.11}$$

satisfies the same conditions as $\psi(z)$. Accordingly, [5, pp. 307, 308]

$$K_1(z) \triangleq Q_1 - jzP_1 \tag{3.12}$$

obeys

$$\left|\frac{K_1(z)}{\bar{K}_1(z)}\right| < 1 \qquad \text{for } \mathrm{Im}[z] > 0 \tag{3.13}$$

Now P_1 and Q_1 have no common roots and so Q_1 and zP_1 have no common roots. Hence K_1 and $\bar{K}_1$ have no common roots, so that $\bar{K}_1(z)$ has no roots in $\mathrm{Im}\, z \geqslant 0$, i.e., $K_1(z)$ has all its roots in $\mathrm{Im}\, z > 0$. This means the function is of class HB [5, p. 307] and accordingly, using the representations of such functions in [5, p. 318], we see that $K_1(z)$ is of class H_p for some p [4, p. 249].

Now the function $zg_1 - jh_1$ can be written as

$$\begin{aligned} zg_1 - jh_1 &= z[-E_1(z) P_1(z)] - jE_1(z) Q_1(z) \\ &= E_1(z)[jK_1(z)] \end{aligned} \tag{3.14}$$

and because $jK_1(z)$ is of class H_p for some p, it follows [4, th. 11′] that $B(zg_1, h_1; \xi_0, \xi_1, \cdots]$ is positive.

The converse of the theorem follows by reversing the above arguments.

Remarks: 1) Equivalent statements can obviously be made to that of the theorem which involve the positivity of the quadratic forms associated with the Krein criterion, or positivity of the Hurwitz determinants. For example, with all the f_i positive, stability of $f(z)$ will follow from positivity of either all odd or all even leading principal minors of H in (2.30).

2) It might be argued that unless $f(z)$ is a polynomial, the reduced order tests offer little or no advantage, since they still require the checking of an infinite number of conditions. Nevertheless, if one regards examination of successive leading principal minors of the Bezoutians as involving examination of successively more f_i, progression through the f_i is more rapid with the reduced order tests.

IV. Conclusions

We have shown how reduced order tests available for checking the stability of prescribed real polynomials carry over to checking the stability of prescribed real exponential polynomials. These reduced tests pin down stability regions more rapidly than the unreduced tests. However, because they involve for exponential polynomials the satisfaction of an infinite number of conditions, they should in the first instance be regarded as providing necessary conditions for stability.

It would of course be of interest to develop a sequence of finite numbers of sufficient conditions, approaching in the limit necessary and sufficient conditions. However, it appears unlikely that the ideas of this paper could be readily adapted towards this end.

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Realization of Switched-Capacitor Voltage-Wave Filters Containing Zeros of Transmission

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***Abstract*—This paper presents a novel approach to the design of filters which are suitable for eventual realization in monolithic form. The method is based on the use of sampled-data-analog signals and is related to the wave digital filter in its design techniques. Designs for a prototype sixth-order Chebyshev bandpass filter and a third-order low-pass filter, containing zeros of transmission at dc and infinity, are presented. Good agreement with theory is obtained for the practical circuits realized in discrete form.**

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I. Introduction

This paper describes the implementation of two wave filters which use the voltage-wave sampled-data technique already applied [1]–[3] to integrated circuit realizations of low-pass filters which have no zeros of transmission.

In wave filters computations are performed on wave variables, rather than on the more conventional voltage–current variables. They were first reported by Fettweis [4] in 1971 and have been thought of as primarily a type of digital filter. However, recent work [1]–[3] has shown that analog voltages may be used as an alternative to digital numbers to represent the quantities present within the wave structure. This can result in a more efficient means of performing the calculations required within the filter which takes advantage of the lack of sensitivity to coefficient variation present in this class of filter. It also removes the necessity for $A-D$ and $D-A$ converters in the system. Calculations can be performed in parallel at potentially high speed, only two clock phases are required and the circuit power consumption is low. Past work described the implementation of low-pass filters which were based on cascaded transmission-line reference filters but, while these filters are adequate for some purposes, they do not provide real zeros of transmission. However, filter responses which do include zeros of transmission can be constructed from *RLC* ladder filters and these circuits can be modeled by wave filters [5], [6].

Direct wave filter representations of *RLC* ladder filters, while easily found, are not optimum for some realizations due to the fact that the calculations required during one sample period must be carried out in a set sequence. This results in a circuit in which all calculations are performed serially, rather than in parallel. While this is acceptable for some digital implementations in which a single multiplier is multiplexed throughout the circuit, it is a great limitation in an implementation in which parallel arithmetic is possible. However, it will be shown by example that Kuroda transforms [7] can be used to interpose unit elements between each of the reactive elements in the circuit without changing the amplitude response of the filter, or adversely affecting its lack of sensitivity to coefficient variation. This permits the use of a simple two-phase nonoverlapping clocking arrangement which is very suitable for an analog system.

An example of the structure of a reference bandpass filter suitable for transformation into a two-phase wave filter is shown in Fig. 1.

The wave filter is formed directly from this reference filter according to the methods given in [4] and is comprised of three series and three parallel adaptors, each with three ports. Series adaptors are used for the three series inductors and capacitors and parallel adaptors are used for the three shunt capacitors and inductors. The unit elements provide the time delays required for the operation of practical adaptor circuits. The problem lies in performing the calculations required by the adaptors in the circuit.

II. Adaptor Implementation

In general, wave filter structures require that the equations for both parallel and series adaptors be solved. However, it has already been shown by Fettweis and Meerkotter [8] that series adaptors can be replaced by modified parallel adaptors. This procedure is equivalent to a transformation of the reference filter using gyrators and transformers. For this reason we shall con-

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