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PORT PROPERTIES OF NONLINEAR RECIPROCAL NETWORKS*

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Abstract. Time-invariant n-port networks are considered which are constructed from linear resistors and transformers, and nonlinear inductors and capacitors. A port property is sought which is analogous to the symmetry of the impedance matrix obtained in the linear case. Such a property is found using a stochastic characterization: with all resistors assumed to be at the one temperature and modelled to include the usual thermal noise, a fluctuating noise voltage will be observed at the ports of the networks. With some reasonable restrictions on the nature of the nonlinearities in the reactive elements, the vector of such noise voltages is shown to constitute what is known as a reversible stochastic process; in the linear case, reversibility holds if and only if the impedance matrix is symmetric.

1. Introduction

Suppose a finite number of linear, time-invariant passive resistors, inductors and capacitors are connected together to form a two port network which has an impedance matrix $Z(s)$. It is normally established in elementary circuit theory texts that $Z_{12}(s) = Z_{21}(s)$, and this property is termed reciprocity. In effect, what is happening is that a property of the *internal* structure of the network, viz. that it is composed of restricted classes of elements, gives rise to a port property, or *external* property of the network, viz. $Z_{12}(s) = Z_{21}(s)$. One can then ask questions of the following type. Suppose that the inductors and capacitors are allowed to become nonlinear; what is the port property, if any, of the network? This paper is an attempt to come to grips with this question.

There are several approaches that might be tried. For example, one might assume that arbitrary but predetermined current inputs are applied at the

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two ports, giving rise to certain voltage responses. Then one could, by linearization around the resulting trajectories, seek to prove some sort of reciprocity behavior linking small signal perturbations in the current inputs with small signal perturbations in the voltage responses. In general, this is not effective, since the impulse responses linking current perturbations to voltage perturbations are not time-invariant, and this is a prerequisite for reciprocal behavior. To be sure, if one linearizes around the origin, or any DC operating point, so that the impulse responses are time-invariant, one will get a reciprocity property, i.e. symmetry of the impulse response matrix; but this is a very restricted result, conveying a property of the network when the latter is operating only in a very restricted regime. Again, one might attempt to exploit Tellegen's theorem [1], which is not restricted to linear networks and which can nevertheless be used to prove the linear network results; this did not prove successful.

Now it turns out that there is another way of stating the reciprocity result for linear networks, using not deterministic ideas but stochastic ideas. Suppose the network is unexcited at its ports, but we analyze the effects of the thermal noise associated with the resistors in terms of the fluctuations produced at the ports. As we review later in more detail, this fluctuation is Gaussian, and the spectral matrix associated with the noise at the two ports is symmetric (as opposed to just being Hermitian symmetric, a property which all spectral matrices have). So here we have a port property, this time associated with the stochastic behavior of the network. Can it be generalized to yield a port property when the reactive elements are nonlinear? The answer is yes.

Naturally, once the network contains nonlinear elements, one cannot expect the port fluctuations to be Gaussian, and therefore spectral characterizations, even if they could be found (and usually they cannot) are inadequate. Thus joint probability densities of the processes (or of increments of the processes when they contain a white noise component) are relevant. And the property that collapses to symmetry of the spectrum matrix in the Gaussian case is termed reversibility.

Another approach to a discussion of reciprocity using fluctuation ideas is taken in unpublished work of Penfield [2]. In this reference, it is argued that the deterministic behavior of the network when linearized around the origin can be linked to the fluctuation properties of the network, assuming certain approximations are valid, and that reciprocity of the network is reflected in symmetry of the impedance matrix associated with the linearization. We have already commented above that if a network with nonlinear reactive elements, linear resistors and transformers is linearized around the origin, then the linearized impedance matrix will be symmetric – essentially the same argument that works in the truly linear case works here, and there need be no appeal to stochastic processes.

We now sketch the layout of the paper. In the next section, we develop the noise model of an n-port network containing resistors, transformers, and nonlinear inductors and capacitors. Of course, it is a nonlinear model, in fact a nonlinear stochastic differential equation with white noise inputs.

Next, the concepts of reversibility and an extension thereof, dynamic reversibility, are reviewed briefly, and then we can show fairly quickly, by appealing to a recently established result [3] on the reversibility properties of the capacitor charges in a network of the type under consideration, that if the port voltages are linear combinations of capacitor voltages, they too are reversible.

Following this, we set up the machinery necessary to examine reversibility when the port voltages are not as just described, and may instead contain a white noise component. This leads us into a discussion of reverse time models of stochastic differential equations, and in turn allows us to characterize reversibility as arising when a reverse time model is almost the same as the normal, forward time model from which it is derived.

We then apply this machinery in Section 6 to the noise model of the network developed in Section 2 to establish our main result. Section 7 describes why, in linear networks, reversibility is equivalent to the property that $Z(s) = Z'(s)$ and Section 8 contains concluding remarks.

The fact that reciprocity in the linear case is related to reversibility has been known for some time, see e.g. [4-8], and as noted above, one result is available for nonlinear networks, [3]. Reversibility is now recognized as an important property of many physical systems; see [9], [10] for an introduction to many of the applications. Reference [11] also contains an interesting system theoretic treatment of reversibility for finite-dimensional linear systems.

2. Network equations

We consider n-port time-invariant networks constructed from linear resistors, transformers, and nonlinear capacitors and inductors. The resistors are all passive. The capacitors and inductors may be coupled and are described by

$$v_C = f_C(q),\ i_L = f_L(\phi) \tag{2.1}$$

Here q, v_C, i_L and ϕ denote charge, voltage, current and flux vectors, $f_C(\ .\ )$ and $f_L(\ .\ )$ are C^1-diffeomorphisms[1] [12], and there exist scalar energy storage functions $E_C(q)$ and $E_L(\phi)$ such that

$$f_C(q) = \nabla E_C(q) \qquad f_L(\phi) = \nabla E_L(\phi) \tag{2.2}$$

Further, $\nabla^2 E_C(q) - \epsilon I$ and $\nabla^2 E_L(\phi) - \epsilon I$ are positive definite for all q, ϕ and some ϵ, while $f_C(0)$ and $f_L(0)$ are zero. These sorts of assumptions are reasonably standard, and have frequently been used elsewhere [13-16]. In the case of a single capacitor, they become very familiar: the charge-voltage characteristic is smooth, monotonic, confined to first and third quadrants, and $|q| \to \infty$ if and only if $|v_C| \to \infty$. More generally, the conditions ensure passivity and losslessness by most definitions.

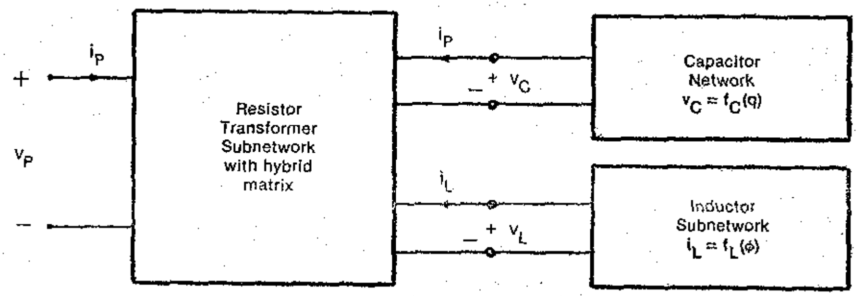

Figure 1. ***n*-port network drawn as resistive network terminated in reactive network.**

We further suppose that the particular n-port of interest, when drawn as depicted in Figure 1, allows definition of a hybrid matrix mapping $[i'_P \; i'_L \; v'_C]'$ into $[v'_P \; v'_L \; i'_C]'$:

$$\begin{bmatrix} v_P \\ v_L \\ i_C \end{bmatrix} = \begin{bmatrix} H_{PP} & H_{PL} & H_{PC} \\ H'_{PL} & H_{LL} & H_{LC} \\ -H'_{PC} & -H'_{LC} & H_{CC} \end{bmatrix} \begin{bmatrix} i_P \\ i_L \\ v_C \end{bmatrix} \tag{2.3}$$

Existence of the hybrid matrix is not normally a restrictive assumption; see [17] for a full discussion. The reciprocal character of the resistor-transformer subnetwork accounts for the structure of the matrix in (2.3).

If the resistor-transformer network were disconnected from the capacitors and inductors, and not externally excited, the thermal motion of the electrons in the resistors of the resistor transformer subnetwork would reflect itself in fluctuations in v_P, v_L, i_C, of zero mean and with covariance matrix [18]

$$E\left\{ \begin{bmatrix} v_P(t) \\ v_L(t) \\ i_C(t) \end{bmatrix} [v'_P(s) \;\; v'_L(s) \;\; i'_C(s)] \right\} = 2kT \begin{bmatrix} H_{PP} & H_{PL} & 0 \\ H_{PP} & H_{LL} & 0 \\ 0 & 0 & H_{CC} \end{bmatrix} \delta(t-s) \tag{2.4}$$

Here, k is Boltzmann's constant and T is the absolute temperature of all the resistors. One can think of the fluctuation as arising from Thevenin voltage sources and Norton current sources (the latter at the port to which capacitors will be connected), and it is normal to assume that such fluctuations can continue to be modelled in this way, even in the presence of external excitations and interconnections; accordingly, noting that the reactive element interconnections force $\dot{\phi} = -v_L$, $\dot{q} = -i_C$, we have the following equations describing the behavior of the network:

$$\begin{bmatrix} v_P \\ -\dot{\phi} \\ -\dot{q} \end{bmatrix} = \begin{bmatrix} H_{PP} & H_{PL} & H_{PC} \\ H'_{PL} & H_{LL} & H_{PC} \\ -H'_{PC} & -H'_{LC} & H_{CC} \end{bmatrix} \begin{bmatrix} i_P \\ f_L(\phi) \\ f_C(q) \end{bmatrix} + \sqrt{2kT} \begin{bmatrix} K_{11} & K_{12} & 0 \\ K'_{12} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \begin{bmatrix} dw_1 / dt \\ dw_2 / dt \\ dw_3 / dt \end{bmatrix} \tag{2.5}$$

where w_1, w_2, w_3 are three independent vector Wiener processes, each with independent entries, and

$$\begin{bmatrix} K_{11} & K_{12} & 0 \\ K'_{12} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix}^2 = \begin{bmatrix} H_{PP} & H_{PL} & 0 \\ H'_{PL} & H_{PL} & 0 \\ 0 & 0 & H_{CC} \end{bmatrix} \tag{2.6}$$

Without loss of generality, the (K_{ij}) matrix can be assumed symmetric.

In this paper, we shall be interested particularly in the noise in v_P, under the condition that $i_P \equiv 0$, i.e. we shall be interested in the open-circuit port noise. Thus we rearrange (2.5) as

$$\begin{bmatrix} d\phi \\ dq \end{bmatrix} = \begin{bmatrix} -H_{LL} & -H_{LL} \\ H'_{LC} & -H_{CC} \end{bmatrix} \begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix} dt + \sqrt{2kT} \begin{bmatrix} -K'_{12} & -K_{22} & 0 \\ 0 & 0 & -K_{33} \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \end{bmatrix}$$

$$v_P = [H_{PL} \quad H_{PC}] \begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix} + \sqrt{2kT}\, [K_{11} \quad K_{12} \quad 0] \begin{bmatrix} dw_1/dt \\ dw_2/dt \\ dw_3/dt \end{bmatrix} \tag{2.7}$$

This is the fundamental set of equations which we shall study. Throughout the paper, we shall assume that the network is maintained at constant temperature T.

3. Reversibility and dynamic reversibility

Let $x(\cdot)$ be a stationary random vector process for which all joint densities exist. Following the standard definition, see e.g. [10], we say that $x(\cdot)$ is reversible if and only if

$$\begin{aligned} &p[x(t_1) = a_1, x(t_2) = a_2, \ldots, x(t_n) = a_n] \\ &= p[x(T-t_1) = a_1, x(T-t_2) = a_2, \ldots, x(T-t_n) = a_n] \end{aligned} \tag{3.1}$$

for all n, $t_1, t_2, \ldots, t_n$, $a_1, \ldots, a_n$ and T. Actually, (3.1) implies stationarity.

In case $x(\cdot)$ is a Markov process, it is necessary and sufficient to have (in addition to an explicit assumption of stationarity)

$$p[x(t_1) = a_1, x(t_2) = a_2] = p[x(t_1) = a_2, x(t_2) = a_1] \tag{3.2}$$

for all t_1, t_2, a_1, a_2.

Now let $x(\cdot) = [y'(\cdot) \;\; z'(\cdot)]'$ be a pair of jointly stationary random vector processes for which all joint densities exist. Following a definition that appears to have first been highlighted by Casimir, [5], but is now common in statistical thermodynamics, we call $x(\cdot)$ dynamically reversible if and only if

$$\begin{aligned} p[y(t_1) = a_1, z(t_1) = b_1, y(t_2) = a_2, z(t_2) = b_2, \ldots, y(t_n) = a_n, z(t_n) = b_n] \\ = p[y(T-t_1) = a_1, z(T-t_1) = -b_1, y(T-t_2) = a_2, z(T-t_2) = \\ -b_2, \ldots, y(T-t_n) = a_n, z(T-t_n) = -b_n] \end{aligned} \tag{3.3}$$

for all $n, t_1, t_2, \ldots, t_n, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ and T.

The main result of [3] is that for networks of the type described in Section 2, the process $[\phi'(\cdot) \;\; q'(\cdot)]'$ is dynamically reversible provided that for all q, $f_C(-q) = -f_C(q)$. Also, by taking $x(\cdot) = [q'(\cdot) \;\; \phi'(\cdot)]'$, we have dynamic reversibility if and only if for all ϕ, $f_L(-\phi) = f_L(\phi)$.

Our main task in this paper is to consider reversibility properties of the port process $v_P(\cdot)$ defined in (2.7), as opposed to the processes $\phi(\cdot)$ and $q(\cdot)$ which are internal to the network.

4. Port fluctuations: A special case

Suppose in the model discussed in Section 2 that H_{PP} is zero. (This means that if current generators were connected at the n ports of the network depicted in Figure 1, and all noise generators were zero, there would be no "direct feedthrough" term to the response v_P). Because the resistor network is passive, the symmetric part of the hybrid matrix in (2.3) is positive definite and so $H_{PL} = 0$ also. [This is also evident from the covariance property exhibited by (2.4)]. Using (2.6), we conclude further that $K_{11} = 0$, $K_{12} = 0$. Accordingly, the second equation of (2.7) becomes simply

$$v_P = H_{PC} f_C(q) \tag{4.1}$$

This says that the port voltage is a linear combination of capacitor voltages. From the definition of reversibility, it is clear that if $\ell: y(t) \to \ell[y(t)]$ is a smooth, time-invariant transformation of $y(t)$, then the process $\ell[y(\cdot)]$ will be reversible. Accordingly, with $f_L(-\phi) = f_L(\phi)$, it follows that v_P is reversible.

When H_{PP} is nonzero, the question becomes more difficult. It is no longer true that in (2.7), $K_{11} = 0$, $K_{12} = 0$ and so v_p contains a white noise component. In this case, the definition of reversibility of the last section is no longer applicable, since the probability densities used in the definition will not exist. In the next section, we describe the tool which allows us to address this difficulty.

It might be thought that one approach to dealing with this problem would be to introduce temporarily a capacitance ϵ across every port, (which would immediately imply the reversibility of the port voltages for every $\epsilon > 0$), and then to let $\epsilon \to 0$. There is however a structural discontinuity in the stochastic differential equations describing the arrangement when ϵ changes from a positive quantity to zero, and in the linear case, the average stored energy on the reactive elements would change instantaneously (by an amount $\frac{1}{2}kT$) as ϵ changed from $0+$ to 0. For these reasons, it seems unsatisfactory to regard the $\epsilon = 0$ situation as a limit (as $\epsilon \to 0$) of situations applicable when $\epsilon > 0$.

5. Reverse time models, reversibility and dynamic reversibility

Following is a slightly simplified version of a theorem proved in [19].

Theorem 1: Consider the vector process defined by

$$dx_t = f(x_t, t)dt + G dw_t \tag{5.1}$$

where with (Ω, A, P) a fixed probability space, $\{A_t, -\infty < t < \infty\}$ is an increasing family of sub-σ-algebras on A, and $\{w_t, -\infty < t < \infty\}$ is an r-vector Brownian motion process such that w_t is A_t-measurable for each t, and with $s<0$, $w_{t+s} - w_s$ is independent of A_s, with $E[w_{t+s} | A_t] = w_t$ and $E[(w_{t+s} - w_t)(w_{t+s} - w_t)' | A_t] = sI$; x_t is an n-vector process and $f(\cdot, \cdot)$ has appropriate smoothness and growth properties so that the form of $f(\cdot,\cdot)$ and G imply the existence and uniqueness of a solution to (5.1), together with the existence and uniqueness of the probability density $p(x_t, t)$ for $t_0 \leq t \leq T$ as a smooth solution of the associated Kolmogorov equation. Define an r-vector process $\bar{w}_t$ by $\bar{w}_{t_0} = 0$ and

$$d\bar{w}_t = dw_t + \frac{1}{p(x_t, t)} G' \nabla p(x_t, t) \tag{5.2}$$

[Here, $\nabla p(x_t, t)$ has i-th entry equal to $\frac{\partial}{\partial x_t^i} p(x_t, t)$].

Then

i) x_t and $\bar{w}_t - \bar{w}_s$ are independent for all $t \geq s \geq t_0$.

(ii) with $\bar{A}_t$ the minimal σ-algebra with respect to which x_s for $s \geq t$ and $\bar{w}_s$ for $s \geq t$ are measurable, there holds for $s > 0$

$$E[\bar{w}_t | \bar{A}_{t+s}] = \bar{w}_{t+s} \text{ and } E[(\bar{w}_t - \bar{w}_{t+s})(\bar{w}_t - \bar{w}_{t+s})' | \bar{A}_{t+s}] = sI$$

(iii) a reverse time model for x_t is defined by

$$dx_t = \bar{f}(x_t, t)dt + G d\bar{w}_t \tag{5.3}$$

where

$$\bar{f}(x_t, t) = f(x_t, t) - \frac{1}{p(x_t, t)} GG' \nabla p(x_t, t) \tag{5.4}$$

Roughly, the theorem states that if (5.1) is a model of a random x_t, driven by white noise, evolving forwards in time, so that future noise is independent of past state, then we can define an alternative model of that same random process, viz. (5.3), again driven by white noise, but evolving backwards in time so that now past noise is independent of future state. The recipe for defining the reverse time model is given by (5.2) and (5.4); construction of the reverse time model depends on knowing the probability density $p(x_t, t)$.

The construction of reverse time models for processes with linear (5.1) is discussed in for example, [20].

If x_t has a reversibility or dynamic reversibility property, it is possible to prove this using the idea of a reverse time model. For if $p(x_t, t)$ is independent of time, and $f(x_t, t) = -\bar{f}(x_t, t)$, then it is clear that forward evolution of x_t [via. (5.1)] is mirrored by its backward evolution [via (5.3)], i.e. x_t is reversible. This sort of idea was used in [3] to discuss the charge process in a resistor-capacitor network where excitations stemmed from the thermal noise. Dynamic reversibility is slightly more complicated. Writing $x_t = [y'_t \; z'_t]'$, rewrite (5.3) as

$$\begin{bmatrix} dy_t \\ dz_t \end{bmatrix} = \begin{bmatrix} \bar{f}^1(y_t, z_t) \\ \bar{f}^2(y_t, z_t) \end{bmatrix} dt + \begin{bmatrix} G^1 \\ G^2 \end{bmatrix} d\bar{w}_t \tag{5.5}$$

(Time dependence of $\bar{f}$ is omitted since it is assumed now that processes are stationary.) This implies, with $\tilde{z}_t = -z_t$,

$$\begin{bmatrix} dy_t \\ d\tilde{z}_t \end{bmatrix} = \begin{bmatrix} \bar{f}^1(y_t, -\tilde{z}_t) \\ -\bar{f}^2(y_t, -\tilde{z}_t) \end{bmatrix} dt + \begin{bmatrix} G^1 \\ -G^2 \end{bmatrix} d\bar{w}_t \tag{5.6}$$

Now the matrices $\begin{bmatrix} G^1 \\ G^2 \end{bmatrix}$ and $\begin{bmatrix} G^1 \\ -G^2 \end{bmatrix}$ are only significant to within multiplication on the right by an arbitrary orthogonal matrix, in the sense that if either is so multiplied, the joint probability associated with the x_t process is unchanged. Equivalently, we can say that these probabilities are determined in the case of (5.1) by $f(x_t)$ and GG'. [This fact can also be seen from the Fokker Planck equation for $p(x_t, t \mid x_{t_0}, t_0]$. It follows that dynamic reversibility will hold if and only if

$$\bar{f}^1(y_t, -z_t) = -f^1(y_t, z_t) \tag{5.7}$$

$$\bar{f}^2(y_t, -z_t) = f^2(y_t, z_t) \tag{5.8}$$

and

$$\begin{bmatrix} G^1 \\ \\ -G^2 \end{bmatrix} [G^{1\prime} \ -G^{2\prime}] = \begin{bmatrix} G^1 \\ \\ G^2 \end{bmatrix} [G^{1\prime} \ G^{2\prime}]$$

i.e.

$$G^1 G^{2\prime} = 0 \tag{5.9}$$

This sort of idea was used in [3] to discuss the dynamic reversibility of the charge-flux process in a network defined in Section 2, given that certain of the nonlinear characteristics were odd.

Now suppose that we have associated with (5.1) a further equation

$$d\alpha_t = h(x_t, t)\,dt + J\,dw_t \tag{5.10}$$

Under the definition (5.2), it follows that

$$d\alpha_t = \bar{h}(x_t, t) + J\,d\bar{w}_t \tag{5.11}$$

where

$$\bar{h}(x_t, t) = h(x_t, t) - \frac{1}{p(x_t, t)} JG' \nabla p(x_t, t) \tag{5.12}$$

Of course, it is natural to associate (5.11) with (5.3), just as (5.10) is associated with (5.1). We can also consider the question of reversibility or dynamic reversibility of $[\alpha_t' \ x_t']'$.

In one case, this is easily dealt with. Suppose $J = 0$. Then $\frac{d\alpha_t}{dt} = h(x_t) = \bar{h}(x_t)$ and $\frac{d\alpha_t}{dt}$ will be reversible if x_t is reversible. Also if $x_t = [y_t' \ z_t']'$ is dynamically reversible, and $h(x_t) = h(y_t)$, then again $\frac{d\alpha_t}{dt}$ will be reversible. This was the situation effectively encountered in the last section.

Suppose therefore $J \neq 0$. At once, there is a problem defining reversibility or dynamic reversibility for $[\alpha_t' \ x_t']'$ because α_t is no longer stationary, or for $[\frac{d\alpha_t'}{dt} \ x_t']'$, because $\frac{d\alpha_t}{dt}$ contains white noise, and therefore cannot have probability densities associated with it. However, we can skirt these problems by using not α_t itself, but *increments* of α_t in our definition. These are of course stationary. By abuse of language, we say that $[\alpha_t' \; x_t']' = [\beta_t' \; \gamma_t' \; y_t' \; z_t']'$ is dynamically reversible if

$$p\left(\begin{bmatrix} \beta_{t_1} - \beta_{t_0} = a_1 \\ \alpha_{t_1} - \alpha_{t_0} = b_1 \\ y_{t_1} \quad\quad = c_1 \\ z_{t_1} \quad\quad = d_1 \end{bmatrix}, \begin{bmatrix} \beta_{t_2} - \beta_{t_0} = b_2 \\ \alpha_{t_2} - \alpha_{t_0} = b_2 \\ y_{t_2} \quad\quad = c_2 \\ z_{t_2} \quad\quad = d_2 \end{bmatrix}, \ldots, \begin{bmatrix} \beta_{t_n} - \beta_{t_0} = a_n \\ \alpha_{t_n} - \alpha_{t_0} = b_n \\ y_{t_n} \quad\quad = c_n \\ z_{t_n} \quad\quad = d_n \end{bmatrix} \right)$$

$$= p\left(\begin{bmatrix} \beta_{T-t_1} - \beta_{T-t_0} = a_1 \\ \alpha_{T-t_1} - \alpha_{T-t_0} = -b_1 \\ y_{T-t_1} \quad\quad = c_1 \\ z_{T-t_1} \quad\quad = -d_1 \end{bmatrix}, \begin{bmatrix} \beta_{T-t_2} - \beta_{T-t_0} = a_2 \\ \alpha_{T-t_2} - \alpha_{T-t_0} = -b_2 \\ y_{T-t_2} \quad\quad = c_2 \\ z_{T-t_2} \quad\quad = -d_2 \end{bmatrix}, \ldots, \right.$$

$$\left. \begin{bmatrix} \beta_{T-t_n} - \beta_{T-t_0} = a_n \\ \alpha_{T-t_n} - \alpha_{T-t_0} = -b_n \\ y_{T-t_n} \quad\quad = c_n \\ z_{T-t_n} \quad\quad = -d_n \end{bmatrix} \right)$$

for all n, $t_0, t_1, \ldots, t_n$, T and a_i, b_i, c_i, d_i, for $i = 1, 2, \ldots, n$.

Let us rewrite (5.11) as

$$\begin{bmatrix} d\beta_t \\ d\gamma_t \end{bmatrix} = \begin{bmatrix} \bar{h}^1(y_t, z_t) \\ \bar{h}^2(y_t, z_t) \end{bmatrix} dt + \begin{bmatrix} J^1 \\ J^2 \end{bmatrix} d\bar{w}_t$$

or, with $\tilde{\gamma}_t = -\gamma_t$ and $\tilde{z}_t = -z_t$ as before,

$$\begin{bmatrix} d\beta_t \\ d\tilde{\gamma}_t \end{bmatrix} = \begin{bmatrix} \bar{h}^1(y_t, -\tilde{z}_t) \\ -\bar{h}^2(y_t, -\tilde{z}_t) \end{bmatrix} dt + \begin{bmatrix} J^1 \\ -J^2 \end{bmatrix} d\bar{w}_t \tag{5.13}$$

For dynamic reversibility, we require that (5.6) and (5.13) should do backwards what (5.1) and (5.10) do forwards, i.e., in addition to (5.7) and (5.8), we need

$$\bar{h}^1(y_t, -z_t) = -h^1(y_t, z_t) \tag{5.14}$$

$$\bar{h}^2(y_t, -z_t) = \bar{h}^2(y_t, -z_t) \tag{5.15}$$

and

$$\begin{bmatrix} G^1 \\ -G^2 \\ J^1 \\ -J^2 \end{bmatrix} [G^{1\prime} \; -G^{2\prime} \; J^{1\prime} \; -J^{2\prime}] = \begin{bmatrix} G^1 \\ G^2 \\ J^1 \\ J^2 \end{bmatrix} [G^{1\prime} \; G^{2\prime} \; J^{1\prime} \; J^{2\prime}]$$

whence (5.9) follows together with

$$\begin{aligned} G^1 J^{2\prime} &= 0 \\ G^2 J^{1\prime} &= 0 \\ J^1 J^{2\prime} &= 0 \end{aligned} \tag{5.16}$$

We summarize the result as follows:

Theorem 2: Consider a random process x_t defined as described in Theorem 1 by (5.1), and suppose x_t is stationary. Suppose that from x_t a further process α_t is formed as in (5.10). Assuming that $p(x_t)$ exists and is differentiable, define $\bar{f}$ and $\bar{h}$ by (5.4) and (5.12). Then $(\alpha_t' \, x_t')' = (\beta_t' \, \gamma_t' \, y_t' \, z_t')'$ is dynamically reversible in the sense described previously if and only if

$$\bar{f}(y_t, z_t) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} f\left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix}\right) \tag{5.17a}$$

$$\bar{h}(y_t, z_t) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} h\left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix}\right) \tag{5.17b}$$

and

$$G^1 G^{2\prime} = 0, \; G^1 J^{2\prime} = 0, \; J^1 G^{2\prime} = 0, \; J^1 J^{2\prime} = 0 \tag{5.17c}$$

6. Reversibility of the port fluctuations

Theorem 2 of section 5 provides us with the tool for examining the reversibility of the port fluctuations, which we found in Section 2 to satisfy equation (2.7), repeated for convenience as

$$\begin{bmatrix} d\phi \\ dq \end{bmatrix} = \begin{bmatrix} -H_{LL} & -H_{LC} \\ H'_{LC} & -H_{CC} \end{bmatrix} \begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix} dt + \sqrt{2kT} \begin{bmatrix} -K'_{12} & -K_{22} & 0 \\ 0 & 0 & -K_{33} \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \end{bmatrix} \tag{6.1}$$

$$v_P = [H_{PL} \quad H_{PC}] \begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix} + \sqrt{2kT} \, [K_{11} \quad K_{12} \quad 0] \begin{bmatrix} dw_1/dt \\ dw_2/dt \\ dw_3/dt \end{bmatrix} \tag{6.2}$$

We shall also assume that the capacitor characteristics are odd: $f_C(q) = -f_C(-q)$.

To make contact with Theorem 2, we need to identify ϕ with y_t, q with z_t and $\int_{t_0}^{t} v_p dt$ with β_t. There is no γ_t process. The identifications of G^i and J^i are obvious, and (5.17c) is immediately verifiable. The identifications of f and h are also obvious. Furthermore, as was shown in [3], we have the steady state density

$$p(\phi,q) = N \exp\left[-\frac{E_L(\phi)}{kT} - \frac{E_C(q)}{kT}\right] \tag{6.3}$$

where N is a normalizing constant. Recalling the relations between E_C, f_C and E_L, f_L in (2.2), an easy calculation gives

$$\frac{1}{p(\phi,q)}\begin{bmatrix}\nabla_q[p(\phi,q)]\\ \\ \nabla_\phi[p(\phi,q)]\end{bmatrix} = -\frac{1}{kT}\begin{bmatrix}f_L(\phi)\\ \\ f_C(q)\end{bmatrix} \tag{6.4}$$

Accordingly, (5.4) yields $\bar{f}(\phi, q)$

$$= \begin{bmatrix}-H_{LL} & -H_{LC}\\ \\ H'_{LC} & -H_{CC}\end{bmatrix}\begin{bmatrix}f_L(q)\\ \\ f_C(q)\end{bmatrix} + 2\begin{bmatrix}-K'_{12} & -K_{22} & 0\\ \\ 0 & 0 & -K_{33}\end{bmatrix}\begin{bmatrix}-K_{12} & 0\\ -K_{22} & 0\\ 0 & -K_{33}\end{bmatrix}\begin{bmatrix}f_L(\phi)\\ f_C(q)\end{bmatrix}$$

$$= \begin{bmatrix}-H_{LL} & -H_{LC}\\ H'_{LC} & -H_{CC}\end{bmatrix}\begin{bmatrix}f_L(\phi)\\ f_C(q)\end{bmatrix} + 2\begin{bmatrix}H_{LL} & 0\\ 0 & H_{CC}\end{bmatrix}\begin{bmatrix}f_L(\phi)\\ f_C(q)\end{bmatrix} \quad \text{using (2.6)}$$

$$= \begin{bmatrix}H_{LL} & -H_{LC}\\ H'_{LC} & H_{CC}\end{bmatrix}\begin{bmatrix}f_L(\phi)\\ f_C(q)\end{bmatrix} \tag{6.5}$$

$$= \begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}-H_{LL} & H_{LC}\\ H'_{LC} & H_{CC}\end{bmatrix}\begin{bmatrix}f_L(\phi)\\ f_C(q)\end{bmatrix}$$

$$= \begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}-H_{LL} & -H_{LC}\\ H_{LC} & -H_{CC}\end{bmatrix}\begin{bmatrix}f_L(\phi)\\ f_C(-q)\end{bmatrix} \quad \text{on using the oddness of the capacitor characteristics}$$

$$= \begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix} f\left(\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}\begin{bmatrix}\phi\\ q\end{bmatrix}\right)$$

This verifies (5.17a).

Similarly, (5.12) yields

$$\bar{h}(\phi, q) = [H_{PL}\ \ H_{PC}]\begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix} + 2[K_{11}\ \ K_{12}\ \ 0]\begin{bmatrix} -K_{12} & 0 \\ -K_{22} & 0 \\ 0 & -K_{33} \end{bmatrix}\begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix}$$

$$= [H_{PL}\ \ H_{PC}]\begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix} + 2[-H_{PL}\ 0]\begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix} \quad \text{using (2.6)}$$

$$= [-H_{PL}\ H_{PC}]\begin{bmatrix} f_L(\phi) \\ f_C(q) \end{bmatrix}$$

$$= -h\left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} \phi \\ q \end{bmatrix}\right) \quad \text{(by capacitor characteristic oddness)} \tag{6.6}$$

Hence we have established that $[\int_{t_0}^t v_P' d\tau \ \ \phi'(t)\ q'(t)]'$ *is dynamically reversible, where* $\int_{t_0}^t v'_P d\tau$ and $\phi'(t)$ *have the same "polarity". In case there is no direct feedthrough i.e.* $J=0$, *then we can work with* v_P *instead of* $\int_{t_0}^t v'_P d\tau$, *and then* $[v'_P\ \phi'\ \ q']'$ *is dynamically reversible, where* v_P *has the same polarity as* q.

An obvious variation on the above follows if we assume that inductor characteristics, rather than capacitor characteristics, are odd. Also, an obvious extension follows if we consider an n port network with n_1 ports open-circuit (defining v_{P1}) and n_2 ports short-circuit (defining i_{P2}). Then it will follow that $[\int_{t_0}^t v'_{P1} dt\ \ \int_{t_0}^t i'_{P2} dt\ \ \phi'(t)\, q'(t)]'$ will be dynamically reversible. We sum this up as follows.

Theorem 3: Consider a time-invariant network constructed from a finite number of linear passive resistors and transformers, and nonlinear inductors and capacitors with characteristics restricted as in Section 2, and with capacitor characteristics odd. Suppose that all resistors are at the one temperature and the usual thermal noise property for the resistors applies. Suppose further that n_1 of the ports are open-circuit, allowing observation of a noise voltage vector v_{P1} and $n_2 = n - n_1$ are short-circuit, allowing observation of a noise current vector i_{P2}. Finally, suppose that the topology of the network is such that a certain hybrid matrix exists, permitting a network description of the form

$$\begin{bmatrix} d\phi \\ dq \end{bmatrix} = f(\phi, q)\, dt + dw$$
$$\begin{bmatrix} v_{P1}\, dt \\ i_{P2}\, dt \end{bmatrix} = h(\phi, q)\, dt + J\, dw \tag{6.7}$$

Then $[\int_{t_0}^{t} v'_{P1}\, d\tau \quad \int_{t_0}^{t} i'_{P2}\, d\tau \quad \phi'(t) \quad q'(t)]'$ for any fixed t_0 is a dynamically reversible random process.

The above characterization includes ϕ and q. To obtain a port-variable-only characterization, we obtain by immediate application of the dynamic reversibility definition:

Corollary 4: Under the same hypotheses as Theorem 3, $[\int_{t_0}^{t} v'_{P1}\, d\tau \quad \int_{t_0}^{t} i'_{P2}\, d\tau]'$ is a dynamically reversible random process.

It is interesting to consider the following specialized implication of this corollary. Suppose $n = n_1 = 2$. Reversibility then implies

$$p[v_p(t_1) = a_1,\, v_P(t_2) = a_2] = p\,[v_P(t_2) = a_1,\, v_P(t_1) = a_2]$$

(but this alone does not imply reversibility). Now integrate over the second entry of a_1 and the first entry of a_2. There results

$$p[v_P^1(t_1) = a_1^1,\, v_P^2(t_2) = a_2^2] = p[v_P^2(t_1) = a_2^2,\, v_P^1(t_2) = a_2^1]$$

Now suppose $t_1 < t_2$. Then in a rough sense the quantity on the left measures the probability that $v_P^1(t_1) = a_1^1$ will "cause" $v_P^2(t_1) = a_2^2$, while that on the right measures the probability that $v_P^2(t_1) = a_2^2$ will "cause" $v_P^1(t_2) = a_2^1$. In this way we capture in a vague sort of way the idea familiar from the linear version of reciprocity that excitation at port 1 produce responses at port 2 in the same manner as excitations at port 2 produce responses at port 1.

7. Rapprochement with linear networks

If a two port network comprising linear, time-invariant passive resistors, transformers, inductors and capacitors has an impedance matrix $Z(s)$, the reciprocity property as we know is described by $Z_{21}(s) = Z_{12}(s)$.

How is this idea most clearly seen to cohere with the preceding material? The answer lies in the fluctuation-dissipation theorem [5], see also [18], which states that the thermal noise perceived at the ports of the network is gaussian, and to within a scaling constant, has spectrum

$$\phi(j\omega) = Z(j\omega) + Z'(-j\omega) \tag{7.1}$$

(Let us assume the network is such that $Z(\cdot)$ has no $j\omega$-axis poles). This result is valid for reciprocal and nonreciprocal passive networks, i.e., it remains valid if the network is permitted to contain gyrators. If now the network is reciprocal, so that $Z(s) = Z'(s)$, this forces $\phi_{12}(j\omega) = \phi_{21}(j\omega)$, so that the noise at the network ports is gaussian, with symmetric (not just hermitian symmetric) spectrum. Now for *a stationary vector gaussian process,* *reversibility of the process is equivalent to symmetry of the spectrum.* (For a discussion and various network interpretations of this, see e.g. [6]). Consequently, we have the chain: reciprocity $\Leftrightarrow$ $Z_{21}(s) = Z_{12}(s)$ $\Leftrightarrow$ by fluctuation dissipation theorem, port thermal noise has a symmetric spectrum $\Leftrightarrow$ reversibility, using gaussian character of noise. This chain shows how the nonlinear result given in the last section is a generalization of the linear result.

Of course, the n-port generalization of these ideas is straight-forward, as is the generalization to cover short-circuited as well as open-circuited ports. Also, the fact that in the linear situation, the port noise, capacitor voltages and inductor fluxes can all be grouped to form a dynamically reversible process is discussed in [6].

It is curious to observe that for one port linear networks, the reversibility of the port noise voltage is automatically guaranteed by the fact that any scalar stationary gaussian process is reversible, yet reversibility in the nonlinear case is a nontrivial property.

8. Conclusions

The main result of this paper has been that the noise voltages (or more strictly, their integrals) at the ports of a network of resistors, transformers, inductors and capacitors, with the reactive elements permitted to be nonlinear and all elements time-invariant and passive, constitute a vector reversible process (given certain reasonable conditions on the nonlinearities and the network topology). In the linear case, this reversibility is precisely equivalent to symmetry of the impedance matrix, provided this exists.

In the general context of developing nonlinear analogues of known linear results, involving random processes in networks, several problems still remain. For example, how can one handle nonlinear resistors? The first difficulty is to obtain a noise model of a nonlinear resistor, and preliminary work suggests that although the nonlinear resistor noise can be modelled by a series white noise source, the intensity depends both on the resistance characteristic and on the network to which the resistance is connected, rather than on the resistance characteristic alone – a notion which in some ways is an unsatisfactory generalization of the linear resistor situation.

Note

1. Thus $f_c(\cdot)$ is continuously differentiable, the Jacobian determinant is nonzero for all q, and $\|f_c\| \to \infty$ if $\|q\| \to \infty$. These properties ensure that the map $q \to f_c(q)$ has an inverse which is also a C^1-diffeomorphism.

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