3) Using a convergence theorem in Rao [18]

$$Q_N^{-1}[t_N + s_N + r_N] \xrightarrow{p} 0$$

is a direct consequence of (A2.3) and (A2.4). Together with (A2.2), this implies that

$$\hat{\theta}_N \xrightarrow{p} \theta$$

and completes the proof of the theorem.

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Technical Notes and Correspondence

Generic Pole Assignment: Preliminary Results

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Abstract—**Constructive procedures are developed for generically assigning, with constant output feedback, the closed-loop poles of 2-output, m-input, $2m$-dimensional linear systems, for the cases $m = 2$ and $m = 3$.**

I. Introduction

It has long been of interest to find explicit conditions for a real canonical linear system $(C_{p\times n}, A_{n\times n}, B_{n\times m})$ to be *completely assignable*, i.e., to have the property that for each real, monic, nth degree polynomial

$$\alpha(\lambda) = \lambda^n + \sum_{i=1}^{n} a_i\lambda^i \tag{1}$$

there exists a real constant output-feedback matrix $F_{m\times p}$ for which

$$\det(\lambda I - A - BFC) = \alpha(\lambda). \tag{2}$$

It has also been of interest to identify those systems which are "generically assignable": (C, A, B) is *generally assignable* if the set of all coefficient vectors $(a_1, a_2, \cdots, a_n)'$ associated with (1) for which there exist real F satisfying (2) is open and dense in $\mathbb{R}^n$.

Apart from the cases when either C has independent columns or B has independent rows (which can be dealt with using the state-feedback theory), little is known about either complete or generic assignability. Perhaps the sharpest result to date, due to Kimura [1] and others, asserts that for "almost every" linear system, generic pole assignment is possible provided $n \leqslant m + p - 1$. Willems and Hesselink [2] show that for almost every system with $m = p = 2$ and $n = 4$, generic pole assignment is *not* possible, even though the number of free parameters in F equals n. Using a version of the dominant morphism theorem, Hermann and Martin [3] prove that for almost every linear system whose first n Markov matrices $CB, CAB, \cdots, CA^{n-1}B$ are linearly independent, generic pole assignment is possible provided F is allowed to be complex-valued. Brockett and Byrnes [4] take advantage of certain classical ideas based upon the elimination theory to develop a formula which gives values of m, n, and p for which almost every linear system is generically assignable—but their development is not constructive.

The purpose of this correspondence is to develop a useful formulation of the assignment problem for $p = 2$ (Section II), and to provide constructive solutions for some special cases. For $m = 2$ and $n = 4$ we characterize in Section III the classes of generically assignable and completely assignable systems. In Section IV we show that for $m = 3$ and $n = 6$, almost every linear system is generically assignable; we do this by reducing the assignment problem to the problem of finding a real root of a real fifth degree polynomial in one variable. That such a polynomial should exist is agreement with the Brockett–Byrnes formula derived in [4].

Notation: In the sequel $\mathbb{R}$ denotes the real field and $\mathbb{R}^m$ is real m-space.

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If x and y are in $\mathbb{R}^m$, we write $x \wedge y$ for their exterior product; we often represent this product by the m^*-vector $[d_{12}, d_{13}, \cdots, d_{1m}, d_{23}, \cdots, d_{2m}, \cdots, d_{m-1,m}]^T$ where $m^* = m(m-1)/2$ and d_{ij} is the determinant of the 2×2 matrix consisting of rows i and j of $[x, y]_{m\times 2}$. In this framework we note that if $H_{r\times m}$ is a matrix, then $Hx \wedge Hy = H^*(x \wedge y)$ where $H^* = [h_1 \wedge h_2, h_1 \wedge h_3, \cdots, h_1 \wedge h_m, h_2 \wedge h_3, \cdots, h_{m-1} \wedge h_m]_{r^* \times m^*}$ and h_i is the ith column of H.

II. Formulation

We begin with the observation that

$$\det(\lambda I - A - BFC) = \alpha_0(\lambda)\det\left(I - BFC(\lambda I - A)^{-1}\right) = \alpha_0(\lambda)\det\left(I - \frac{H(\lambda)F}{\alpha_0(\lambda)}\right) \tag{3}$$

where $\alpha_0(\lambda) = \det(\lambda I - A)$, and $H_{2\times m}(\lambda)$ is the polynomial matrix $\alpha_0(\lambda)C(\lambda I - A)^{-1}B$. If we write f_i, $(i = 1,2)$ for the ith column of $F_{m\times 2}$ and e_i $(i = 1,2)$ for the ith unit vector in $\mathbb{R}^2$, then (3) can be expanded further by noting that

$$\det\left(I - \frac{H}{\alpha_0}F\right) = \left(e_1 - \frac{H}{\alpha_0}f_1\right) \wedge \left(e_2 - \frac{H}{\alpha_0}f_2\right) = 1 - \frac{1}{\alpha_0}\left(h_1 f_1 + h_2 f_2 + h_3(f_1 \wedge f_2)\right) \tag{4}$$

where h_i $(i = 1,2)$ is the ith row of H, h_3 is the polynomial matrix $-1/\alpha_0[\bar{h}_1 \wedge \bar{h}_2, \bar{h}_1 \wedge \bar{h}_3, \cdots, \bar{h}_1 \wedge \bar{h}_m, \bar{h}_2 \wedge \bar{h}_3, \cdots, \bar{h}_2 \wedge \bar{h}_m, \cdots, \bar{h}_{m-1} \wedge \bar{h}_m]_{1\times m^*}$, and $\bar{h}_i$ is the ith column of H.[1] Hence, if we define the linear transformation $L: \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{m^*} \to$ space of all polynomials of degree $< n$, so that $L(f_1, f_2, f_3) = -(h_1 f_1 + h_2 f_2 + h_3 f_3)$, then we can combine (4) with (3) to obtain the following.

Lemma 1: c. p. $(A + BFC) = \alpha_0 + L(f_1, f_2, f_1 \wedge f_2)$.

In view of Lemma 1 we see that if $\alpha(\lambda)$ is a real, monic, nth degree polynomial to be assigned with $F = [f_1, f_2]$, then F must be chosen to satisfy $L(f_1, f_2, f_1 \wedge f_2) = \alpha(\lambda) - \alpha_0(\lambda)$. It is easy to see that this single nonlinear equation in f_1 and f_2 is equivalent to the simultaneous equations

$$L(f_1, f_2, f_3) = \alpha(\lambda) - \alpha_0(\lambda) \tag{5a}$$

$$f_3 = f_1 \wedge f_2. \tag{5b}$$

Since $L(f_1, f_2, f_3)$ is linear in f_1, f_2, and f_3 and $L(f_1, f_2, f_1 \wedge f_2)$ is continuous in f_1 and f_2, (C, A, B) cannot be completely assignable or even generically assignable unless L is an epimorphism. Thus, if $L_{n\times(2m+m^*)}$ is a matrix representation of L, then we can state the following.

Proposition 1: A necessary condition for (C, A, B) to be either generically or completely assignable is that

$$\operatorname{rank} L = n. \tag{6}$$

Remark: It is worth noting that (6) is weaker than the requirement that the first n Markov matrices $CB, CAB, \cdots, CA^{n-1}B$ be linearly independent.

To proceed with our formulation, let us observe that if (6) holds, then $\dim(\ker L) = n^*$ where $n^* = 2m + m^* - n$. Thus, if (f_1^i, f_2^i, f_3^i), $i = 1, 2, \cdots, n^*$ is a basis for $\ker L$, then a typical element in the kernel is of the form $(M_1 x, M_2 x, M_3 x)$ where $M_i = [f_i^1, f_i^2, \cdots, f_i^{n^*}]_{m\times n^*}(i = 1,2)$, $M_3 = [f_3^1, f_3^2, \cdots, f_3^{n^*}]_{m^*\times n^*}$, and $x \in \mathbb{R}^{n^*}$.

Remark 1: Note that the columns of $[M_1^T, M_2^T, M_3^T]^T$, which are actually a representation of the basis of $\ker L$, are linearly independent. Use will be made of this fact in the sequel.

To continue, observe that any solution (f_1, f_2, f_3) to (5a) for any $\alpha(\lambda)$, must be of the form

$$(f_1, f_2, f_3) = (f_{10} + M_1 x, f_{20} + M_2 x, f_{30} + M_3 x)$$

[1]The elements of $\alpha_0 h_3$ are in fact nothing more than a lexigraphically ordered set of independent 2×2 minors of H.

where $x \in \mathbb{R}^{n^*}$ and (f_{10}, f_{20}, f_{30}) is a particular solution to (5a). Thus, to solve our problem [i.e., *both* (5a) and (5b)], we must find, if possible, $x \in \mathbb{R}^{n^*}$ for which

$$(f_{10} + M_1 x) \wedge (f_{20} + M_2 x) = f_{30} + M_3 x. \tag{7}$$

Assuming (6) holds, we know that for any $\alpha(\lambda)$, (5a) must have a particular solution (f_{10}, f_{20}, f_{30}). Conversely, for any choice of (f_{10}, f_{20}, f_{30}), (5a) uniquely defines a polynomial $\alpha(\lambda) = \alpha_0(\lambda) + L(f_{10}, f_{20}, f_{30})$. From this it follows that for complete assignability (7) must be solvable for all possible $(f_{10}, f_{20}, f_{30}) \in \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{m^*}$ while for generic solvability (7) must be solvable for all (f_{10}, f_{20}, f_{30}) in some open dense subset of $\mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{m^*}$. For the remainder of this paper then, we assume that (6) holds, and study when (7) might have one or the other of these properties.

Remark: One simple condition which ensures that (7) will always have a solution x no matter what f_{10}, f_{20}, f_{30} are, is that

$$\operatorname{rank}\begin{bmatrix} M_1 \\ M_3 \end{bmatrix}_{(m+m^*)\times n^*} = m + m^*.$$

For if this is so, then the linear equations $M_1 x = -f_{10}$, $M_3 x = -f_{30}$ will always be solvable and any such solution will automatically satisfy (7). Now the preceding rank condition will hold (at least generically) if $m + m^* \leqslant n^*$, but using the definition of n^* we see that this is equivalent to $m \geqslant n$, which of course is a case which could be handled using the state-feedback theory.

III. Case $m = 2, n = 4$

In this case $m^* = n^* = 1$, and x is a scalar. Direct expansion of (7) thus yields

$$(M_1 \wedge M_2)x^2 + (f_{10} \wedge M_2 + M_1 \wedge f_{20} - M_3)x + (f_{10} \wedge f_{20} - f_{30}) = 0 \tag{8}$$

which is a quadratic equation in x. For (8) to have a real solution x for either all or almost all (f_{10}, f_{20}, f_{30}), it is clearly necessary that

$$M_1 \wedge M_2 = 0. \tag{9}$$

Now if (9) is true, the set of (f_{10}, f_{20}, f_{30}), for which (8) does not have a solution, is given by all (f_{10}, f_{20}, f_{30}) satisfying

$$f_{10} \wedge M_2 + M_1 \wedge f_{20} - M_3 = 0 \tag{10a}$$

$$f_{10} \wedge f_{20} - f_{30} \neq 0. \tag{10b}$$

Since by Remark 1, M_1, M_2, and M_3 cannot all be zero, the set of (f_{10}, f_{20}, f_{30}) satisfying (10a) must be either empty or a proper variety in $\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}$; therefore, the set of (f_{10}, f_{20}, f_{30}) satisfying (10) must be the compliment of an open-dense set in $\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}$. In other words, if (9) holds, then (8) is solvable for almost all (f_{10}, f_{20}, f_{30}), so (9) together with (6) are necessary and sufficient for generic pole assignment.

For complete pole assignment, (8) must be solvable for all (f_{10}, f_{20}, f_{30}) so the set of such elements satisfying (10) must be empty. In view of Remark 1, this will be true just in case $M_1 = M_2 = 0$, which of course implies (9). We summarize as follows.

Proposition 2: Let $m = p = 2$ and $n = 4$. Then (C, A, B) is generically assignable if and only if

$$\operatorname{rank} L = 4 \tag{11}$$

and

$$M_1 \wedge M_2 = 0. \tag{12}$$

(C, A, B) is completely assignable if and only if (11) is true and, in addition

$$M_1 = M_2 = 0. \tag{13}$$

It is not difficult to see that (12) will almost certainly fail to hold if (C, A, B) is chosen at random. From this it follows that generic pole assignment is not possible for "almost every" system with $m = p = 2$ and $n = 4$, as was observed previously in [2].

It is possible to interpret the conditions of Proposition 2 in more familiar terms. For this, first note from (4) that $h_1 f_1 + h_2 f_2 = \text{trace}(HF)$ and $h_3 = -(1/\alpha_0)\det H$. Since the transfer matrix $T(\lambda)$ of (C, A, B) equals H/α_0, we can write

$$L(f_1, f_2, f_3) = \alpha_0(f_3 \det T - \text{trace}(TF))$$

where $F = [f_1, f_2]$. Condition (11) is thus equivalent to the requirement that the *numerator polynomials* of T (i.e., $h_{11}, h_{12}, h_{21}, h_{22}$), together with the *transmission polynomial* $h_t = \alpha_0 \det T$, span the linear space of real polynomials of degree less than 4. Assuming that this is so, a basis for the kernel of L can be represented by a single pair (G, g) where g is a real scalar and G is a real 2×2 matrix satisfying

$$(G, g) \neq 0 \tag{14a}$$

$$\text{trace}(TG) = g \det T. \tag{14b}$$

In this framework (13) is equivalent to $G = 0$, which in turn is equivalent to $\det T = 0$. We arrive at the following result, obtained previously in [4].

Corollary 1: (C, A, B) is completely assignable if and only if the transfer matrix T of (C, A, B) is singular, and the numerator polynomials of T span the linear space of all real polynomials of degree less than 4.

Since complete assignability implies generic assignability and since the spanning property of $\{h_{11}, h_{12}, h_{21}, h_{22}, h_t\}$ is necessary in either case, the spanning property of $\{h_{11}, h_{12}, h_{21}, h_{22}\}$ in Corollary 1 is necessary and sufficient for generic assignability in the case when $T(\lambda)$ is singular.

Suppose that $T(\lambda)$ is nonsingular, then (14) implies $G \neq 0$. For if $G = 0$, then (14b) would imply $g = 0$ contradicting (14a). Condition (12) is equivalent to $\det G = 0$, which in turn is equivalent to rank $G = 1$. Thus, (12) is equivalent to the existence of *nonzero* vectors g_1 and g_2 such that $\text{trace}(Tg_1 g_2') = g \det T$, but $\text{trace}(Tg_1 g_2') = g_2' T g_1$. Therefore, we can state the following.

Corollary 2: If the transfer matrix T of (C, A, B) is nonsingular, then generic pole assignment is possible if and only if there exist real nonzero vectors g_1 and g_2 and a real scalar g such that

$$g_2' T g_1 = g \det T \tag{15}$$

and, in addition, the numerator polynomials of T, together with the transmission polynomial of T span the linear space of all real polynomials of degree less than 4.

It is easy to see that if at least one of the entries in T is zero, then nonzero g_1 and g_2 necessarily exist for which (15) holds with $g = 0$. It is easy to show that if all entries of T are nonzero, and if (15) holds with nonzero g_1 and g_2, then it is possible to construct an "initializing" output-feedback matrix F_0, so that after suitable input and output coordinate transformations, the transfer matrix T_{F_0} of the resulting transformed closed-loop system has at least one entry equal to zero. In other words, the first condition of Corollary 2 is equivalent to the existence of input-coordinate, output-coordinate, and output-feedback transformations which when applied to (C, A, B) result in a new system whose transfer matrix has at least one zero entry.

IV. $m = 3, n = 6$

In this case $m^* = n^* = 3$, so $x \in \mathbb{R}^3$ and M_1, M_2, and M_3 are 3×3 matrices. We make the following assumption.

Assumption 1: There exists a real scalar μ such that $M_1 + \mu M_2$ is nonsingular and $M \equiv [M_1 + \mu M_2]^{-1} M_2$ has at least two distinct eigenvalues.

In view of Remark 1, it is easy to see that Assumption 1 is "generic" in the sense that "almost every" system with $p = 2$, $m = 3$, and $n = 6$ has the required property.

Our approach will be to first make a judicious change of variables in (7), then expand (7), and finally to reduce what results to a fifth degree polynomial equation in one variable. Since M is 3×3, it must have at least one real eigenvalue. If the remaining two eigenvalues are complex, let w denote the real eigenvalue. If all eigenvalues are real, let w denote any one distinct from the remaining two. In either case let g be an eigenvector for w and let $G_{3\times 2}$ be a real matrix whose columns span the two-dimensional M-invariant subspace associated with the two eigenvalues distinct from w. Then

$$MG = GW \tag{16a}$$

$$Mg = gw \tag{16b}$$

$$\det[G, g] \neq 0 \tag{16c}$$

$$\det[wI - W] \neq 0 \tag{16d}$$

where W is a matrix representation of the restriction of M to column span G. Next define

$$T = [t_1, t_2] = [M_1 + \mu M_2] G \tag{17a}$$

$$t_3 = [M_1 + \mu M_2] g. \tag{17b}$$

Remark 2: Using (16c) and the nonsingularity of $[M_1 + \mu M_2]$ it is easy to see that t_1, t_2, and t_3 are linearly independent.

From (15), (16a), and (17a) it follows that $M_2 G = TW$. Similarly from (15), (16b), and (17b), we obtain $M_2 g = t_3 w$. Thus, if we introduce the new variables $[y^T, a]^T = [G, g]^{-1} x$ where $y \in \mathbb{R}^2$, then $\begin{bmatrix} y \\ z \end{bmatrix} = [G, g]^{-1} x$, $x = Gy + gz$, $[M_1 + \mu M_2]x = Ty + t_3 z$, and $M_2 x = TWy + t_3 wz$. With reference to (7) we can write

$$\begin{aligned}(f_{10} + M_1 x) \wedge (f_{20} + M_2 x) &= (f_{10} + \mu f_{20} + (M_1 + \mu M_2)x) \wedge (f_{20} + M_2 x) \\ &= (f_{10} + \mu f_{20} + Ty + t_3 z) \wedge (f_{20} + TWy + t_3 wz).\end{aligned}$$

Thus, in our new variables, (7) becomes

$$(f_{10} + \mu f_{20} + Ty + t_3 z) \wedge (f_{20} + TWy + t_3 wz) = f_{30} + M_3 Gy + M_3 gz.$$

Expanding and collecting terms we obtain

$$(t_1 \wedge t_2)(y \wedge Wy) + (L_1 + L_2 z) y = L_3 z + d \tag{18}$$

where

$$L_1 y = Ty \wedge f_{20} + (f_{10} + \mu f_{20}) \wedge (TWy) - M_3 Gy,$$

$$d = f_{30} - (f_{10} + \mu f_{20}) \wedge f_{20},$$

and

$$L_2 y = Ty \wedge (t_3 w) + t_3 \wedge (TWy) = [t_1 \wedge t_3, t_2 \wedge t_3][wI - W] y \tag{19}$$

$$L_3 = M_3 g - (f_{10} + \mu f_{20}) \wedge (t_3 w) - t_3 \wedge f_{20}. \tag{20}$$

To proceed we need the following.

Lemma 2: $t_1 \wedge t_3$, $t_2 \wedge t_3$ and $t_1 \wedge t_2$ are linearly independent.

Proof: Suppose $a_1 t_1 \wedge t_2 + a_2 t_1 \wedge t_3 + a_3 t_2 \wedge t_3 = 0$. To prove $a_1 = 0$, write $a_1 t_1 \wedge t_2 \wedge t_3 + a_2 t_1 \wedge t_3 \wedge t_3 + a_3 t_2 \wedge t_3 \wedge t_3 = 0$. But $t_1 \wedge t_3 \wedge t_3 = t_2 \wedge t_3 \wedge t_3 = 0$ and since by Remark 2, $t_1 \wedge t_2 \wedge t_3 \neq 0$, it follows that $a_1 = 0$. By similar reasoning, $a_2 = a_3 = 0$. □

Since $[wI - W]$ is nonsingular [cf. (16d)], it follows from Lemma 2 and (19) that $[t_1 \wedge t_2, L_2]$ is nonsingular as well. Define $[q^T, P^T]^T = [t_1 \wedge t_2, L_2]^{-1}$ where q is a row vector. Then $P(t_1 \wedge t_2) = 0$, $PL_2 = I$ and

$$q(t_1 \wedge t_2) = 1$$

$$qL_2 = 0. \tag{21}$$

Application of P to (18) yields $(PL_1 + Iz)y = PL_3 z + Pd$ or $y = (PL_1 + Iz)^{-1}(PL_3 z + Pd)$. Thus,

$$y = \frac{\bar{y}}{\beta(z)} \tag{22}$$

where $\beta(z) = \det(PL_1 + Iz)$ and $\bar{y}$ is a vector of polynomials in z, each polynomial having degree no greater than 2. Application of q to (18) and using (21) yields $y \wedge Wy + qL_1 y = qL_3 z + qd$. From (22) we obtain

$$\frac{1}{\beta^2}(\bar{y} \wedge W\bar{y}) + \frac{1}{\beta} qL_1 \bar{y} = qL_3 z + qd \text{ or}$$

$$\bar{y} \wedge W\bar{y} + \beta qL_1 \bar{y} = \beta^2 qL_3 z + \beta^2 qd. \tag{23}$$

Now $\bar{y} \wedge W\bar{y}$, $\beta qL_1 \bar{y}$, and $\beta^2 qd$ all must be polynomials in z of degree not exceeding 4, while $\beta^2 qL_z z$ will be of degree 5 if $qL_3 \neq 0$. In other words, if $qL_3 \neq 0$, then (23) is a polynomial equation in z of degree 5.

From (20), $qL_3 = qM_3 g - q((f_{10} + \mu f_{20}) \wedge (t_2 w) + t_3 \wedge f_{20})$. Since by Remark 2, $t_3 \neq 0, qL_3 \neq 0$ for almost all (f_{10}, f_{20}, f_{30}). We conclude the following.

Proposition 3: For the case $p = 2, n = 6, m = 3$, almost every linear system is generically assignable.

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Finite Spectrum Assignment Problem for Systems with Delay in State Variables

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***Abstract*—This correspondence is concerned with the finite spectrum assignment problem for linear systems with delay in state variables. A systematic procedure for constructing the feedback law is presented on the basis of advanced elementary row operations on matrices with two variables s and $z = e^{-sh}$.**

I. INTRODUCTION AND PRELIMINARIES

This correspondence is concerned with the finite spectrum assignment problem for the system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) - A_1 x(t-h) + bu(t) \qquad t \geq 0 \\ x(\tau) &= \varphi(\tau)\ \tau \in [-h, 0] \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R$, $A_0, A_1 \in R^{n \times n}$, and $b \in R$. The following control is considered.

$$u(t) = \sum_{i=1}^{N} \eta_i x(t - ih) + \int_{-Nh}^{0} \xi(\theta) e^{s\theta} x(t+\theta)\, d\theta + u_1(t) \tag{2}$$

where N is a positive integer, $\eta_i \in R^{1 \times n}$, and $\xi(\cdot) \in L_2([-Nh, 0], R^{1 \times n})$. Taking the Laplace transform of (1) and (2) yields the closed-loop system

$$x(s) = \left(sI - A_0 - A_1 e^{-sh} - bF\right)^{-1} bu_1(s) \tag{3}$$

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where the initial function $\varphi(\cdot)$ is assumed to be zero in order to simplify the equation, and

$$F = \sum_{I=1}^{N} \eta_i e^{-sih} + \int_{-Nh}^{0} \xi(\theta) e^{s\theta}\, d\theta. \tag{4}$$

The system (1) is said to be finite spectrum assignable if, for arbitrary conjugate complex numbers $\beta_1, \beta_2, \cdots, \beta_n$ there exists an F such that

$$|sI - A_0 - A_1 e^{-sh} - bF| = \sum_{i=1}^{n} (s - \beta_i) \tag{5}$$

or equivalently, if for arbitrary real numbers $\gamma_1, \cdots, \gamma_n$ there exists an F which satisfies

$$|sI - A_0 - A_1 e^{-sh} - bF| = s^n + \gamma_1 s^{n-1} +, \cdots, + \gamma_n. \tag{6}$$

The idea of the finite spectrum assignment was found in a paper by Kamen [5] and developed by Manitius and Olbrot [1].

The characteristic polynomial of the closed-loop system is expanded into

$$\begin{aligned} &|sI - A_0 - A_1 z - bF| \\ &\quad = |sI - A_0 - A_1 z| - F\operatorname{adj}(sI - A_0 - A_1 z) b \\ &\quad = s^n + \alpha_1 s^{n-1} +, \cdots, + \alpha_n - F\operatorname{adj}(sI - A_0 - A_1 z) b \end{aligned} \tag{7}$$

where $z = e^{-sh}$ and $|sI - A_0 - A_1 z| = s^n + \alpha_1 s^{n-1} +, \cdots, \alpha_n$. If there exists an $n \times n$ matrix $K(z)$ over polynomials z and the finite Laplace transform such that

$$K(z) \operatorname{adj}(sI - A_0 - A_1 z) b = v(s) \tag{8}$$

where $v(s) = [1\ s, \cdots, s^{n-1}]'$, then the system (1) is finite spectrum assignable. That is to say, let $F = [\alpha_n - \gamma_n, \cdots, \alpha_1 - \gamma_1] K(z)$, then we have (6) via (7). The necessary condition for the existence of such a $K(z)$ is that the system (1) is spectrally controllable [1]. Spectrum controllability presumably implies the existence of $K(z)$. This implication has not been proven. However, there are some procedures for constructing $K(z)$ [1], [3], [4]–[6]. If a systematic procedure for the construction of $K(z)$ is presented, the sufficient condition may be clarified.

As seen in [1], [2], $\operatorname{adj}(sI - A_0 - A_1 z) b$ is expanded into

$$\operatorname{adj}(sI - A_0 - A_1 z) b = M(z) v(s) = P(s) v(z) \tag{9}$$

where

$$M(z) = [A_0 + A_1 z | b] \begin{bmatrix} \alpha_{n-1} & \cdots & \alpha_1 & 1 \\ \vdots & & ⋰ & \\ \alpha_1 & ⋰ & & \\ 1 & & & 0 \end{bmatrix} \tag{10}$$

$$[A_0 + A_1 z | b] = \left[b, (A_0 + A_1 z) b, \cdots, (A_0 + A_1 z)^{n-1} b\right] \tag{11}$$

and $P(s)$ is the $n \times n$ polynominal matrix. If the system (1) is spectrally controllable, $[A_0 + A_1 z | b]$ is nonsingular [7], [8]. If $[A_0 + A_1 z | b]$ is unimodular, $K(z)$ which satisfies (8) is given [3], [4] by

$$K(z) = \begin{bmatrix} \alpha_{n-1} & \cdots & \alpha_1 & 1 \\ \alpha_1 & ⋰ & & \\ 1 & & & 0 \end{bmatrix}^{-1} [A_0 + A_1 z | b]^{-1}. \tag{12}$$

If $[A_0 + A_1 z | b]$ is not unimodular, then $K(z)$ contains rational functions of z. Sontag [4] and Mayeda and Yamada [10] showed that $K(z)$ can be realized by using the feedback loop around delays. However, this realization of $K(z)$ introduces extra poles in the closed loop, which cannot be assigned as desired. This method is out of the finite spectrum assignment problem. Introducing the finite Laplace transform, this problem can be solved. If $P(s)$ is nonsingular, then $K(z)$ which consists of polynomials of z and the finite Laplace transform may be constructed by using [1, Theorem 3.3]. If $P(s)$ is singular, [1, Remark 3.6] is required. If the assumption (H_3) in [1] is not met, [1, Remark 3.5] is required. Reference