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INTERNAL AND EXTERNAL STABILITY OF LINEAR TIME-VARYING SYSTEMS*

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Abstract. Linear, finite-dimensional, time-varying systems are studied. State variable representations of systems with a bounded-input, bounded-output stability property which are uniformly stabilizable and detectable are shown to have their associated homogeneous state-variable systems exponentially stable.

1. Introduction. This paper connects input–output and Lyapunov stability results for finite-dimensional linear systems.

To put the result in perspective, recall that for a finite-dimensional time-invariant linear system in state-variable form: first, if the homogeneous state variable equation is asymptotically stable, the system is bounded-input, bounded-output stable; second, if the system is bounded-input, bounded-output stable and its state variable realization is controllable and observable, then the homogeneous state-variable equation is asymptotically stable. Of course, in the time-invariant case, asymptotic stability is equivalent to exponential stability.

Our aim here is to generalize these results, first by considering time-varying systems, and, second, by relaxing controllability and observability to a form of stabilizability and detectability.

As background, we note first [1], [2] which consider connections between input-output and Lyapunov stability for time-varying systems when the input and output vectors have the same dimension as, and are linked in a uniformly nonsingular way to, the state vector. Reference [3] discusses results for time-varying systems with state-variable realizations in phase-variable form, while [4], [5], [6] provide a major generalization by showing for time-varying systems that input–output stability together with uniform controllability and observability imply exponential stability of the associated homogeneous state variable equation. (In time-varying systems, it is possible to have asymptotic, but nonexponential, stability.)

The advance on [4]–[6] provided in this paper is the weakening of the uniform controllability and observability conditions to uniform stabilizability and detectability. For one of the first uses of these concepts for time-varying systems, see [7] and for a general treatment, see [8]. The result of this paper is of course trivial in the time-invariant case. But in the time-varying case, one cannot, via a Lyapunov basis change, break the system up into a uniformly controllable and observable part which is input–output equivalent to the original system, and an exponentially stable part—hence the nontrivial nature of the problem.

Background information is reviewed in § 2. In § 3, the connection between bounded-input, bounded-output behavior and bounded-input, bounded-state behavior is examined. The main result is proved in § 4. We elect to present the results in discrete time, thus simplifying application of the results of [8], which has results expressed in discrete time. Doubtless with minor adjustment, the results are valid for continuous time. Interestingly, the proof is not just a minor adjustment of that applying in the controllable/observable situation.

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2. Detectability, stabilizability and BIBO behavior. Consider the linear finite-dimensional system

$$x_{k+1}=F_kx_k+G_ku_k, \tag{2.1a}$$

$$y_k=H'_kx_k, \tag{2.1b}$$

where $x_k \in R^n$ is the state, $u_k \in R^m$ is the input, $y_k \in R^p$ is the output, and F_k, G_k, H_k are matrices of appropriate dimension. The state transition matrix is denoted $\phi_{k,l}$ for $k \geqq l$, where $\phi_{k,k}=I$, $\phi_{k+1,k}=F_k$ and $\phi_{k,l}=\phi_{k,k-1}\phi_{k-1,l}$.

Standing Assumption. The sequences F_k, G_k and H_k are bounded.

We now define the concepts of uniform detectability and stabilizability [7], [8].

DEFINITION 2.1. The pair $[F_k, H_k]$ (regarded as a sequence indexed by k) is *uniformly detectable* if there exist integers $s \geqq t \geqq 0$ and constants d, b with $0 \leqq d < 1$, $0 < b < \infty$ such that, whenever

$$\|\phi_{k+t,k}\xi\| \geqq d\|\xi\| \tag{2.2}$$

for some ξ and k, then

$$\xi' M_{k+s,k}\xi \geqq b\xi'\xi, \tag{2.3}$$

where

$$M_{k+s,k}=\sum_{i=k}^{k+s}\phi'_{i,k}H_iH'_i\phi_{i,k}. \tag{2.4}$$

(The idea is that when a zero-input trajectory starting at $x_k=\xi$ fails to converge much towards the origin (see (2.2)), then x_k should be observable to a minimum level (see (2.3)).)

DEFINITION 2.2. The pair $[F_k, G_k]$ (regarded as a sequence indexed by k) is *uniformly stabilizable* if there exist integers $s \geqq t \geqq 0$ and constants d, b with $0 \leqq d < 1$, $0 < b < \infty$, such that, whenever

$$\|\phi_{k+1,k+1-t}\xi\| \geqq d\|\xi\| \tag{2.5}$$

for some ξ, k, then

$$\xi' Y_{k,k-s}\xi \geqq b\xi'\xi, \tag{2.6}$$

where

$$Y_{k,k-s}=\sum_{i=k-s}^{k}\phi_{k+1,i+1}G_iG'_i\phi'_{k+1,i+1}. \tag{2.7}$$

We shall need several results established in [7]. The most important are stated in the following lemma.

LEMMA 2.1. *With the definitions*

$$\hat{F}_k=F'_{-k}, \qquad \hat{H}_k=G_{-k}, \tag{2.8}$$

$[\hat{F}_k, \hat{H}_k]$ is uniformly detectable if and only if $[F_k, G_k]$ is uniformly stabilizable, and $x_{k+1}=F_kx_k$ is exponentially stable if and only if $\hat{x}_{k+1}=\hat{F}_k\hat{x}_k$ is exponentially stable.

In the statement of the lemma, the definition of exponential stability of $x_{k+1}=F_kx_k$ is standard: the transition matrix must satisfy $\|\phi_{k,l}\| \leqq \alpha\beta^{k-l}$ for some $\alpha \in [1,\infty)$, and $\beta \in [0,1)$ and all $k \geqq l$. The key idea in proving the lemma is to show that $\phi_{k,l}=\hat{\phi}'_{-l+1,-k+1}$; application of Definitions 2.1 and 2.2 and the definition of exponential stability then yields the result.

We also need to define bounded-input, bounded-output stability. This is done in a standard way [4], [5], [9].

DEFINITION 2.3. The linear finite-dimensional system (2.1) has the bounded-input, bounded-output (BIBO) (l^p) property for $1 \leqq p \leqq \infty$ if, with W_{kl} the impulse response matrix of the system, the input bound

$$\sum_{k=-\infty}^{+\infty} [\|u_k\|^p]^{1/p} \leqq \alpha \tag{2.9}$$

implies the output bound

$$\sum_{k=-\infty}^{+\infty} [\|y_k\|^p]^{1/p} \leqq \beta\alpha \tag{2.10}$$

for some constant β, independent of $\{u_k\}$ and α, with $\{y_k\}$ related to $\{u_k\}$ by

$$y_k = \sum_{l<k} W_{kl}u_l. \tag{2.11}$$

We shall use the following readily established fact. The result in question is virtually established in [9, see pp. 113–114]; the key idea is to note first that $\|W_{kl}\|$ is exponentially bounded in the same manner as $\phi_{k,l}$.

LEMMA 2.2. *If the homogeneous equation $x_{k+1} = F_k x_k$ associated with* (2.1) *is exponentially stable, then* (2.1) *is* BIBO (l^p) *for all $p \leqq [1, \infty]$.*

Our task in the next two sections is to prove a converse, given uniform detectability and stabilizability.

3. Connection between BIBO and BIBS behavior. Prior to obtaining the main result in the next section, we establish here that, given a uniform detectability condition, BIBO behavior implies bounded-input bounded-state (BIBS) behavior. The result is intuitively reasonable, and is known when detectability is replaced by observability [4], [5]. We use two preliminary lemmas.

LEMMA 3.1. *Let T_k be an orthogonal matrix and suppose that state-variable equations for $\bar{x}_k = T_k x_k$ are constructed from* (2.1). *Then, in obvious notation,*

$$\bar{M}_{k+s,k} = T_k M_{k+s,k} T'_k. \tag{3.1}$$

The proof is obvious. It is clear from this lemma that there is no loss of generality in establishing that BIBO behavior implies BIBS behavior for a system (2.1) for which, for each k, $M_{k+s,k}$ is diagonal, with diagonal entries ordered in decreasing magnitude. Suppose this is done. Let us also write, for a uniformly detectable system,

$$M_{k+s,k} = M^1_{k+s,k} \dotplus M^2_{k+s,k} \tag{3.2}$$

with $\dotplus$ denoting direct sum, where the diagonal entries of $M^1_{k+s,k}$ are all greater than or equal to b, and those of $M^2_{k+s,k}$ are all less than b. Note that the dimension of $M^1_{k+s,k}$ may not be constant with k. Then we can state:

LEMMA 3.2. *Suppose* (2.1) *is uniformly detectable, and let W_{kl} denote the impulse response of* (2.1). *Let k_0 be an arbitrary but fixed integer, suppose notation is as defined in* § 2, *and let $M^1_{k+s,k}$ be as defined above. Let the input sequence $\{u_k\}$ and x_{k_0-t+1} be* *arbitrary. Define a sequence* $x_k^* \in R^n$ *for* $k \geqq k_0 - t + 1$ *by*

$$Hx_{k_0}^* = x_{k_0-1}^* = \cdots = x_{k_0-t+1}^* \tag{3.3}$$

$$x_{k+t}^* = \phi_{k+t,k}\left\{\begin{bmatrix}(M_{k+s,k}^1)^{-1} & 0\\ 0 & 0\end{bmatrix}\sum_{i=k}^{k+s}\phi'_{i,k}H_i\left[y_i - \sum_{l=k}^{i-1}W_{il}u_l\right] + \begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}x_k^*\right\} + \sum_{l=k}^{k+t-1}\phi_{k+t-1,l}G_lu_l. \tag{3.4}$$

Then for $k \geqq k_0 - t + 1$

$$x_{k+t}^* - x_{k+t} = \phi_{k+t,k}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}(x_k^* - x_k). \tag{3.5}$$

The partitioning of the matrix $\left[\begin{smallmatrix}0 & 0\\ 0 & I\end{smallmatrix}\right]$ *in* (3.4) *and* (3.5) *is like that of* $M_{k+s,k}$ *in* (3.2).

Proof. Observe that

$$y_i - \sum_{l=k}^{i-1} W_{il}u_l = H_i'\phi_{ik}x_k.$$

Recalling the definition of $M_{k+s,k}$ in (2.4), we see that (3.4) implies

$$\begin{aligned}
x_{k+t}^* &= \phi_{k+t,k}\left\{\begin{bmatrix}(M_{k+s,k}^1)^{-1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}M_{k+s,k}^1 & 0\\ 0 & M_{k+s,k}^2\end{bmatrix}x_k + \begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}x_k^*\right\} + \sum_{l=k}^{k+t-1}\phi_{k+t-1,l}G_lu_l\\
&= \phi_{k+t,k}\left\{\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}x_k + \begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}x_k^*\right\} + \sum_{l=k}^{k+t-1}\phi_{k+t-1,l}G_lu_l\\
&= \phi_{k+t,k}x_k + \sum_{l=k}^{k+t-1}\phi_{k+t-1,l}G_lu_l + \phi_{k+t,k}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}(x_k^* - x_k)\\
&= x_{k+t} + \phi_{k+t,k}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}(x_k^* - x_k).
\end{aligned}$$

Equation (3.5) is immediate.

Equations (3.3) and (3.4) in effect are defining a type of state estimator, with x_k^* supposed to estimate x_k. In case observability is present, one has $x_{k+t}^* = x_{k+t}$. If it is lacking, (3.5) holds, and as argued below, see (3.7), this ensures that the observation error asymptotically is zero.

Now we can establish the main result of this section, which relates BIBO and BIBS stability.

PROPOSITION 3.1. *Consider the system* (2.1) *and suppose it is uniformly detectable. Then for any one* $p \in [1, \infty]$, *the system is* BIBS (l^p) *if and only if it is* BIBO (l^p).

Proof. Because H_k is bounded, BIBS obviously implies BIBO. So we must prove BIBO implies BIBS.

Assume that $M_{k+s,k}$ is diagonal with $M_{k+s,k}^1$ as described above Lemma 3.2. Equation (3.4) can be reorganized as

$$x_{k+t}^* = \phi_{k+t,k}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}x_k^* + w_k, \tag{3.6}$$

where w_k is a finite length moving average of $\{u_k\}$ and $\{y_k\}$ terms, with bounded weights. The BIBO hypothesis implies that if $[\sum_{k \geqq k_0}\|u_k\|^p]^{1/p} \leqq \alpha$, then $[\sum_{k \geqq k_0}\|w_k\|^p]^{1/p} \leqq \eta\alpha$ for some $\eta > 0$, independent of α, $\{u_k\}$ and k_0. Also, recalling the special form of $M_{k+s,k}$ in (3.2), we see from the detectability definitions that any vector of the form $[0, \xi_2']'$ has the property

$$\left\|\phi_{k+t,k}\begin{bmatrix}0\\ \xi_2\end{bmatrix}\right\| < d\|\xi_2\|$$

or for all ξ_1, ξ_2 of appropriate dimension

$$\left\|\phi_{k+t,k}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}\right\| < d\left\|\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}\right\|,$$

so that

$$\left\|\phi_{k+t,k}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\right\| < d < 1. \tag{3.7}$$

This means that (3.6) is BIBO (l^p) by Lemma 2.2. Using the $\{w_k\}$ bound, we see that x_k^* is such that $[\sum_{k \geqq k_0} \|x_k^*\|^p]^{1/p} < \delta\alpha$ for some δ, independent of α, $\{u_k\}$ and k_0. The boundary conditions on x_k^* in Lemma 3.2 and on x_k in the BIBO Definition 2.3 imply $x_k^* - x_k = 0$ for $k = k_0 - t + 1, \cdots, k_0$ and so by (3.5), $x_k^* = x_k$ for all k. The result is then immediate.

4. Connection between BIBO behavior and exponential stability. In the previous section, we have argued that BIBO behavior and uniform detectability imply BIBS behavior. Here, we shall argue that BIBS behavior and uniform stabilizability imply exponential stability of the homogeneous state equation of the system. We shall use two preliminary lemmas. The main idea of the proof of the main result is to combine the use of duality established in the lemmas with the BIBO/BIBS connection.

LEMMA 4.1. *Let W_{kl} be the impulse response of the system* (2.1) *and $\hat{W}_{kl}$ the impulse response of*

$$\hat{x}_{k+1} = F'_{-k}\hat{x}_k + H_{-k}\hat{u}_k, \tag{4.1a}$$

$$\hat{y}_k = G'_{-k}\hat{x}_k. \tag{4.1b}$$

Then $\hat{W}_{kl} = W'_{-l,-k}$.

The proof is trivial by direct calculation, and is omitted.

LEMMA 4.2. *The system* (2.1) *is* BIBO (l^p) *for p satisfying $1 \leqq p \leqq \infty$ if and only if the system* (4.1) *is* BIBO (l^q) *for q satisfying $p^{-1} + q^{-1} = 1$.*

Proof. Using the Hölder inequality, the duality of l^p and l^q, and simple manipulation, we obtain the following set of equivalences.

System (2.1) is BIBO (l^p)

$$\Leftrightarrow \forall\{u_k\} \in l^p, \{\hat{u}_k\} \in l^q, \left\|\sum_{k=-\infty}^{+\infty} \hat{u}_k'\left(\sum_{l=-\infty}^{k-1} W_{kl}u_l\right)\right\| \leqq c[\sum \|u_k\|^p]^{1/p}[\sum \|\hat{u}_k\|^q]^{1/q}$$

for some constant c, independent of $\{u_k\}$, $\{\hat{u}_k\}$

$$\Leftrightarrow \left\|\sum_{l=-\infty}^{+\infty}\left(\sum_{k=l+1}^{\infty} \hat{u}_k' W_{kl}\right)u_l\right\| \leqq c[\sum \|u_k\|^p]^{1/p}[\sum \|\hat{u}_k\|^q]^{1/q}$$

$$\Leftrightarrow \left\|\sum_{m=-\infty}^{+\infty} u'_{-m}\left(\sum_{n=-\infty}^{m-1} W'_{-n,-m}\hat{u}_{-n}\right)\right\| \leqq c[\sum \|u_{-m}\|^p]^{1/p}[\sum \|\hat{u}_{-n}\|^q]^{1/q}$$

(on transposing and setting $m = -l$, $n = -k$)

$$\Leftrightarrow \left\|\sum_{-\infty}^{\infty} u'_{-m}\left(\sum_{n=-\infty}^{m-1} \hat{W}_{mn}\hat{u}_{-n}\right)\right\| \leqq c[\sum \|u_{-m}\|^p]^{1/p}[\sum \|\hat{u}_{-n}\|^q]^{1/q}$$

$\Leftrightarrow$ (4.1) is BIBO (l^q).

Now we can state the main result.

THEOREM 4.1. *Suppose for the system* (2.1), *the pairs* $[F_k, G_k]$ *and* $[F_k, H_k]$ *are uniformly stabilizable and detectable. Then if* (2.1) *is* BIBO (l^p) *for any one* $p \in [1, \infty]$, $x_{k+1} = F_k x_k$ *is exponentially stable.*

Proof. If (2.1) is BIBO (l^p), then (2.1) is BIBS (l^p) by Proposition 3.1. Then (2.1a) in conjunction with $y_k = x_k$ is BIBO (l^p) and so by Lemma 4.2, the following system is BIBO (l^q):

$$\hat{x}_{k+1} = F'_{-k}\hat{x}_k + \hat{u}_q, \tag{4.2a}$$

$$\hat{y}_k = G'_{-k}\hat{x}_k. \tag{4.2b}$$

By Proposition 3.1, (4.2a) is BIBS (l^q), so (4.2a) together with $\hat{y}_k = \hat{x}_k$ is BIBO (l^q). By Lemma 4.2 again,

$$x_{k+1} = F_k x_k + u_k, \tag{4.3a}$$

$$y_k = x_k \tag{4.3b}$$

is BIBO (l^p), and in particular BIBS (l^p). By a standard result (see [1], [2]), the associated homogeneous equation is exponentially stable.

The following is an immediate consequence of Lemma 2.2 and Theorem 4.1.

COROLLARY 4.1. *Suppose that for the system* (2.1), *the pairs* $[F_k, G_k]$ $[F_k, H_k]$ *are uniformly stabilizable and detectable. Then* (2.1) *is* BIBO (l^p) *for any one* $p \in [1, \infty]$ *if and only if it is* BIBO (l^p) *for all* $p \in [1, \infty]$.

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