INT. J. CONTROL, 1982, VOL. 35, NO. 3, 397–416

Structural controllability and matrix nets†

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We establish structural controllability results for matrix pairs $[A, B]$ where $A = A_0 + \Sigma\, \mu_i A_i$, $B = B_0 + \Sigma\, \mu_i B_i$, with the A_i, B_i fixed, and the μ_i free scalar parameters. The results characterize structural controllability in several ways, via tests involving the checking of the rank or the evaluation of the determinant of various constant matrices formed from the A_i, B_i. A number of the results used as intermediate results tests for a full rank property of matrix nets, i.e. tests that check if $M = M_0 + \Sigma\, \mu_i M_i$, M_i prescribed, μ_i variable, has full rank for almost all μ_i.

1. Introduction

Structural controllability results to this point (Lin 1974, 1977, Shields and Pearson 1976, Glover and Silverman 1976, Morari and Stephanopoulos 1978, Hosoe and Matsumoto 1979) have almost all been of the following type. One is given an $n \times n$ matrix A and $n \times r$ matrix B in which each entry is either zero or is free and can be set at an arbitrary non-zero value, irrespective of the values at which other free entries are set. Conditions are then found for the pair $[A, B]$ to be completely controllable for almost all values of the free parameters.

One other result is that of Corfmat and Morse (1976), in which one postulates

$$\left.\begin{aligned} A &= A_0 + \sum_{i=1}^{k} B_i P_i C_i \\ B &= B_0 + \sum_{i=1}^{k} B_i P_i D_i \\ C &= C_0 + \sum_{i=1}^{k} G_i P_i C_i \end{aligned}\right\} \qquad (1.1)$$

where all matrices consist of fixed elements except the P_i, and all entries of each of the P_i are independently free. Then one seeks conditions on the fixed matrices such that $[A, B]$ is completely controllable and $[A, C]$ completely observable for almost all P_i.

Similar results to those for structural controllability have recently been developed for structural decentralized fixed modes (Sezer and Šiljak 1981).

In this paper, we aim to consider a more general situation than that considered in the first six references cited above.

Received 4 September 1981.

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0020-7179/82/3503 0397 $04·00 © 1982 Taylor & Francis Ltd

First, those elements of A, B known to be constant may be non-zero (it is easy to conceive of situations where the constant 1 would be guaranteed to enter). Second, the free elements are not necessarily independent of one another, but rather, we permit collections of entries in the A, B pairs to vary linearly with some parameter. Consider for example the state-variable equations defined by a linear R-L-C electric network. Then the reciprocity of the network forces certain symmetries and skew symmetries among the entries of A for a particular, natural choice of state variable vector.

This means that we study $[A, B]$ pairs where

$$A = A_0 + \sum_{i=1}^{k} \mu_i A_i \tag{1.2 a}$$

$$B = B_0 + \sum_{i=1}^{k} \mu_i B_i \tag{1.2 b}$$

in which the A_i, B_i should be thought of as fixed matrices, and the μ_i free real scalars. Of course it may be that $A_i \neq 0$ implies $B_i = 0$, and $B_i \neq 0$ implies $A_i = 0$. We seek a procedure for determining whether the pair $[A, B]$ is completely controllable for some (and thus almost all) μ_i, $i = 1, 2, \ldots, k$.

The collection of matrices

$$A_0 + \sum_{i=1}^{k} \mu_i A_i$$

obtained for fixed A_0, A_i and varying μ_i is termed a matrix net. Matrix nets are generalizations of matrix pencils, when $k = 1$. The key ideas of our approach to structural controllability involve matrix nets, and are these.

(*a*) One can express the requirement that $[A, B]$ be controllable for almost all μ_i as a condition that a matrix net should have full rank for almost all μ_i. Call this net $M_0 + \sum_{i=1}^{k} \mu_i M_i$.

(*b*) Following an approach which is available for matrix pencils (Gantmakher 1959) (or other approaches set out in this paper), one can express the condition that the matrix net have full rank in terms of only the constant matrices M_i which define the net.

As it turns out, the approach most closely tied to matrix pencils can lead to very large dimension matrices having to be checked. So we also explore alternative equivalent conditions (again involving constant matrices). These alternative conditions involve large numbers of smaller matrices rather than one large matrix. Next we review several systems problems which can be resolved using matrix nets where the M_i are constrained to have rank 1 ; we present yet another procedure for testing the rank of such nets, and re-examine structural controllability problems, including the particular problem originally studied by this tool.

2. Structural controllability as a condition on a matrix net

Recall that $[A, B]$ with A $n \times n$ and B $n \times r$ is completely controllable if and only if (Rosenbrock 1970)

$$M = \begin{bmatrix} I & & & & & & B \\ -A & I & & & & \cdot^{\cdot^{\cdot}} & \\ & -A & I & & & & \\ & & & \ddots & & & \\ & & & -A & I & B & \\ & & & & -A & B & \end{bmatrix} \tag{2.1}$$

[an $n^2 \times n(n-1+r)$ matrix] has rank n^2. If A and B are individually matrix nets (with or without the same μ_i), M is also a matrix net.

Checking complete controllability of $[A, B]$ involves checking whether M has full row rank.

3. Checking the full rank property of a matrix net

Suppose that we are given a matrix net

$$M = M_0 + \sum_{i=1}^{k} \mu_i M_i \tag{3.1}$$

with each M_i of dimension $q \times r$, $q \leqslant r$. We aim to check whether M has rank q generically.

The following result is a straightforward generalization and sharpening of a result known in the matrix pencil ($k=1$) case (Gantmakher 1959).

Lemma 3.1

With M as defined above, suppose that rank $M < q$ for all μ_i. Then there exists a row q-vector $x^{\mathrm{T}}(\mu_1, \ldots, \mu_k)$, written $x^{\mathrm{T}}(\mu)$, polynomial in the μ_i, such that

$$x^{\mathrm{T}}(\mu) M = 0 \tag{3.2}$$

for all μ_i and the degree of $x(\mu)$ in $\mu_i \leqslant$ rank M_i.

Proof

Suppose that rank $M = p$. Temporarily re-order the rows and columns of M so that the top left $p \times p$ corner of M is non-singular for almost all μ_i. [The $p \times p$ minors of M are polynomial in the μ_i and are therefore either identically zero, or non-zero for almost all μ_i; since M has rank p, at least one $p \times p$ minor cannot be identically zero.] Write

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where M_{11} is $p\times p$. Then for arbitrary polynomial $y_2(\mu)$ there exists a rational $y_1(\mu)$ such that

$$y_1^{\mathrm{T}}(\mu)M_{11}+y_2^{\mathrm{T}}(\mu)M_{21}=0$$

(Take $y_1^{\mathrm{T}}(\mu)=-y_2^{\mathrm{T}}(\mu)M_{21}M_{11}^{-1}$.) Now set

$$x_1(\mu)=|M_{11}|y_1(\mu),\quad x_2(\mu)=|M_{11}|y_2(\mu)$$

Then

$$x_1^{\mathrm{T}}(\mu)M_{11}+x_2^{\mathrm{T}}(\mu)M_{21}=0 \tag{3.3}$$

with $x_1(\mu)$, $x_2(\mu)$ polynomial. Further, if $y_2(\mu)$ has degree zero, then $x_1(\mu)$, the entries of which are linear combinations of various $p\times p$ minors of

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}=M\begin{bmatrix} I \\ 0 \end{bmatrix}=M_0\begin{bmatrix} I \\ 0 \end{bmatrix}+\sum \mu_i M_i\begin{bmatrix} I \\ 0 \end{bmatrix}$$

has degree in μ_i at most equal to the rank of $M_i\begin{bmatrix} I \\ 0 \end{bmatrix}$, which is overbounded by the rank of M_i.

Finally, since M has rank p, and the first p columns are linearly independent, the last $p-r$ columns of M are expressible as linear combinations of the first. Hence (3.3) implies

$$x_1^{\mathrm{T}}(\mu)M_{12}+x_2^{\mathrm{T}}(\mu)M_{22}=0$$

which, together with (3.3), is equivalent to (3.2).

Remark

If rank $M=q-p$, then it is easily seen that there exist p linearly independent left null vectors, all satisfying the degree restriction.

With this lemma in hand, we can consider the derivation of a constant matrix whose rank indicates whether or not M has full rank. To illustrate the idea, we shall discuss the case when the μ_i are two in number [$k=2$ in (3.1)]; then we shall outline the result for arbitrary k. The arguments generalize arguments in Gantmakher (1959) for the matrix pencil case.

Let rank $M_i=r_i$, and define

$$x^{\mathrm{T}}(\mu)=\sum_{\alpha=0}^{r_1}\sum_{\beta=0}^{r_2} x_{\alpha\beta}^{\mathrm{T}}\,\mu_1^{\alpha}\,\mu_2^{\beta} \tag{3.4}$$

Then $x^{\mathrm{T}}(\mu)M=0$ for all μ_1, μ_2 if and only if there holds a series of equations independent of μ_1, μ_2, which are obtained from this equation by equating the coefficients of different powers $\mu_1^{\alpha}\,\mu_2^{\beta}$ to zero.

Now

$$x^{\mathrm{T}}(\mu)M=\sum_{\alpha=0}^{r_1+1}\sum_{\beta=0}^{r_2+1}(x_{\alpha\beta}^{\mathrm{T}}\,M_0+x_{\alpha-1,\beta}^{\mathrm{T}}\,M_1+x_{\alpha,\beta-1}^{\mathrm{T}}\,M_2)\,\mu_1^{\alpha}\,\mu_2^{\beta}$$

so that $x^{\mathrm{T}}(\mu)M=0$ if and only if

$$x_{\alpha\beta}^{\mathrm{T}}\,M_0+x_{\alpha-1,\beta}^{\mathrm{T}}\,M_1+x_{\alpha,\beta-1}^{\mathrm{T}}\,M_2=0,\quad 0\leqslant\alpha\leqslant r_1+1,\quad 0\leqslant\beta\leqslant r_1+1 \tag{3.5}$$

It is understood that $x_{\alpha\beta}=0$ if $\alpha\notin[0,r_1]$, $\beta\notin[0,r_2]$.

We can consider these equations in the following systematic order. All equations with $\beta=0$, α increasing from 0 to r_1; then all equations with $\beta=1$, α increasing from 0 to r_1; and so on. The end result is, with

$$\left.\begin{array}{l} X_2^{\mathrm{T}}=[x_{00}^{\mathrm{T}}\, x_{10}^{\mathrm{T}} \ldots x_{r_1 0}^{\mathrm{T}}\, x_{01}^{\mathrm{T}}\, x_{11}^{\mathrm{T}} \ldots x_{r_1 1}^{\mathrm{T}} \ldots x_{r_1 r_2}^{\mathrm{T}}] \\ P_1=\underbrace{\begin{bmatrix} M_0 & M_1 & \cdot & \cdot & \cdot & \cdot \\ & M_0 & M_1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & \cdot & M_0 & M_1 \end{bmatrix}}_{\longleftarrow (r_1+2)\text{ blocks} \longrightarrow} \\ Q_1=\underbrace{\begin{bmatrix} M_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & M_2 & & & & \\ \cdot & 0 & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ & & & & M_2 & 0 \end{bmatrix}}_{\longleftarrow (r_1+2)\text{ blocks} \longrightarrow} \end{array}\right\} \tag{3.6}$$

$$X_2^{\mathrm{T}}\underbrace{\begin{bmatrix} P_1 & Q_1 & \cdot & \cdot & \cdot & \cdot & 0 \\ & P_1 & Q_1 & & & & \\ & & P_1 & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & P_1 & Q_1 \end{bmatrix}}_{\longleftarrow (r_2+2)\text{ blocks} \longrightarrow}=0 \tag{3.7}$$

By reversing the above argument, it is easy to see that if the large matrix in (3.7) has a non-zero left null vector X_2^{T} then M has a non-trivial left null vector $x^{\mathrm{T}}(\mu)$, the relation between the two vectors being defined by (3.4).

The matrix appearing in (3.7) may clearly be a large matrix—with each M_i of dimension $q\times r$, it has dimension

$$q(r_1+1)(r_2+1)\times r(r_1+2)(r_2+2)$$

However, its structure may allow the rank of the matrix to be more easily derived than usual for a matrix of this size. The matrix $[P_1, Q_1]$ is block Toeplitz, and so can be reduced to a modified row echelon form by elementary operation on the left: using this form to replace $[P_1, Q_1]$ in (3.7), the same modified row echelon reduction procedure can be undertaken on the large matrix. As explained in Bitmead *et al.* (1978), the Toeplitz structure allows the speeding up of this process.

Let us now briefly describe the arrangements applying when $k>2$. We define iteratively matrices P_i, Q_i for $i=2, 3, \ldots$, by

$$P_i = \underbrace{\begin{bmatrix} P_{i-1} & Q_{i-1} & & & \\ & P_{i-1} & Q_{i-1} & & \\ & & \ddots & \ddots & \\ & & & P_{i-1} & Q_{i-1} \end{bmatrix}}_{(r_i+2)\text{ blocks}}, \quad Q_i = \left[\begin{array}{c:c} & 0 \\ \text{copies of } M_{i+1} & 0 \\ \text{arranged} & \vdots \\ \text{on diagonal} & 0 \end{array}\right]$$

with the dimension of Q_i identical to that of P_i. Then for a k net, the condition that M have full row rank is that P_k have full row rank.

Example

Suppose

$$A = \begin{bmatrix} 0 & \mu_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \mu_2 \end{bmatrix}$$

This pair is obviously structurally controllable. The matrix M, see (2.1), is

$$M = \left\{\begin{array}{ccc:ccc:c:c:c} \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{3}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{3}} & \overbrace{\quad}^{1} & \overbrace{\quad}^{1} & \overbrace{\quad}^{1} \\ 1 & 0 & 0 & & & & & & \\ 0 & 1 & 0 & & 0 & & 0 & 0 & 0 \\ 0 & 0 & 1 & & & & & & \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & & & \\ \hdashline & & & 0 & 0 & 0 & & & \\ & 0 & & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & & & \end{array}\right\}$$

$$
+\mu_1 \left[\begin{array}{c:c:c:c:c}
0 & 0 & 0 & 0 & 0 \\ \hdashline
\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 & 0 & 0 & 0 \\ \hdashline
0 & \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 & 0 & 0
\end{array}\right]
$$

$$
+\mu_2 \left[\begin{array}{c:c:c:c:c}
0 & 0 & 0 & 0 & \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} \\ \hdashline
0 & 0 & 0 & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & 0 \\ \hdashline
0 & 0 & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & 0 & 0
\end{array}\right]
$$

$$
= M_0 + \mu_1 M_1 + \mu_2 M_2
$$

Evidently, $r_1 = \text{rank } M_1 = 2$ and $r_2 = \text{rank } M_2 = 3$. This means that the number of rows in the constant matrix whose rank is to be checked is 108 ! The reader is spared the details.

4. Matrix net rank-checking via determinantal expansion

Suppose that

$$
M = M_0 + \sum_{i=1}^{k} \mu_i M_i \tag{4.1}
$$

with, temporarily, the M_i all square. Suppose that a dyadic decomposition is known for each M_i, $i \geqslant 1$:

$$
M_i = \sum_{j=1}^{r_i} b_j{}^i \, c_j{}^{i\mathrm{T}} \tag{4.2}
$$

with $r_i = \text{rank } M_i$. Then we claim :

Lemma 4.1

$$
\det \left[M_0 + \mu_1 \sum_{j=1}^{r_1} b_j{}^1 \, c_j{}^{1\mathrm{T}} + \ldots + \mu_k \sum_{j=1}^{r_k} b_j{}^k \, c_j{}^{k\mathrm{T}} \right] = 0, \quad \text{for all } \mu_i \tag{4.3}
$$

if and only if for all integer $s_1 \in [0, r_1], \ldots, s_k \in [0, r_k]$

$$\sum \det \begin{bmatrix} M_0 & b_{j(11)}{}^1 & \cdots & b_{j(1s_1)}{}^1 & \cdots & b_{j(k1)}{}^k & \cdots & b_{j(ks_k)}{}^k \\ c_{j(11)}{}^{1\mathrm{T}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \\ c_{j(1s_1)}{}^{1\mathrm{T}} & & & & & & & \\ \vdots & & & & & & & \\ c_{j(k1)}{}^{k\mathrm{T}} & & & & & & & \\ \vdots & & & & & & & \\ c_{j(ks_k)}{}^{k\mathrm{T}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} = 0 \tag{4.4}$$

where the summation is over the sets of values

$$\{j(11), j(12), \ldots, j(1s_1)\} \subseteq \{1, \ldots, r_1\} \ldots \{j(k1), j(k2), \ldots, j(ks_k)\} \subseteq \{1, \ldots, r_k\}$$

and

$$j(\alpha\beta) \neq j(\alpha\gamma) \quad \text{for } \beta \neq \gamma$$

Before proving the lemma, we note: the number of separate equalities is $(r_1+1)(r_2+1) \ldots (r_k+1)$, and the number of summands in that equality associated with $s_1, \ldots, s_k$ is $\binom{r_1}{s_1}, \binom{r_2}{s_2}, \ldots, \binom{r_k}{s_k}$. Also, the total number of summands which must be evaluated is

$$\sum_{s_1=0}^{r_1} \sum_{s_2=0}^{r_2} \cdots \sum_{s_k=0}^{r_k} \binom{r_1}{s_1}\binom{r_2}{s_2} \cdots \binom{r_k}{s_k} = 2^{r_1+r_2+\cdots+r_k}$$

So this method is not free of computational burden.

Example

$$A = \begin{bmatrix} 0 & \mu_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \mu_2 \end{bmatrix}$$

We found earlier that

$$M_0 = \left[\begin{array}{c:c:c} I & 0 & 0 \\ \hdashline \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & I & 0 \\ \hdashline 0 & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & 0 \end{array}\right], \quad M_1 = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 & 0 \\ \hdashline 0 & \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 \end{array}\right]$$

$$M_2=\left[\begin{array}{c|c|ccc} & & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ & & 0 & 0 & 1\\ \hline & & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ & & 0 & 1 & 0\\ \hline & & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ & & 1 & 0 & 0 \end{array}\right], \quad r_1=2, \quad r_2=3$$

Denote a 9-dimensional unit vector with 1 in position i by e_i. Then

$$M_1=e_4e_2^{\mathrm{T}}+e_7e_5^{\mathrm{T}}, \quad M_2=e_3e_9^{\mathrm{T}}+e_6e_8^{\mathrm{T}}+e_9e_7^{\mathrm{T}}$$

For this problem, there are 12 separate equalities to check. Consider that obtained by taking $s_1=1$, $s_2=3$. For the net to not have full rank, we require

$$\det\begin{bmatrix} M_0 & e_4 & e_3 & e_6 & e_9\\ e_2^{\mathrm{T}} & 0 & 0 & 0 & 0\\ e_9^{\mathrm{T}} & 0 & 0 & 0 & 0\\ e_8^{\mathrm{T}} & 0 & 0 & 0 & 0\\ e_7^{\mathrm{T}} & 0 & 0 & 0 & 0 \end{bmatrix}+\det\begin{bmatrix} M_0 & e_7 & e_3 & e_6 & e_9\\ e_5^{\mathrm{T}} & 0 & 0 & 0 & 0\\ e_9^{\mathrm{T}} & 0 & 0 & 0 & 0\\ e_8^{\mathrm{T}} & 0 & 0 & 0 & 0\\ e_7^{\mathrm{T}} & 0 & 0 & 0 & 0 \end{bmatrix}=0$$

However, the first determinant evaluates as 1 and the second as zero.

Proof of Lemma 4.1

Recognize that

$$\det\left(M_0+\mu_1\sum_{j=1}^{r_1} b_j^1 c_j^{1\mathrm{T}}+\dots+u_k\sum_{j=1}^{r_k} b_j^k c_j^{k\mathrm{T}}\right)$$

$$=(-1)^{\eta}\det\begin{bmatrix} M_0 & \mu_1 b_1^1 & \mu_1 b_2^1 & \cdots & \mu_1 b_{r_1}^1 & \cdots & \mu_k b_1^k & \cdots & \mu_k b_{r_k}^k\\ c_1^{1\mathrm{T}} & -1 & & & & & & & \\ \vdots & & \ddots & & & & & & \\ c_{r_1}^{1\mathrm{T}} & & & & -1 & & & & \\ \vdots & & & & & \ddots & & & \\ c_1^{k\mathrm{T}} & & & & & & -1 & & \\ \vdots & & & & & & & \ddots & \\ c_{r_k}^{k\mathrm{T}} & & & & & & & & -1 \end{bmatrix} \tag{4.5}$$

where $\eta = r_1 + r_2 + \dots + r_k$. The derivative of the determinant s_1 times with respect to μ_1, s_2 times with respect to μ_2, ..., is the sum of a number of determinants obtained by replacing all possible choices of s_1 of $\mu_1 b_1^1, \dots, \mu_1 b_{r_1}^1$ by $b_1^1, \dots, b_{r_1}^1$, all possible choice of s_2 of $\mu_2 b_1^2, \dots, \mu_2 b_{r_2}^2$ by $b_1^2, \dots, b_{r_1}^2$, and so on. The value of this derivative at the origin is then given by the left hand side of (4.4). Since $\det(M_0 + \mu_i M_i) = 0$ for all μ_i if and only if it and its derivatives of all orders with respect to the μ_i vanish, the lemma is proved.

Let us now consider the modification required when M is not square. Let M be $q \times r$, with $q < r$, and suppose we are testing for rank $M < q$. Then one could apply the technique of Lemma 4.1 to check that all selections of q columns for M gave a non-singular determinant.

One can also develop an approach based on the testing of MM^{T}, which is square, for singularity. The matrix MM^{T} of course involves the μ_i in a more complicated way than does M.

One can think of this section as providing a tool for decomposing one problem of very large dimension (checking the rank of P_k defined in § 3) into a number of problems of smaller dimension. The matrix P_k had a number of rows equal to $q(r_1+1) \dots (r_k+1)$ where M has q rows. On the other hand the largest determinant to be evaluated by the method of this section has size $q + r_1 + r_2 + \dots + r_k$.

The decomposition just referred to is far from obvious. For example, to have

$$\det(A + \mu bc^{\mathrm{T}}) = 0$$

for all μ and some vector b, c and A $q \times q$, the matrix pencil methods give

$$\operatorname{rank} \begin{bmatrix} A & bc^{\mathrm{T}} & 0 \\ 0 & A & bc^{\mathrm{T}} \end{bmatrix} < 2q \tag{4.6}$$

and (by considering the transposed problem)

$$\operatorname{rank} \begin{bmatrix} A & 0 \\ bc^{\mathrm{T}} & A \\ 0 & bc^{\mathrm{T}} \end{bmatrix} < 2q \tag{4.7}$$

while the procedure of this section gives

$$\det A = 0 \quad \text{and} \quad \det \begin{bmatrix} A & b \\ c^{\mathrm{T}} & 0 \end{bmatrix} = 0 \tag{4.8}$$

The equivalence of all these conditions is not obvious, and indeed takes some lines of algebra to prove. Neither is it obvious that further equivalent conditions, which we discuss in the next section, are provided by

$$\operatorname{rank}[A, b] < q \quad \text{or} \quad \operatorname{rank} \begin{bmatrix} A \\ c^{\mathrm{T}} \end{bmatrix} < q \tag{4.9}$$

5. Systems problems yielding matrix nets with further rank restrictions

We shall now describe systems problems leading to matrix nets

$$M = M_0 + \sum_{i=1}^{k} \mu_i M_i$$

where M_i for $i \geqslant 1$ has rank 1. Then we shall indicate yet another procedure than those of §§ 3 and 4 for checking whether M has full row rank.

While the structural controllability problem apparently does not lead to a net with the rank restrictions described here, we shall see in the next section that the ideas described here are nevertheless relevant to some structural controllability problems.

Consider a triple $\{A, B, C\}$ with fixed A $n \times n$, B $n \times p$, C $m \times n$. Then it is known that $A + BKC$ has an eigenvalue λ_0 that is independent of K if and only if either $[A, B]$ is not completely controllable or $[A, C]$ is not completely observable. Equivalently

$$\det\left(\lambda_0 I - A + \sum_{\alpha\beta} k_{\alpha\beta} b_\alpha c_\beta^{\mathrm{T}}\right) = 0 \tag{5.1}$$

for all k_{ij}, where $B = [b_1, b_2, \ldots, b_p]$ and $C^{\mathrm{T}} = [c_1 \ldots c_m]$. With $M_0 = \lambda_0 I - A$, $k_{\alpha\beta}$ for $\alpha = 1, \ldots, p$, $\beta = 1, \ldots, m$ corresponding to the μ_i and $b_\alpha c_\beta^{\mathrm{T}}$ to the M_i, (5.1) is an equation of the form studied earlier.

The procedure of § 3 would lead to an equivalent condition to (5.1) without the $k_{\alpha\beta}$, and the condition would involve a matrix with $n2^{mp}$ rows. The procedure of § 4 yields 2^{mp} separate simultaneous equalities equivalent to (5.1); actually, this number can immediately be reduced, since a number are trivially seen to be true; one gets the simultaneous set

$$\det\begin{bmatrix} \lambda_0 I - A & b_{i_1} & \ldots & b_{i_k} \\ c_{j_1}^{\mathrm{T}} & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ c_{j_k}^{\mathrm{T}} & 0 & \ldots & 0 \end{bmatrix} = 0 \tag{5.2}$$

for all subsets $\{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, m\}$ and $\{j_1, \ldots, j_k\}$ of $\{1, 2, \ldots, p\}$, and $k = 0, 1, \ldots, \min(m, p)$. Nevertheless, neither type of condition remotely resembles a condition recognizable as requring $[A, B]$ not controllable, or $[A, C]$ not observable. Indeed, neither condition even suggests decomposition into two alternatives.

Again, consider a system

$$\left.\begin{aligned} \dot{x} &= Ax + B_1 u_1 + B_2 u_2 \\ y_1 &= C_1 x, \quad y_2 = C_2 x \end{aligned}\right\} \tag{5.3}$$

with A $n \times n$, B_1 $n \times p_1$, etc. The system is said to have a decentralized fixed mode λ_0 if

$$\det(\lambda_0 I - A - B_1 K_1 C_1 - B_2 K_2 C_2) = 0 \tag{5.4}$$

independently of K_1, K_2. It is clear that this condition is again of the form

$$\det\left(M_0 + \sum_{1=i}^{k} \mu_i M_i\right) = 0 \tag{5.5}$$

with rank $M_i = 1$ for all $i \geqslant 1$. Equivalent conditions can be found via the procedures of the preceding sections; neither procedure leads to the conditions, known from e.g. Sezer and Šiljak (1981): at least *one* of the following holds:

$$\operatorname{rank}\,[\lambda_0 I - A,\ B_1,\ B_2] < n \tag{5.6 a}$$

$$\operatorname{rank}\begin{bmatrix} \lambda_0 I - A \\ C_1 \\ C_2 \end{bmatrix} < n \tag{5.6 b}$$

$$\operatorname{rank}\begin{bmatrix} \lambda_0 I - A & B_1 \\ C_2 & 0 \end{bmatrix} < n \tag{5.6 c}$$

$$\operatorname{rank}\begin{bmatrix} \lambda_0 I - A & B_2 \\ C_1 & 0 \end{bmatrix} < n \tag{5.6 d}$$

The above two examples suggest that there might be another way of handling nets with rank $M_i = 1$, $i \geqslant 1$, which would lead in the case of the examples to more familiar conditions. The key to handling such nets is contained in the following observation.

Lemma 5.1

Consider the matrix pencil $M = M_0 + \mu b_1 c_1^{\mathrm{T}}$ where M_0 is $q \times r$, $q \leqslant r$ and b_1 and c_1 are non-zero vectors. Suppose that rank $M = p < q$ for all μ. Then either there exist $q-p$ linearly independent constant left nullvectors of M or $r-p$ linearly independent constant right nullvectors of M. Equivalently, either

$$\operatorname{rank}\,[M_0,\ b_1] = p \quad \text{or} \quad \operatorname{rank}\begin{bmatrix} M_0 \\ c_1^{\mathrm{T}} \end{bmatrix} = p \tag{5.7}$$

Proof

Without loss of generality, suppose the coordinate basis is such that $b_1^{\mathrm{T}} = [1, 0, \ldots, 0]$, $c_1^{\mathrm{T}} = [1, 0, \ldots, 0]$. If the last $(q-1)$ rows of M have rank $(p-1)$, then they have $q-p$ linearly independent constant left nullvectors (of length $q-1$). Prefixing these nullvectors with a zero gives $q-p$ linearly independent constant left nullvectors for M. With x^{T} a constant vector the equation $x^{\mathrm{T}} M = 0$ implies $x^{\mathrm{T}} M_0 = 0$ and $x^{\mathrm{T}} b_1 = 0$ on equating the constant term and linear-in-μ term to zero. Thus each left nullvector for M is a left nullvector for $[M_0, b_1]$ and conversely. Hence the first alternative in (5.7) holds.

Now suppose the last $(q-1)$ rows of M have rank p. Then the last $(q-1)$ rows have $r-p$ linearly independent constant right nullvectors. The first row must be linearly dependent on the last $q-1$ since M has rank p and so the right nullvectors of the last $q-1$ rows are also nullvectors for the first row and thus for all of M. Thus M has $r-p$ linearly independent constant right nullvectors. The second alternative in (5.7) now follows easily.

Now we can state the main result.

Lemma 5.2

Consider the matrix pencil

$$M = M_0 + \sum_{i=1}^{k} \mu_i b_i c_i^{\mathrm{T}}$$

where M_0 is $q \times r$, $q \leqslant r$ and b_i, c_i are non-zero vectors. Suppose that rank $M = p < q$ for all μ_i. Then at least *one* of the following alternatives holds :

$$\operatorname{rank}\begin{bmatrix} M_0 & b_{i_1} & \dots & b_{i_\alpha} \\ c_{i_{\alpha+1}}^{\mathrm{T}} & 0 & \dots & 0 \\ \vdots & \vdots & & \\ c_{i_k}^{\mathrm{T}} & 0 & \dots & 0 \end{bmatrix} = p \tag{5.8}$$

where $\{i_1 \dots i_k\}$ is any permutation of $\{1, \dots, k\}$.

Proof

We apply Lemma 4.1 repeatedly :

$$\left(M_0 + \sum_{i=1}^{k-1} \mu_i b_i c_i^{\mathrm{T}} \right) + \mu_k b_k c_k^{\mathrm{T}}$$

has rank p and so either

$$\operatorname{rank}\left[M_0 + \sum_{i=1}^{k-1} \mu_i b_i c_i^{\mathrm{T}}, b_k \right] = p \quad \text{or} \quad \operatorname{rank}\begin{bmatrix} M_0 + \sum_{i=1}^{k-1} \mu_i b_i c_i^{\mathrm{T}} \\ c_k^{\mathrm{T}} \end{bmatrix} = p$$

Now

$$\left[M_0 + \sum_{i=1}^{k-1} \mu_i c_i c_i^{\mathrm{T}}, b_k \right] = \left[M_0 + \sum_{i=1}^{k-2} \mu_i b_i c_i^{\mathrm{T}}, b_k \right] + \mu_{k-1} b_{k-1} [c_{k-1}^{\mathrm{T}}, 0]$$

and this has rank p for all μ_{k-1} if and only if either

$$\operatorname{rank}\left[M_0 + \sum_{i=1}^{k-2} \mu_i b_i c_i^{\mathrm{T}}, b_k, b_{k-1} \right] = p$$

or

$$\operatorname{rank}\begin{bmatrix} M_0 + \sum_{i=1}^{k-2} \mu_i b_i c_i^{\mathrm{T}} & b_k \\ c_{k-1}^{\mathrm{T}} & 0 \end{bmatrix} = p$$

Continuation of this procedure leads to (5.8).

Assuming no two of the b_i are the same nor any two of the c_i, the number of alternatives is 2^k. When the conjunctive conditions of § 4 are used, the number of such conditions is also 2^k (at least in the square case). The dimension of the matrix examined in the procedure of § 3 is $q2^k \times r3^k$.

Let us now observe how this result applies to the centralized and decentralized fixed mode problems considered earlier. For the centralized problem, assume (5.1) holds. Using the ideas of Lemma 5.2, we see that an equivalent condition to (5.1), based on (5.8), will involve a matrix such that for each selection of $\alpha\beta$, $\alpha = 1, \dots, p$, $\beta = 1, \dots, m$, we include either b_α or c_β^{T} in the block first row or block first column, respectively. Suppose that for the selections corresponding to $\alpha = i$, $\beta = 1, \dots, m$, we never select b_i for inclusion in the first block row. Then

$c_1^{\mathrm{T}}, \ldots, c_m^{\mathrm{T}}$ all appear in the first block column. Hence either the first block row contains all of $b_1, \ldots, b_p$ or the first block column contains all $c_1^{\mathrm{T}}, \ldots, c_m^{\mathrm{T}}$, with each alternative condition corresponding to (5.8) being of the form

$$\operatorname{rank}\begin{bmatrix} \lambda_0 I - A & \text{Selection of } b_i \\ \text{Selection of } c_j^{\mathrm{T}} & 0 \end{bmatrix} < n$$

Hence every alternative implies and is implied by at least one of

$$\operatorname{rank}\,[\lambda_0 I - A, b_1, \ldots, b_p] < n \quad \text{or} \quad \operatorname{rank}\begin{bmatrix} \lambda_0 I - A \\ c_1^{\mathrm{T}} \\ \vdots \\ c_m^{\mathrm{T}} \end{bmatrix} < n$$

i.e. $[A, B]$ is not completely controllable, or $[A, C]$ is not completely observable.

In a similar manner, Lemma 4.2 can be used to derive the alternatives (5.6) for the decentralized fixed mode problem.

6. Structural controllability with rank restrictions

In this section we shall consider pairs $[A, B]$, to be tested for structural controllability in which

$$[sI - A, B] = [sI - A_0, B_0] + \sum_{i=1}^{k} \mu_i \alpha_i [\beta_i^{\mathrm{T}}, \gamma_i^{\mathrm{T}}] \tag{6.1}$$

This arrangement is similar to that considered in Corfmat and Morse (1976). We shall see that, provided one knows the eigenvalues of A_0, one can check structural controllability fairly efficiently. We observe first :

Lemma 6.1

Suppose

$$\operatorname{rank}\begin{bmatrix} sI - A_0 & B_0 & \alpha_{i_1} \ldots \alpha_{i_l} \\ \beta_{i_{l+1}}^{\mathrm{T}} & \gamma_{i_{l+1}}^{\mathrm{T}} & 0 \ldots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ \beta_{i_k}^{\mathrm{T}} & \gamma_{i_k}^{\mathrm{T}} & 0 \ldots 0 \end{bmatrix} \leqslant \dim A_0 \tag{6.2}$$

for all s and some partition of $\{1, \ldots, k\}$ into sets $\{i_1, \ldots, i_l\}$ and $\{i_{l+1}, \ldots, i_k\}$, the second non-empty. Then $[A, B]$ is structurally controllable if and only if

$$\left.\begin{aligned} A_1 &= A + \mu_{i_1}\alpha_{i_1}\beta_{i_1}^{\mathrm{T}} + \ldots + \mu_{i_l}\alpha_{i_l}\beta_{i_l}^{\mathrm{T}} \\ B_1 &= B_0 + \mu_{i_1}\alpha_{i_1}\gamma_{i_1}^{\mathrm{T}} + \ldots + \mu_{i_l}\alpha_{i_l}\gamma_{i_l}^{\mathrm{T}} \end{aligned}\right\} \tag{6.3}$$

is structurally controllable.

Remark

The condition (6.2) is equivalent to the condition

$$\begin{bmatrix} \gamma_{i_{l+1}}^{\mathrm{T}} & 0 \ldots 0 \\ \vdots & \\ \gamma_{i_k}^{\mathrm{T}} & 0 \ldots 0 \end{bmatrix} - \begin{bmatrix} \beta_{i_{l+1}}^{\mathrm{T}} \\ \vdots \\ \beta_{i_k}^{\mathrm{T}} \end{bmatrix} (sI - A_0)^{-1}[B_0, \alpha_{i_1}, \ldots, \alpha_{i_l}] \equiv 0 \tag{6.4}$$

and it is checkable by selecting any dim A_0 values of s differing from eigenvalues of A_0 and checking (6.2) only at these values. (Of course, the inequality sign in (6.2) is impossible for such s.) The equivalence between (6.2) and (6.4) can be seen by noting that for square non-singular W

$$\operatorname{rank}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I & 0 \\ -YW^{-1} & I \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$
$$= \operatorname{rank} W + \operatorname{rank}(Z - YW^{-1}X)$$

Hence

$$\operatorname{rank}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \operatorname{rank} W$$

if and only if $Z - YW^{-1}X = 0$.

Proof of Lemma

Making use of the well known feedback formula

$$H(sI - F - KH)^{-1}G = [I + H(sI - F)^{-1}K]^{-1}H(sI - F)^{-1}G$$

we have

$$\begin{bmatrix} \beta_{i_{l+1}}{}^{\mathrm{T}} \\ \vdots \\ \beta_{i_k}{}^{\mathrm{T}} \end{bmatrix} [sI - A]^{-1}B$$
$$= \begin{bmatrix} \beta_{i_{l+1}}{}^{\mathrm{T}} \\ \vdots \\ \beta_{i_k}{}^{\mathrm{T}} \end{bmatrix} \left(sI - A_0 - \sum_{j=1}^{l} \mu_{i_j}\alpha_{i_j}\beta_{i_j}{}^{\mathrm{T}} - \sum_{j=l+1}^{k} \mu_{i_j}\alpha_{i_j}\beta_{i_j}{}^{\mathrm{T}} \right)^{-1} \times \left(B_0 + \sum_{j=1}^{k} \mu_{i_j}\alpha_{i_j}\gamma_{i_j}{}^{\mathrm{T}} \right)$$
$$= \left\{ I + \begin{bmatrix} \beta_{i_{l+1}}{}^{\mathrm{T}} \\ \vdots \\ \beta_{i_k}{}^{\mathrm{T}} \end{bmatrix} \left(sI - A_0 - \sum_{j=1}^{l} \mu_{i_j}\alpha_{i_j}\beta_{i_j}{}^{\mathrm{T}} \right)^{-1} [\mu_{i_{l+1}}\alpha_{i_{l+1}}, \ldots, \mu_{i_k}\alpha_{i_k}] \right\}^{-1}$$
$$\begin{bmatrix} \beta_{i_{l+1}}{}^{\mathrm{T}} \\ \vdots \\ \beta_{i_k}{}^{\mathrm{T}} \end{bmatrix} \left(sI - A_0 - \sum_{j=1}^{l} \mu_{i_j}\alpha_{i_j}\beta_{i_j}{}^{\mathrm{T}} \right)^{-1} \left(B_0 + \sum_{j=1}^{k} \mu_{i_j}\alpha_{i_j}\gamma_{i_j}{}^{\mathrm{T}} \right)$$

Now using (6.4), it is not hard to verify that the second term on the right hand side is zero and thus the left hand side is zero. Now if $[A, B]$ is structurally controllable, the zero nature of the left side implies $\beta_{i_{l+1}}, \ldots, \beta_{i_k}$ are all zero, while (6.4) implies $\gamma_{i_{l+1}}, \ldots, \gamma_{i_k}$ are all zero. The conclusion of the lemma is now obvious since $[A_1, B_1] = [A, B]$. Conversely, if $[A, B]$ is not structurally controllable, *a fortiori* $[A_1, B_1]$ cannot be, being obtained for $[A, B]$ by setting some μ_i to zero.

The lemma allows us, via a preliminary checking procedure, to possibly reduce the number of parameters μ_i entering the picture. Let us suppose therefore this has been done, i.e. without loss of generality, let us suppose no condition such as (6.2) holds.

Now suppose that in (6.1), for any choice of μ_i, there exists a particular s, in general dependent on the choice of μ_i, such that $[sI - A, B]$ has less than full row rank. Temporarily regard μ_k as the only variable. Then every $n \times n$ minor of $[sI - A, B]$ is of the form $\kappa(s) + \mu_k \lambda(s)$. At least one is not zero identically in s, and if B is not identically zero, which we shall assume, more than one minor is not identically zero.

The question of common zeros of polynomials which involve a parameter linearly is addressed in the following lemma, the proof of which is straightforward.

Lemma 6.2

Let $\alpha(s)$, $\beta(s)$, $\gamma(s)$ and $\delta(s)$ be real polynomials in s, and such that for all μ, $\alpha(s) + \mu\beta(s)$ and $\gamma(s) + \mu\delta(s)$ have a common zero. Then either this zero is independent of μ and is a common zero of α, β, γ and δ, or else there exists a real rational $\epsilon(s)$, independent of μ, for which $(\alpha + \mu\beta) = \epsilon(\gamma + \mu\delta)$.

Hence if all $n \times n$ minors of $[sI - A, B]$ have a common zero for each value of μ_k, either this zero is independent of μ_k and is then a zero of every $\kappa(s)$ and $\lambda(s)$, or else there exist $k_{ij}(s, \mu_1, \ldots, \mu_{k-1})$ such that two minors $\kappa_i(s) + \mu_k \lambda_i(s)$ and $\kappa_j(s) + \mu_k \lambda_j(s)$ are related by

$$\kappa_i(s) + \mu_k \lambda_i(s) = k_{ij}[\kappa_j(s) + \mu_k \lambda_j(s)] \tag{6.5}$$

We rule out this last possibility as follows. With p = number of columns of B_0, it implies that (6.1) has a set of p linearly independent right nullvectors which are independent of μ_k. In fact, using Cramer's rule and (6.5), one can organize this set of nullvectors into a matrix of the form

$$\begin{bmatrix} n \times p \text{ matrix of } k_{ij} \\ I_p \end{bmatrix}$$

It is trivial to see that any nullvector of (6.1) when independent of μ_k must also be a nullvector of

$$\begin{bmatrix} sI - A_0 - \sum_{i=1}^{k-1} \mu_i \alpha_i \beta_i^{\mathrm{T}} & B_0 + \sum_{i=1}^{k-1} \mu_i \alpha_i \gamma_i^{\mathrm{T}} \\ \beta_k^{\mathrm{T}} & \gamma_k^{\mathrm{T}} \end{bmatrix}$$

which accordingly has rank $\leqslant \dim A_0$. Following the ideas of the previous section, see Lemma 5.3, we then conclude that (6.2) holds, with the set $\{i_{l+1}, \ldots, i_k\}$ including k. This contradicts the assumption that a preliminary checking procedure has been applied so that (6.2) does not hold.

Accordingly, any zero of all $n \times n$ minors of (6.1) must be independent of μ_k. By symmetry, it is independent of all μ_i. Hence, taking all $\mu_i = 0$, we see it is a zero of $|sI - A_0|$. The checking of structural controllability can proceed by checking for each eigenvalue λ_i of A_0 whether

$$\operatorname{rank} \left\{ [\lambda_i I - A_0, B_0] + \sum_{i=1}^{k} \mu_i \alpha_i [\beta_i^{\mathrm{T}}, \gamma_i^{\mathrm{T}}] \right\} = \dim A_0$$

for almost all μ_i. This is straightforward given the results of the last section. We require simply, see Lemma 5.2, that for all partitions $\{i_1, \ldots, i_l\} \cup \{i_{l+1}, \ldots, i_k\}$ of $\{1, \ldots, k\}$

$$\operatorname{rank}\begin{bmatrix} \lambda_i I - A_0 & B_0 & \alpha_{i_1} & \ldots & \alpha_{i_l} \\ \beta_{i_{l+1}}{}^{\mathrm{T}} & \gamma_{i_{l+1}}{}^{\mathrm{T}} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \\ \beta_{i_k}{}^{\mathrm{T}} & \gamma_{i_k}{}^{\mathrm{T}} & 0 & \ldots & 0 \end{bmatrix} = \dim A_0 \tag{6.6}$$

Summing up :

Lemma 6.3

With notations as above, suppose that no condition of the type (6.2) holds. Then $[A, B]$ is generically controllable if for all eigenvalues λ_i of A_0

$$\operatorname{rank}\,[\lambda_i I - A_0, B] = \dim A_0 \tag{6.7}$$

or equivalently, (6.1) holds for all partitions $\{i_1, \ldots, i_l\} \cup \{i_{l+1}, \ldots, i_k\}$ of $\{1, 2, \ldots, k\}$.

Example

Consider as before

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \mu_1 e_1 e_2{}^{\mathrm{T}}, \quad B = \mu_2 e_3$$

We consider first (6.2) with $\{i_1\} = \{1\}$, we see

$$\operatorname{rank}\begin{bmatrix} s & 0 & 0 & 0 & 1 \\ 0 & s & -1 & 0 & 0 \\ 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} > 3$$

and with $\{i_1\} = \{2\}$, we see

$$\operatorname{rank}\begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 \\ 0 & 0 & s & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} > 3$$

This means that it is enough to study (6.6). We find for the four cases $\{i_1, \ldots, i_l\} = \{1, 2\}, \{1\}, \{2\}, \{\emptyset\}$, with $\lambda_1 = 0$,

$$\operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \dim A$$

This establishes generic controllability.

The ideas of this section also allows rapid derivation of the original result on structural controllability, where the entries of A and B are either non-zero or totally free. Suppose that the non-zero entries of an $n \times n$ A are in position $i_1 j_1, \ldots, i_p j_p$ and the non-zero entries of an $n \times m$ B are in positions $k_1 l_1, \ldots, k_q l_q$. Thus

$$A = \sum_{\alpha=1}^{p} \mu_{i_\alpha j_\alpha} e_{i_\alpha} e_{j_\alpha}^{\mathrm{T}}, \quad B = \sum_{\beta=1}^{q} \mu_{k_\beta l_\beta} e_{k_\beta} e_{l_\beta}^{\mathrm{T}} \tag{6.8}$$

(Evidently A_0 and B_0 are zero.) First, we consider the possibilities dealt with in Lemma 6.1, i.e. we ask whether it is possible that, for all s,

$$\operatorname{rank}\begin{bmatrix} sI & 0 & e_{i_1} & \ldots & e_{i_\gamma} & e_{k_1} & \ldots & e_{k_\delta} \\ e_{j_{\gamma+1}}^{\mathrm{T}} & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ e_{j_p}^{\mathrm{T}} & 0 & 0 & & & & & 0 \\ 0 & e_{l_{\delta+1}}^{\mathrm{T}} & 0 & & & & & 0 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & e_{l_q}^{\mathrm{T}} & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix} \leqslant n \tag{6.9}$$

with the set $\{j_{\gamma+1}, \ldots, j_p, l_{\delta+1}, \ldots, l_q\}$ non-empty. Clearly, (6.9) cannot hold unless the set $\{l_{\delta+1}, \ldots, l_q\}$ is empty. And even then, by virtue of (6.4), (6.9) will hold if and only if

$$\{j_{\gamma+1}, \ldots, j_p\} \cap \{i_1, \ldots, i_\gamma, k_1, \ldots, k_q\} = \emptyset \tag{6.10}$$

Now $\{i_1, \ldots, i_\gamma\}$ is a strict subset of the indices of those rows of A with non-zero elements, and $\{k_1, \ldots, k_q\}$ is the set of rows of B with non-zero elements. Suppose that A, B are replaced by PAP^{-1}, PB with P a permutation matrix such that

$$\{k_1, \ldots, k_q\} = \{r+1, r+2, \ldots, s\}$$
$$\{i_1, \ldots, i_\gamma\} = \{t, \ldots, n\}, \quad t \leqslant s$$

Then (6.10) requires that any non-zero entries in rows of A other than $t, \ldots, n$ cannot occur in columns $r+1$ to n, i.e. we have the general pattern

$$PAP^{-1} = \begin{array}{c} \begin{array}{cc} \quad r+1 & \quad n \end{array} \\ \left[\begin{array}{c:c} X & 0 \\ \hdashline 0 & 0 \\ \hdashline X & X \end{array}\right] \end{array} \begin{array}{l} \\ r+1 \\ t \\ \\ \end{array}, \quad PB = \left[\begin{array}{c} 0 \\ \hdashline X \\ \hdashline X \\ \hdashline 0 \end{array}\right] \begin{array}{l} r+1 \\ t \\ s \end{array} \tag{6.11}$$

This pattern is a particular case of the following pattern :

$$PAP^{-1} = \begin{array}{cc} \begin{array}{cc} r+1 & n \end{array} & \\ \left[\begin{array}{c|c} X & 0 \\ \hline X & X \end{array}\right] & \begin{array}{l} r+1, \\ n \end{array} \end{array} \qquad PB = \begin{array}{cc} \left[\begin{array}{c} 0 \\ \hline X \end{array}\right] & \begin{array}{l} r+1 \\ n \end{array} \end{array} \tag{6.12}$$

Moreover, if this pattern holds, lack of controllability is evident.

We have just shown that if the possibility dealt with in Lemma 6.1 holds, there is lack of structural controllability, evidenced by the existence of a permutation matrix P implying (6.12).

The second possible way for having lack of generic controllability arises when we consider rank $[\lambda I - A, B]$ with λ an eigenvalue of A_0, i.e. $\lambda = 0$ here. Thus lack of generic controllability will follow if

$$\operatorname{rank}[A, B] < n \tag{6.13}$$

In summary, if and only if either (6.12) or (6.13) holds, there is lack of structural controllability.

The results of Sezer and Šiljak (1981) can also be obtained by this approach.

7. Conclusions

The main ideas of the paper are the following.

1. General structural controllability questions can be posed as questions about the rank of matrix nets.
2. A matrix net can be checked for its full rank property by
   (*a*) examining the rank of a very large matrix,
   (*b*) examining the determinants of a number of smaller matrices (conjunctive conditions), and,
   (*c*) for those nets with rank restrictions on the coefficient matrices multiplying the free parameters, by examining the ranks of a number of smaller matrices (disjunctive conditions).
3. Other system problems than structural controllability, for example, checking for decentralized fixed modes, can be examined using matrix nets.
4. Structural controllability of a pair $[A, B]$ can be tested efficaciously when the matrix $[sI - A, B]$ for any fixed s has all rank 1 matrices multiplying the free parameters. The method allows derivation, as a special case, of the original structural controllability results.

It is interesting to note that for nets with coefficient matrices of rank 1, three logically equivalent sets of conditions for less than full rank have been given, which all involve constant matrices—that involving a single large matrix, the conjunctive conditions and the disjunctive conditions. The first two sets are available when the rank restriction on the coefficient matrices is relaxed, the third is not ; the authors have sought and obtained partial results providing disjunctive conditions when the rank restriction is relaxed, but they are not convinced they are the appropriate ones.

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