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USEFUL TIME-VARIABLE CIRCUIT-ELEMENT EQUIVALENCES†

Because time-variable circuit elements have properties not possessed by time-invariant ones, such as the ability to modulate or amplify with low noise, it is useful to have different ways of looking at them. Here we present general equivalences for time-variable inductors, capacitors, resistors and gyrators, using time-invariant elements and time-variable transformers, which allow various properties to be determined and interpreted on physical grounds.

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We first consider the inductor of inductance $l(t)$, whose defining constraint is

$$v = d[li]/dt \quad \ldots \ldots \quad (1a)$$
$$= l'i + li' \quad \ldots \ldots \quad (1b)$$

where v is the voltage across and i is the current through the inductor; a prime denotes differentiation. Since no physical measurement can prove otherwise, we assume l to be an infinitely differentiable real-valued function of time t. Consequently, we can reasonably assume that, over any finite interval $[a, b]$, where $a \leqslant t \leqslant b$, l is a function of bounded variation.[2] Hence we write, over $[a, b]$,

$$l(t) = l_+(t) - l_-(t) \quad \ldots \ldots \quad (2a)$$

where l_+ and l_- are the positive and negative variations,[2] except that $l(a)$ is contained in l_+ if $l(a) \geqslant 0$ or in l_- if $l(a) < 0$ to obtain two nonnegative, nondecreasing functions which are, in fact, differentiable; i.e.

$$l_+ \geqslant 0, \quad l_+' \geqslant 0, \quad l_- \geqslant 0, \quad l_-' \geqslant 0 \,. \quad (2b)$$

Moreover, it is possible for eqns. 2 to take $b = +\infty$ and often $a = -\infty$ [but not always $a = -\infty$, as $l(t) = \cos t$, and $l(t) = \varepsilon^{-t}$ show]. In any event, we can choose a (dependent on i) such that $v = i = 0$ for $t < a$, which rigorously[3] must be possible for every i (since we assume all physical i to be zero before a finite time).

With these preliminaries, consider the circuit of Fig. 1*a*, which represents a time-variable transformer[1] of turns ratios $t_{ij}(t)$ loaded in passive and active time-invariant inductors and resistors. Using the definition of a time-variable transformer, we easily find

$$v = (t_{11}t_{11}' + t_{31}^2 - t_{21}t_{21}' - t_{41}^2)\, i + (t_{11}^2 - t_{21}^2)\, i' \quad . \; . \quad (3)$$

Equating coefficients of eqns. 3 with eqn. 1, and using eqns. 2, we arrive at

$t_{11}^2 = l_+$	$t_{21}^2 = l_-$
$2t_{31}^2 = l_+'$	$2t_{41}^2 = l_-'$

. . (4)

We conclude that over the interval $[a, \infty]$, for every finite a, any time-variable inductor is equivalent to the circuit of Fig. 1*a* with parameters chosen by eqn. 4.

At this point, one can apply physical reasoning to deduce the consequences of various properties. If an inductor is passive, the negative elements of Fig. 1*a* are necessarily absent, requiring

$$t_{21} = t_{41} \equiv 0,$$

or

$$l_- \equiv 0 \text{ (passive } l) \quad \ldots \ldots \quad (5)$$

If an inductor is passive and lossless, then, besides the negative elements being absent, the positive resistor must not be present, forcing also t_{31} to zero and hence l_+' to zero, or

$$l = l_+ \equiv \text{constant (lossless } l) \quad . \; . \quad (6)$$

From this one concludes that the time variation must be absent from all lossless inductors. The equivalence also shows that one is naturally led to the consideration of time-variable transformers for synthesis of time-variable networks.[4]

An entirely dual procedure yields the equivalence of Fig. 1*b* for the linear time-variable capacitor of capacitance $c(t)$. We have

$$i = c'v + cv' \quad \ldots \ldots \quad (7a)$$
$$c = c_+ - c_- \quad \ldots \ldots \quad (7b)$$

$t_{11}^2 = c_+$	$t_{21}^2 = c_-$
$2t_{31}^2 = c_+'$	$2t_{41}^2 = c_-'$

. . (8)

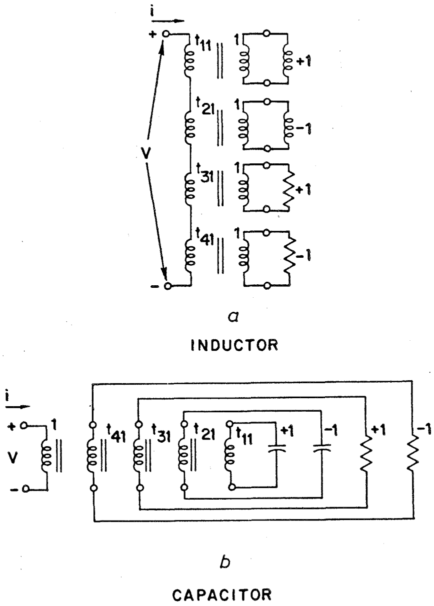

Fig. 1 *Time-variable reactive elements*

An equivalence for the time-variable resistor should now be clear. Denoting the resistance by $r(t)$ with r_+ and r_- nonnegative functions, we have

$$v = ri \quad \ldots \ldots \quad (9a)$$
$$r = r_+ - r_- \quad \ldots \ldots \quad (9b)$$

from which Fig. 2*a* results, with

$t_{11}^2 = r_+$
$t_{21}^2 = r_-$

. . . . (10)

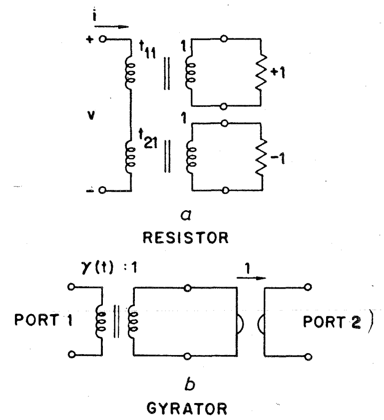

Fig. 2 *Resistive equivalents*

However, this equivalence is not unique; the monotonicity of r_+ and r_- is not required. As with l and c, we can actually add any positive nondecreasing $f(t)$, which is infinitely differentiable, to r_+ and r_-, to obtain a nonunique decomposition. In contrast to the case with l and c, arbitrary infinitely differentiable $f(t)$ can be added to r_+ anc r_- to obtain nonuniqueness. A dual result holds for conductance, while Fig. 2*b* shows the

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equivalence for the gyrator, which follows from the time-variable impedance matrix[5]

$$z(t, \tau) = \delta(t - \tau)\begin{bmatrix} 0 & \gamma(t) \\ -\gamma(t) & 0 \end{bmatrix}$$

$$= \delta\begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} \quad . \quad . \quad (11)$$

Here $\gamma(t)$ is the gyration resistance, and δ is the unit impulse.

As in the time-invariant case, the inductor and capacitor can be interrelated through the gyrator. Such an equivalence is shown in Fig. 3, for which we have

$$v = \gamma\frac{d[\gamma ci]}{dt} = T\frac{d[Tli]}{dt} \quad . \quad . \quad (12)$$

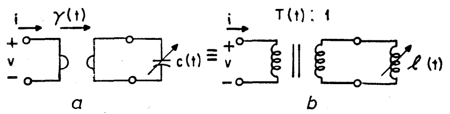

Fig. 3 *Capacitor–inductor equivalent*

To summarise, linear time-variable inductors, capacitors, resistors and gyrators have the equivalences given. Thus any connection of a finite number of such elements can be described by a circuit having transformers as the only time-variable elements. This allows us to consider all networks of this class to be looked upon as a transformer network loaded by a time-invariant inductor, capacitor, resistor, gyrator network to which various physical properties can be ascribed. Because the turns ratios of Figs. 1 and 2 are related to the positive and negative variations of element values, various constraints on the individual elements can be easily obtained by physical reasoning on the equivalent circuits.

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