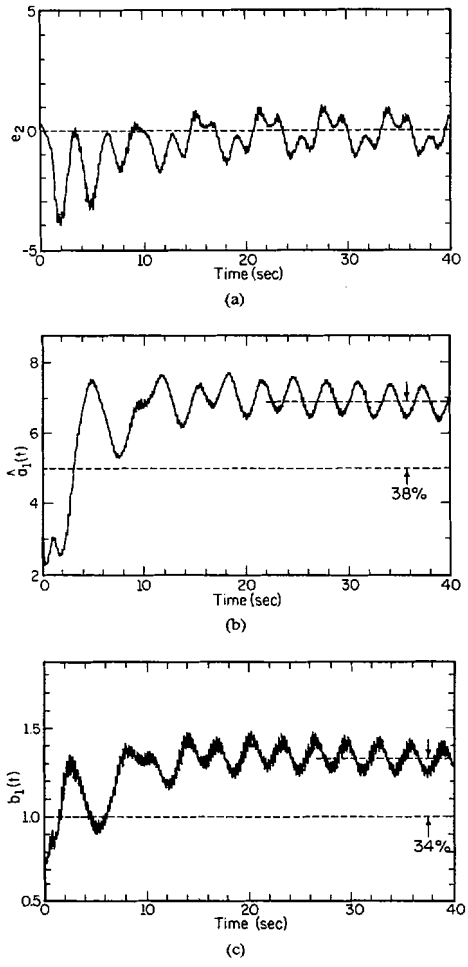

Fig. 5. $\mu = 0.02$, $u = 5\sin t + 5\sin 25t$ ($\gamma = 130$).

where

$$\gamma = \sup|\dot{u}|, \quad \beta = \min|\lambda(A_f)| \geq \|A_f^{-1}\|^{-1}, \quad f_2 = \min|\lambda(\Lambda)|. \tag{A.4}$$

Substituting for $\eta(\tau)$ and $\theta(\tau)$ in (A.1) we obtain

$$\|Z(t)\| \leq p_1 e^{-m_2 t} + p_2 e^{-\beta t/\mu} + p_3 e^{-f_2 t} + \mu\gamma \frac{m_1}{m_2}\frac{\alpha}{\beta}\|A_f^{-1}B_f\| \cdot \left(\|H\| + \frac{f_1}{f_2}\|F\|\|R\|\right) \tag{A.5}$$

where p_1, p_2, and p_3 are independent of t and bounded for all $\mu \in [0, 1]$. Thus as $t \to \infty$, (2.15) follows.

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On Reduced-Order Adaptive Output Error Identification and Adaptive IIR Filtering

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***Abstract*—The reduced-order application of Landau's adaptive output error identifier results in a perturbed error system where the perturbation signal is a moving average of the unmodeled portion of the unknown plant output (or desired signal in adaptive filter parlance). It is proven in this paper that if this perturbation signal is sufficiently small and a reduced-order dimension model is sufficiently excited, then the output and parameter estimates of this adaptive identifier/filter remain bounded. The influence of various operating conditions on this quantitatively defined bound are noted. This robustness property is crucial in all real applications, which due to nonlinearities and distributed effects are subject to reduced-order modeling.**

I. Introduction

The finite-dimensional, linear, differential, or difference equation is a mathematical device that has proved to be a fundamental tool in the study of dynamic systems because of the accuracy with which such descriptions can match the real behavior of numerous physical systems. These finite-dimensional models are ultimately all approximations of infinite-dimensional physical phenomena. Yet, the theory of adaptive identification and control normally requires that the adjustable, linear, finite-dimensional

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model used can *exactly* match, with some setting of its parameters, the underlying physical system being identified or controlled [1]–[4]. This raises the question of how adaptive algorithms will behave in practice and, in particular, how an adaptive algorithm which assumes a certain model order will function when used on a system of higher order.

Since the question of reduced-order model selection—even from (presumed) exact higher order descriptions—for specific purposes such as prediction or control remains incompletely resolved, as witnessed by a continuing flood of papers on this topic [5], it may appear pretentious to resort to the "magic" of an adaptive solution to extract a satisfactory reduced-order model for an incompletely understood system given only input–output operating data. Although specialized search algorithms, e.g., [6], have been shown to have desirable local convergence properties, this paper avoids this philosophical debate by asking a quite different question for a specific algorithm: How does an output error identification scheme, which has been proven [7] to behave "satisfactorily" only when the model order is not underestimated, survive in engineering practice where this condition is undeniably violated? An initial response to this question is obtained by appropriate definition of "satisfactory" behavior. Most studies of convergence properties of adaptive identification schemes involve conversion to asymptotic stability problems [7]–[12] where one proves that all trajectories converge to a certain *point*. This corresponds to *exact* identification. In practice, convergence to a region near this point is satisfactory and occasionally indiscernible from the "exact" solution. This seemingly trivial shift from asymptotic stability to simple stability allows a subtle but significant shift in the methods of evaluation of adaptive identifiers. The desirability of considering such a shift is supported by the practical need for answers to the reduced-order modeling problem. The success of adaptive identifiers in practical usage provides confidence that meaningful results exist. We focus here on a specific class of algorithms developed initially as output error identifiers [7], [8], subsequently recognized as adaptive infinite impulse response filters [9], [13], and recently expanded to include model reference adaptive controllers [14], [15]. Only their reduced-order identification and/or filtering usage will be addressed in this paper.

This paper builds on a qualitative Lyapunov analysis in [16] where the adaptive identifier/filter (AIF) of [9] was shown to maintain a bounded output and parameter error despite reduced-order usage if the AIF dimension and not the full "real" system dimension were sufficiently excited. This concept of reduced-order sufficient excitation is again utilized to establish quantitative relationships between the output and parameter error bound and the unmodelable portion of the real output to be tracked. Section II follows the suggestion in [11] of transforming the AIF of [9] into a perturbed, nonlinear, time-varying state difference equation with the state being a concatenation of several past output and present parameter estimate errors and with the perturbation due to the nonideal, i.e., reduced-order modeling, situation. (The similar parameterization of the identifier in [7], [8] is given in the Appendix.) Section III briefly and qualitatively supports the inclusion of the sufficient excitation requirement. Section IV provides the major theoretical results, first, in proving that no matter how large the initial output and parameter estimate errors, some amount of mismodeling is allowed by retention of identifier stability, and second, in deriving an original quantitative relationship between the amount of mismodeling and the output and parameter estimate bounds. The implications and limitations of these new results are also discussed. The Conclusion notes that these general results may be best extended by incorporating any additional structural knowledge regarding the general nature of the neglected system dynamics.

II. Reduced-Order Adaptive Identifier/Filter Error Model

Before obtaining the equations for a reduced-order identifier/filter, we review the standard equations for an output error identifier/filter. As shown in [12], the adaptive IIR filter of [9]

$$\hat{y}(k)=\sum_{i=1}^{n}\hat{a}_i(k)z(k-i)+\sum_{j=1}^{m}\hat{b}_j(k)u(k-j) \quad (a\ priori\ \text{output}) \quad (2.1)$$

$$\hat{a}_i(k+1)=\hat{a}_i(k)+\mu_i z(k-i)v(k),\ \mu_i>0 \quad (\text{AR parameter update}) \quad (2.2)$$

$$\hat{b}_j(k+1)=\hat{b}_j(k)+\rho_j u(k-j)v(k),\ \rho_j>0 \quad (\text{MA parameter update}) \quad (2.3)$$

$$v(k)=y(k)-z(k)+\sum_{i=1}^{n}c_i[y(k-i)-z(k-i)] \quad (\text{smoothed } a\ posteriori \text{ output error})$$

$$=-\frac{y(k)-\hat{y}(k)+\sum_{i=1}^{n}c_i[y(k-i)-z(k-i)]}{1+\sum_{i=1}^{n}\mu_i z^2(k-i)+\sum_{j=1}^{m}\rho_j u^2(k-j)} \quad (2.4)$$

$$z(k)=\sum_{i=1}^{n}\hat{a}_i(k+1)z(k-i)+\sum_{j=1}^{m}\hat{b}_j(k-1)u(k-j) \quad (a\ posteriori \text{ output}) \quad (2.5)$$

with the desired output of

$$y(k)=\sum_{i=1}^{n}a_i y(k-i)+\sum_{j=1}^{m}b_j u(k-j) \quad (2.6)$$

can be written in the form of the error model of [10]

$$\boldsymbol{e}(k+1)=\boldsymbol{A}\boldsymbol{e}(k)+\boldsymbol{b}w(k) \quad (2.7)$$

$$v(k)=\boldsymbol{h}^T\boldsymbol{e}(k)+dw(k) \quad (2.8)$$

$$w(k)=\boldsymbol{\phi}^T(k)\boldsymbol{x}(k)-\alpha\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)v(k),\quad \alpha>\frac{1}{2},\ \boldsymbol{\Gamma}=\boldsymbol{\Gamma}^T>0 \quad (2.9)$$

$$\boldsymbol{\phi}(k+1)=\boldsymbol{\phi}(k)-\boldsymbol{\Gamma}v(k)\boldsymbol{x}(k) \quad (2.10)$$

given (2.4) and the following definitions:

$$\boldsymbol{e}^T(k)\triangleq[y(k-1)-z(k-1)\quad y(k-2)-z(k-2)\quad\cdots y(k-n)-z(k-n)] \quad (2.11)$$

$$\boldsymbol{A}\triangleq\begin{bmatrix}a_1 & a_2 & \cdots & a_n\\ 1 & 0 & \cdots & 0\\ 0 & & & \\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 1 & 0\end{bmatrix} \quad (2.12)$$

$$\boldsymbol{b}^T\triangleq[1\quad 0\quad\cdots\quad 0] \quad (2.13)$$

$$\boldsymbol{h}^T\triangleq[c_1+a_1\quad c_2+a_2\quad\cdots\quad c_n+a_n] \quad (2.14)$$

$$d\triangleq 1 \quad (2.15)$$

$$w(k)\triangleq\sum_{i=1}^{n}[a_i-\hat{a}_i(k+1)]z(k-i)+\sum_{j=1}^{m}[b_j-\hat{b}_j(k-1)]u(k-j) \quad (2.16)$$

$$\boldsymbol{\phi}^T(k)\triangleq[a_1-\hat{a}_1(k)\quad\cdots\quad a_n-\hat{a}_n(k)\quad b_1-\hat{b}_1(k)\quad\cdots\quad b_m-\hat{b}_m(k)] \quad (2.17)$$

$$\boldsymbol{x}^T(k)\triangleq[z(k-1)\quad\cdots\quad z(k-n)\quad u(k-1)\quad\cdots\quad u(k-m)] \quad (2.18)$$

$$\alpha\triangleq 1 \quad (2.19)$$

$$\boldsymbol{\Gamma}\triangleq\operatorname{diag}[\mu_1\quad\cdots\quad\mu_n\quad\rho_1\quad\cdots\quad\rho_m]. \quad (2.20)$$

Similarly, the adaptive output error identifier combining (1)–(6) in [7] and (1)–(4) in [8] can also be written in the form of (2.7)–(2.10). As shown in [11] and [12], (2.7)–(2.10) can be combined as

$$\begin{bmatrix} e(k+1) \\ \phi(k+1) \end{bmatrix} = F(k) \begin{bmatrix} e(k) \\ \phi(k) \end{bmatrix} \tag{2.21}$$

with

$$F(k) = \begin{bmatrix} A - \dfrac{\alpha x^T(k)\Gamma x(k) b h^T}{1+\alpha d x^T(k)\Gamma x(k)} & \dfrac{b x^T(k)}{1+\alpha d x^T(k)\Gamma x(k)} \\ \dfrac{-\Gamma x(k) h^T}{1+\alpha d x^T(k)\Gamma x(k)} & I - \dfrac{d\Gamma x(k) x^T(k)}{1+\alpha d x^T(k)\Gamma x(k)} \end{bmatrix}. \tag{2.22}$$

The main result for (2.1)–(2.21) is that if

$$H(z) = d + h^T(zI - A)^{-1} b = \frac{1 + \sum_{i=1}^{n} c_i z^{-i}}{1 - \sum_{i=1}^{n} a_i z^{-i}} \tag{2.23}$$

is strictly positive real (SPR), the original plant in (2.6) is exponentially stable, and the input sequence $\{u(k)\}$ is bounded, then the algorithm is convergent in the sense that $y(k) - z(k) \to 0$ and $\phi(k)$ remains bounded. Note that the SPR requirement on (2.23) requires the stability of (2.6). The addition of sufficient excitation by $\{u(k)\}$ and the structural minimality of (2.6), and therefore (2.1), guarantees that e and $\phi \to 0$ exponentially fast [4].

Now for reduced-order identification, we shall assume that (2.6) is replaced by a higher order model, and we seek to identify an nth-order part of this model. Thus, as in [16], (2.6) is replaced by

$$y(k) = y_M(k) + y_U(k) \tag{2.24}$$

where y_M is the modeled portion of y and has the same order as the identifier filter, so that

$$y_M(k) = \sum_{i=1}^{n} a_i y_M(k-i) + \sum_{j=1}^{m} b_j u(k-j) \tag{2.25}$$

and $y_U(k)$ is the unmodeled portion of y. Note that $y_U(k)$ could also represent a noise signal. The first concern addressed in [16] is whether or not $z(k)$, and subsequently $\hat{y}(k)$, will remain bounded with (2.24) and (2.25) rather than (2.6) used in (2.1)–(2.5) if $y_U(k)$ remains bounded. The desired implication that

$$\|y_U(k)\|^2 \leq \delta^2 < \infty \quad \forall k \quad \Rightarrow \|y(k) - z(k)\|^2 \leq \epsilon^2 < \infty \quad \forall k > \bar{k} \tag{2.26}$$

was shown to be true in [16]. The second concern, i.e., whether or not $\phi(k)$ remains bounded, was argued heuristically in [16] to be true given sufficient excitation, i.e., small $|\phi^T x| \Rightarrow$ small $\|\phi\|$ or, conversely, large $\|\phi\| \Rightarrow$ large $|\phi^T x|$.

These results were obtained by examination of the error model resulting from the alterations of (2.7)–(2.10) reflecting the use of (2.24) and (2.25) for (2.6). The altered error model is obtained by first noting the change in (2.4)

$$v(k) = \sum_{i=1}^{n} [c_i + a_i][y_M(k-i) - z(k-i)] + w(k) + p(k) \tag{2.27}$$

where w is defined in (2.16) and the perturbation signal p is defined by

$$p(k) \triangleq y_U(k) + \sum_{i=1}^{n} c_i y_U(k-i). \tag{2.28}$$

With the definition of e in (2.11) altered to

$$e^T(k) \triangleq [y_M(k-1) - z(k-1) \cdots y_M(k-n) - z(k-n)], \tag{2.29}$$

(2.8) becomes

$$v(k) = h^T e(k) + dw(k) + p(k) \tag{2.30}$$

in order to match (2.27). The remainder of the error system, i.e., (2.7), (2.9), (2.10), and (2.12)–(2.20), remains unaltered in accommodating the effects of reduced-order modeling. (A similar set of equations results due to reduced-order application of the identifier in [7], [8]. See the Appendix.) The modification required for (2.21) is

$$\begin{bmatrix} e(k+1) \\ \phi(k+1) \end{bmatrix} = F(k) \begin{bmatrix} e(k) \\ \phi(k) \end{bmatrix} + G(k) p(k) \tag{2.31}$$

with $F(k)$ in (2.22) as before and

$$G(k) = \begin{bmatrix} \dfrac{-\alpha b x^T(k)\Gamma x(k)}{1+\alpha d x^T(k)\Gamma x(k)} \\ \dfrac{-\Gamma x(k)}{1+\alpha d x^T(k)\Gamma x(k)} \end{bmatrix}. \tag{2.32}$$

The perturbed error system of (2.31), given (2.22) and (2.32), now compactly describes the reduced-order AIF behavior.

III. Qualitative Discussion of Sufficient Excitation and Boundedness

We establish in Section IV that with a sufficient (or persistent) excitation condition in $\{u(k)\}$ and a bound on $\{p(k)\}$, $\{e(k)\}$ and $\{\phi(k)\}$ remain bounded; the smaller the bound on $\{p(k)\}$, the smaller the bounds on $\{e(k)\}$ and $\{\phi(k)\}$.

As noted, when there is no undermodeling of order and $p(k) \equiv 0$, persistent excitation by $\{u(k)\}$ as defined in [4] guarantees exponentially fast convergence to zero of $e(k)$ and $\phi(k)$ in (2.21) [4]. Therefore, when $p(k)$ is not equal to zero, bounded-input, bounded-output behavior of (2.31) is to be expected. But one can be deceived concerning the reasoning behind this conclusion. Although (2.31) looks linear, it is not, as explained in Section IV. This makes the proof of bounded-input, bounded-output stability much harder, and implies that for large enough but bounded $\{p(k)\}$, (2.31) might go unstable.

The persistent excitation property is implied by the composition of $\{u(k)\}$ as a linear combination of a certain number of sinusoids. (Note that other $\{u(k)\}$ besides a linear combination of sinusoids can also be sufficiently exciting.) This number, as will be seen, depends upon the dimension of the reduced model, not the dimension of the plant itself. Then, if the reduced-order model has dimension $2p(=n+m)$, $\{u(k)\}$ need contain p sinusoids to be sufficiently exciting. Theoretically, the frequencies of these sinusoids do not matter. But in practice, the convergence rate of the identifier, the amount of modeling error $\{p(k)\}$ which can be tolerated without encountering instability, and the bounds on $\{e(k)\}$ and $\{\phi(k)\}$ resulting from a fixed bound on $\{p(k)\}$ will all depend upon these frequencies, or more generally, the manner in which $\{u(k)\}$ is persistently exciting. Theoretical developments are not yet such as to allow much quantification of these issues.

IV. Output and Parameter Error Boundedness

For notational ease, define

$$\tilde{x}(k) = [y_M(k-1) \quad \cdots \quad y_M(k-n) \quad u(k-1) \quad \cdots \quad u(k-m)]^T. \tag{4.1}$$

Notice that $\tilde{x}(k)$ depends only upon the driving sequence and the initial conditions in the modeled portion of $y(k)$. (Since the latter dependence is exponentially decaying, it will be neglected.) Also note that

$$\tilde{x}(k) = x(k) + [e^T(k) \quad 0]^T. \tag{4.2}$$

Next, define $\tilde{F}(k)$ to be $F(k)$ with $x(k)$ replaced by $\tilde{x}(k)$ in (2.22), and set $\Psi[e(k), k] = F(k) - \tilde{F}(k)$. Notice that the matrix sequence $\tilde{F}(k)$ is truly independent of $e(k)$, whereas $F(k)$ is not; accordingly, the arguments of Ψ display the dependence upon $e(k)$ as well as k. Now write

(2.31) as

$$\begin{bmatrix} e(k+1) \\ \phi(k+1) \end{bmatrix} = \tilde{F}(k) \begin{bmatrix} e(k) \\ \phi(k) \end{bmatrix} + \Psi[e(k), k] \begin{bmatrix} e(k) \\ \phi(k) \end{bmatrix} + G(k)p(k). \tag{4.3}$$

Adopt the abbreviation

$$q(k) = \begin{bmatrix} e(k) \\ \phi(k) \end{bmatrix} \tag{4.4}$$

and regard Ψ as a function of $q(k)$ and k.

To obtain the first result for (4.3), recall the following.

Lemma 1 [4]: With $u(k)$ satisfying a persistently exciting condition as defined in [4],

$$r(k+1) = \tilde{F}(k)r(k) \tag{4.5}$$

is exponentially stable. ■

We remark that the key dimension in formulating the persistently exciting condition is that of $\tilde{x}(k)$, i.e., a dimension associated with the reduced-order model and not the plant. Lemma 1 allows statement of the following.

First Main Theorem: With $\|q(0)\|$ and $\max_k |p(k)|$ suitably small in (4.3), $\|q(k)\|$ remains bounded.

Proof: From the various definitions, it is readily checked that $\tilde{F}$, G, and Ψ are smooth and bounded, and that $\|\Psi\| \to 0$ as $\|e\| \to 0$, implying that (4.5) is the linearized, homogeneous version of (4.3). The result is then an immediate consequence of Lemma 1 and [17]. The fact that $G(k)$ is dependent upon $q(k)$ does not affect the conclusion since [see (2.32)] $G(k)$ is bounded in the region of interest. ■

This result, although clearly important, is easy to obtain. The main interest in the rest of this section is to prove a result which says no matter what $\|q(0)\|$ is, there is an appropriately small bounded $\max_k |p(k)|$ which implies a bound on $\|q(k)\|$. The result, stated below as the Second Main Theorem, subsumes the First Main Theorem. The question can be asked in relation to both of the theorems as to how small is "suitably small" or "appropriately small." The answer is that the bound is always problem specific, but that by following through the calculations of this paper and the references used in proving the Main Theorem, one can sometimes compute a value; simulations, however, suggest that the value is very conservative.

The Second Main Theorem requires two additional lemmas which develop in precise terms the nature of the exponential stability of the homogeneous version of (4.3).

Lemma 2 [7]–[9]: With $p(k) \equiv 0$ in (4.3) and $q(k_0)$ arbitrary, $q(k)$ is bounded for $k \geq k_0$ and

$$\sum_{k=k_0}^{\infty} \|e(k)\|^2 \leq \gamma_1 V[q(k_0)] \tag{4.6}$$

$$\sum_{k=k_0}^{\infty} \|\Psi(q(k), k)\|^2 \leq \gamma_2 V[q(k_0)], \tag{4.7}$$

where $V[q(k)]$ is a positive definite quadratic form in $q(k)$ monotonically decreasing along $q(\cdot)$ trajectories, and γ_1, γ_2 are finite, positive constants. ■

Outline of Proof: Because of the strict positive realness of $d + h^T(zI - A)^{-1}b$, there exist positive definite π and L, positive ν, and a vector k such that

$$A^T \pi A - \pi = -kk^T - L \tag{4.8}$$

$$A^T \pi b = \frac{1}{2}h + \nu k \tag{4.9}$$

$$b^T \pi b = d - \nu^2. \tag{4.10}$$

Then, it turns out that

$$V[q(k)] = q^T(k) \begin{bmatrix} \pi & 0 \\ 0 & \frac{1}{2}\Gamma^{-1} \end{bmatrix} q(k) \tag{4.11}$$

is a Lyapunov function for (4.3) with $p(k) \equiv 0$ which causes

$$\begin{aligned} V[q(k+1)] - V[q(k)] &= -\begin{bmatrix} e^T(k) & w(k) \end{bmatrix} \begin{bmatrix} L + kk^T & -\nu k \\ -\nu k^T & \nu^2 \end{bmatrix} \begin{bmatrix} e(k) \\ w(k) \end{bmatrix} \\ &\quad - \left(\alpha - \frac{1}{2}\right) v^2(k) x^T(k) \Gamma x(k). \end{aligned} \tag{4.12}$$

Also,

$$\begin{aligned} V[q(k_0)] &\geq -\sum_{k=k_0}^{\infty} \{V[q(k+1)] - V[q(k)]\} \\ &\geq \sum_{k=k_0}^{\infty} \left\{ \begin{bmatrix} e^T(k) & w(k) \end{bmatrix} \begin{bmatrix} L + kk^T & -\nu k \\ -\nu k^T & \nu^2 \end{bmatrix} \begin{bmatrix} e(k) \\ w(k) \end{bmatrix} \right. \\ &\quad \left. + \left(\alpha - \frac{1}{2}\right) v^2(k) x^T(k) \Gamma x(k) \right\} \\ &\geq \sum_{k=k_0}^{\infty} \left\{ \lambda_{\min} \begin{bmatrix} L + kk^T & -\nu k \\ -\nu k^T & \nu^2 \end{bmatrix} [\|e(k)\|^2 + \|w(k)\|^2] \right\} \\ &\quad \left(\text{recall } \alpha > \frac{1}{2}\right) \\ &\geq \sum_{k=k_0}^{\infty} \lambda_{\min} \begin{bmatrix} L + kk^T & -\nu k \\ -\nu k^T & \nu^2 \end{bmatrix} \|e(k)\|^2. \end{aligned}$$

Here, $\lambda_{\min}[\cdot]$ denotes the smallest eigenvalue, which is easily verified to be positive. Hence, (4.6) is immediate.

Since $\Psi[e(k); k] = F(k) - \tilde{F}(k)$, where $\tilde{F}(k)$ is obtained from $F(k)$ by replacing $x(k)$ by $x(k) + [e^T(k) \ \ 0]^T$, and in view of the form in (2.22) of $F(k)$, the l_2-bound on $e(k)$ translates into a similar bound on $\Psi[e(k), k]$, as in (4.7). ■

Lemma 3: Under the persistently exciting condition which guarantees stability of (4.5) [the linearized homogeneous version of (4.3)], the homogeneous ($p(k) \equiv 0$) version of (4.3) yields

$$\|q(k)\| \leq \gamma_4 \alpha^{k-k_0} \|q(k_0)\| \tag{4.13}$$

for some $\alpha < 1$, all $\|q(k_0)\|$ with $V[q(k_0)] \leq R$, the constant γ_4 depending on R, which can be arbitrary.

Proof: With $\tilde{\Phi}(l, k)$ denoting the transition matrix of (4.5), define

$$P(k) = \sum_{l=k}^{\infty} \tilde{\Phi}^T(l, k) \tilde{\Phi}(l, k). \tag{4.14}$$

Since $\tilde{F}^T(k)P(k+1)\tilde{F}(k)$

$$= \sum_{l=k+1}^{\infty} \tilde{F}^T(k) \tilde{\Phi}^T(l, k+1) \tilde{\Phi}(l, k+1) \tilde{F}(k),$$

$\tilde{\Phi}(l, k+1)\tilde{F}(k) = \tilde{\Phi}(l, k)$, and $\tilde{\Phi}(k, k) = I$, then

$$P(k) - \tilde{F}^T(k)P(k+1)\tilde{F}(k) = I \tag{4.15}$$

and

$$0 < I < P(k) < \alpha_2 I \tag{4.16}$$

for some $\alpha_2 > 1$. Now for the homogeneous (4.3), try to use $q^T(k)P(k)q(k)$ as a Lyapunov function. It does not quite work, but

yields the desired result. Evidently, in light of (4.15)

$$\begin{aligned} q^T(k)P(k)q(k)-q^T(k+1)P(k+1)q(k+1) \\ =q^T(k)q(k)-q^T(k)\big[2\Psi^T(q(k),k)P(k+1)\tilde{F}(k) \\ -\Psi^T(q(k),k)P(k+1)\Psi(q(k),k)\big]q(k). \end{aligned} \quad (4.17)$$

Now observe that, by virtue of the bounds on $\tilde{F}(k)$ and $P(k)$, for some $\gamma_3>0$,

$$\begin{aligned} \big|q^T(k)\{2\Psi^T(q(k),k)P(k+1)\tilde{F}(k) \\ +\Psi^T(q(k),k)P(k+1)\Psi(q(k),k)\}q(k)\big| \\ \leq \alpha_2\|q(k)\|^2\{\gamma_3\|\Psi(q(k),k)\|+\|\Psi(q(k),k)\|^2\}. \end{aligned} \quad (4.18)$$

Accordingly, given arbitrary $\delta\in(0,1)$, there exists an $\eta>0$ such that if

$$\begin{aligned} \big|q^T(k)\{2\Psi^T(q(k),k)P(k+1)\tilde{F}(k) \\ +\Psi^T(q(k),k)P(k+1)\Psi(q(k),k)\}q(k)\big| \\ >\delta\|q(k)\|^2, \end{aligned} \quad (4.19)$$

then

$$\|\Psi(q(k),k)\|>\eta \quad (4.20)$$

(although the converse is not guaranteed). Note that there may be no k for which (4.19) holds.

Given the bound (4.7), it follows that for fixed R and all $q(k_0)$ with $V[q(k_0)]\leq R$, there is at most a finite set K of values of $k\geq k_0$ for which (4.20), and therefore (4.19), could hold. The cardinality (or number of entries) of K denoted by $|K|$ is bounded by $\gamma_2 R/\eta^2$ since from (4.7), $|K|\eta^2\leq\sum_{k=k_0}^{\infty}\|\Psi(q(k),k)\|^2\leq\gamma_2 R$. Using the fact that (4.19) fails when $k\notin K$, we obtain from (4.17)

$$q^T(k)P(k)q(k)-q^T(k+1)P(k+1)q(k+1)\geq(1-\delta)q^T(k)q(k) \quad (4.21)$$

and so

$$\begin{aligned} \frac{q^T(k+1)P(k+1)q(k+1)}{q^T(k)P(k)q(k)} &\leq 1-\frac{(1-\delta)q^T(k)q(k)}{q^T(k)P(k)q(k)} \\ &\leq 1-\frac{1-\delta}{\alpha_2}. \end{aligned} \quad (4.22)$$

For $k\in K$, based upon the bounds on $P(k)$, $\tilde{F}(k)$, and $\Psi(q(k),k)$, it is clear that we can find a positive constant γ_3, depending monotonically upon R, such that for $k\geq k_0$, $k\in K$,

$$\frac{q^T(k+1)P(k+1)q(k+1)}{q^T(k)P(k)q(k)}\leq\gamma_3. \quad (4.23)$$

Combining (4.22) and (4.23) for $k=k_0, k_0+1,\cdots,l-1$ through multiplication yields

$$\frac{q^T(l)P(l)q(l)}{q^T(k_0)P(k_0)q(k_0)}\leq\gamma_3^{|K|}\left(1-\frac{1-\delta}{\alpha_2}\right)^{k-k_0}. \quad (4.24)$$

Now (4.13) follows on using the bound (4.16) on $P(\cdot)$. Notice that $\gamma_4=[\alpha_2\gamma_3^{|K|}]^{1/2}$ depends monotonically on R, and that $\alpha=[1-(1-\delta)\alpha_2^{-1}]^{1/2}\in(0,1)$ by virtue of the bounds on δ and α_2. ■

Remark: A standard definition of exponential stability for the homogeneous version of (4.3) would require that there exists a positive $\gamma_4>\alpha$ with $\alpha<1$ and with γ_4 independent of $q(k_0)$ or of R. Thus, in the strictest sense, (4.13) is not the same as exponential stability.

Now the main result can be stated as follows.

Second Main Theorem: Consider (4.3) with $q(k_0)$ such that $V[q(k_0)]\leq R$ where R is arbitrary. There exists a constant $T(R)$ such that when $|p(k)|<T$ for all k, $\|q(k)\|$ is bounded, and a constant $S(R)$ exists such that

$$\lim_{k\to\infty}\sup\|q(k)\|<S\lim_{k\to\infty}\sup|p(k)|. \quad (4.25)$$

Remarks:

1) The interpretation is as follows. The theorem guarantees that the algorithm will not blow up provided that $\max_k|p(k)|$ is small enough. The question of how small is small is initial condition dependent; thus, the larger the initial error, the smaller the bound on $|p(k)|$. Provided that $\max_k|p(k)|$ is within such a limit, then the smaller the actual value of $\max_k|p(k)|$, the smaller will be $\lim_{k\to\infty}\sup\|q(k)\|$, and thus the smaller will be the tracking error.

2) One can ask how one can be sure that $|p(k)|$ is small; the answer is that one cannot always be sure, without some knowledge of the plant and further calculation or simulations. Nevertheless, when the algorithm is running on an actual plant, one can verify easily whether or not the algorithm is performing well—one simply compares the model and plant output.

3) The driving term in (4.3) can be thought of as $G(k)p(k)$, rather than $p(k)$. Recalling the form of $G(k)$ [see (2.32)] and $x(k)$, we see that what is really needed to ensure that $G(k)p(k)$ is small is that $\|p(k)\|/\|x(k)\|$ be small, i.e., the unmodeled output should be small, not in absolute terms, but in comparison to the total output. The "percentage" of mismodeling is the critical issue.

4) The proof of the theorem gives some insight into the computation of S. This, nevertheless, remains a very difficult task.

Proof: It is easily verified that the equation obtained by linearizing (4.3) around the trajectory $q_0(\cdot)$ of (4.3) associated with initial condition $q_0(k_0)=q_0$ and $p(k)\equiv 0$ is

$$r(k+1)=\{\tilde{F}(k)+\hat{\Psi}(q_0(k),k)\}r(k)+\tilde{G}(k)p(k) \quad (4.26)$$

where $\tilde{G}(k)$ denotes $G(k)$ in (2.32) with $x(k)$ replaced by $\tilde{x}(k)$ and $\hat{\Psi}$ involves Ψ and derivatives of Ψ with respect to q; further, $\hat{\Psi}\to 0$ as $q_0(k)\to 0$, i.e., as $k\to\infty$. Denote the transition matrix of the homogeneous version of (4.26) by $\Phi_{q_0}(k,k_0)$. For some $\alpha_4>0$, and α as earlier,

$$\|\Phi_{q_0}(k,k_0)\|<\alpha_4\alpha^{k-k_0} \quad (4.27)$$

since $\hat{\Psi}\to 0$ as $k\to\infty$ and $r(k+1)=\tilde{F}(k)r(k)$ is exponentially stable [18]. Also, $\tilde{G}(k)$ is bounded.

Now consider (4.3) with $q_0(k_0)$ as the initial condition under two excitation regimes: $p(k)\equiv 0$, giving a trajectory $q_0(\cdot)$, and $p(k)\not\equiv 0$, giving a trajectory $q(\cdot)$. Given (4.27), appealing to [17] allows the conclusion that there exists $T(q_0)$ such that $|p(k)|<T(q_0)$ for all k implies that $\|q(k)-q_0(k)\|$ is bounded and a constant $S(q_0)$ exists such that

$$\lim_{k\to\infty}\sup\|q(k)-q_0(k)\|<S(q_0)\lim_{k\to\infty}\sup|p(k)|. \quad (4.28)$$

Since $\|q_0(k)\|$ is bounded and approaches zero as $k\to\infty$, $\|q(k)\|$ is bounded and

$$\lim_{k\to\infty}\sup\|q(k)\|<S(q_0)\lim_{k}\sup|p(k)|. \quad (4.29)$$

Clearly, the quantities $T(q_0)$ and $S(q_0)$ will work for initial conditions in a nonzero neighborhood of q_0 as well as for q_0 itself. Hence, the set of values $T(\cdot)$ and $S(\cdot)$ required to cover the region $V[q_0(k_0)]\leq R$ is finite. Let

$$T(R)=\min_{V[q_0(k_0)]\leq R}T(q_0) \quad (4.30)$$

and

$$S(R)=\max_{V[q_0(k_0)]\leq R}S(q_0); \quad (4.31)$$

then the theorem claims follow. ■

Remarks:

1) As previously mentioned, ambiguity of this result is in the designation of y_M in (2.25) and the subsequent error state space origin $\boldsymbol{q} = \mathbf{0}$ from (4.4). Clearly, the designation of y_M affects $\max_k |p(k)|$ and $V(0)$, and therefore, $R, T(R)$ and $S(R)$ in (4.30) and (4.31). Consider the possibility that y_M is parameterized such that after the adaptive transients decay, the trajectory of $\boldsymbol{q}(k)$ as far as possible is evenly distributed within a ball of radius ξ_1. Now, if y_M were reselected such that the new $\boldsymbol{q}(0)$ were shifted by a distance ξ_2 outside the original ξ_1 ball, then the new bound on $\|\boldsymbol{q}(k)\|$ would be $\xi_1 + \xi_2$ with no real change in the adaptive identifier. This coordinate specification problem arises due to the lack of the formula for a nominal (or mean) convergent reduced-order AIF parameterization. The results of this Second Main Theorem suggest that this nominal convergent parameterization could be designated as that y_M which minimizes the bounding constant in (4.25) with the other factors influencing this bound assumed fixed.

2) Since (4.25) clearly suggests that the size of $p(k)$ influences the size of $\|\boldsymbol{q}(k)\|$, consider (2.28). If y_U is the response of the high-frequency component of the plant being identified, then y_U and p can be reduced by limiting the frequency content of u to low-frequency components. Similarly, the error smoothing coefficients c_i may be selected to minimize p while retaining the SPR of (2.23).

3) Since smaller α tightens the bound in (4.27) for $k > k_0$, then from [17], a smaller $S(\boldsymbol{q}_0)$ would result. Therefore, a faster convergence rate for the algorithm, i.e., a smaller α, results in a closer approximation of the reduced-order AIF error time history to that of the asymptotically convergent full-order AIF. The convergence rate is influenced by the selection of Γ and the frequency content of $\boldsymbol{x}(k)$.

4) In [4], the exponential convergence of the full (minimal)-order case was established by showing that $\tilde{\boldsymbol{x}}(k) \to \boldsymbol{x}(k)$ (not necessarily exponentially fast). Therefore, persistently exciting $\{u(k)\}$ implies persistently exciting $\{\tilde{\boldsymbol{x}}(k)\}$, and therefore, persistently exciting $\{\boldsymbol{x}(k)\}$. The results in [4] were then based on this persistently exciting characteristic of $\{\boldsymbol{x}(k)\}$. In this paper, both with $p(k) = 0$ and $p(k) \neq 0$, we have relied upon the condition that $\{\tilde{\boldsymbol{x}}(k)\}$ is persistently exciting without using the persistent excitation of $\{\boldsymbol{x}(k)\}$ as an intermediate requirement. (Persistently exciting $\{\tilde{\boldsymbol{x}}(k)\}$ follows from the persistent excitation by $\{u(k)\}$, defined using the order of the reduced-order model.) Nevertheless, $\{\boldsymbol{x}(k)\}$ will be persistently exciting for a satisfactorily small $\max_k |p(k)|$. For then, $\lim_{k\to\infty} \sup \|e(k)\|$ is small and from (4.2), $\|\Sigma_{k=j}^{j-S} \boldsymbol{x}(k)\boldsymbol{x}^T(k) - \Sigma_{k=j}^{j-S} \tilde{\boldsymbol{x}}(k)\tilde{\boldsymbol{x}}^T(k)\|$ will be small. Since the persistent excitation condition requires the uniform positive definiteness of $\Sigma_{k=j}^{j+S} \tilde{\boldsymbol{x}}(k)\tilde{\boldsymbol{x}}^T(k)$, for p sufficiently small $\{\boldsymbol{x}(k)\}$ will satisfy this same type of condition and also be persistently exciting.

V. Conclusions

The key result of the paper is that reduced-order models can, on occasion, be satisfactory for adaptive output error identification and adaptive infinite impulse response filtering. This result plays a key role in justifying the practical use of many proposed algorithms. For, as is well known, the "true plant" being identified can virtually never be regarded as being able to be described by a finite ARMA model, so that reduced-order modeling is not the atypical, but normal practical situation.

The paper also demands a persistently exciting condition for the results to be valid. This condition can be fulfilled if the excitation signal is a linear combination of a sufficient number of sinusoids; the minimum number depends upon the dimension of the reduced-order model, not the plant itself. There is the potential for the unmodeled part of the plant to destroy the linear independence property associated with the $\{\boldsymbol{x}(k)\}$ sequence; however, the requirement that the unmodeled part of the plant is small ensures that the perturbations introduced into $\boldsymbol{x}(k)$ do not destroy the linear independence property induced by persistently exciting $\{u(k)\}$.

As an alternative to the persistently exciting condition, one can modify the basic algorithm to incorporate a forgetting factor. Under this arrangement, (2.2) and (2.3) are adjusted by the insertion of a multiplication factor $\lambda < 1$ on the right-hand side. This introduces a twofold modification to (2.21): $\boldsymbol{F}(k)$ is replaced by a matrix for which the transition matrix is guaranteed to be bounded by a decreasing exponential, and an additive driving term appears. In (4.3), the change is to $\tilde{\boldsymbol{F}}(k)$, which is now guaranteed in its changed form to have an exponentially decaying transition matrix, and a further additive driving term is inserted. In the event that there are lengthy intervals over which $\{u(k)\} \equiv 0$, all parameter values approach zero, which may not be altogether satisfactory when $\{u(k)\}$ again becomes nonzero.

Note that the permissible amount of "mismodeling" is bounded, the bound being a complex function of many quantities, including the initial parameter estimation error and the convergence rate which would be encountered where there is no modeling error, the latter in turn being dependent upon the degree of sufficiency of the excitation, as already noted. Thus, it is hard, if not impossible, to lay down precise guidelines on the amount of mismodeling which can be tolerated without further problem structure such as exploited in [19]. As a final comment, the assumption of the existence of this bound on the perturbation is the present limiting factor in extending this analysis to the adaptive control case. Since, in the adaptive control case, one objective is to prove the boundedness of $y(= y_M + y_U)$ (e.g., refer to [2]) such an assumption that p or y_U is bounded is inappropriate. Clearly, many questions remain unanswered.

Appendix
Description of Landau's Identifier in the Form of (2.7)–(2.10)

Using the notation of [7], [8] and noting especially (1)–(6) of [7] and (1)–(4) of [8], make the following definitions:

$$\boldsymbol{e}^T(k) \triangleq \left[\theta_p(k-1) - \theta_s(k-1) \quad \cdots \quad \theta_p(k-n) - \theta_s(k-n)\right] = \left[\epsilon_{k-1} \quad \cdots \quad \epsilon_{k-n}\right]. \tag{A.1}$$

$$w(k) \triangleq \sum_{i=1}^{n} \left[a_i - \hat{a}_i(k)\right]\theta_s(k-i) + \sum_{i=0}^{m} \left[b_i - \hat{b}_i(k)\right]\rho(k-i). \tag{A.2}$$

$$\phi^T(k) \triangleq \left[a_1 - \hat{a}_1(k-1) \quad \cdots \quad a_n - \hat{a}_n(k-1) \quad b_0 - \hat{b}_0(k-1) \quad \cdots \quad b_m - \hat{b}_m(k-1)\right] = \boldsymbol{p}^T - \hat{\boldsymbol{p}}^T(k). \tag{A.3}$$

$$\boldsymbol{x}^T(k) \triangleq \left[\theta_s(k-1) \quad \cdots \quad \theta_s(k-n) \quad \rho(k) \quad \cdots \quad \rho(k-m)\right] = \boldsymbol{y}_{k-1}^T. \tag{A.4}$$

$$\Gamma \triangleq \boldsymbol{F}_0 \quad \text{for } \lambda = \infty. \tag{A.5}$$

As is clear from comparison of (2.11)–(2.20) and (A.1)–(A.5), the differences between [9] and [7], [8] are minor: 1) notation: $y(k)$ in [9] $\leftrightarrow \theta_p(k)$ in [7], [8], $z(k) \leftrightarrow \theta_s(k)$, and $u(k) \leftrightarrow \rho(k)$; 2) assumed plant delay: no b_0 in [9]; 3) parameter estimate time indexing: $\hat{a}_i(k+1)$ and $\hat{b}_j(k+1)$ in [9] $\leftrightarrow \hat{a}_i(k)$ and $\hat{b}_j(k)$ in [7], [8]; and 4) adaptive gain matrix structure: constant and diagonal in [9] and possibly time-varying, positive-definite, and symmetric in [7], [8]. The modifications of 2) and 3) for [9] yield a strictly causal algorithm as noted in [9]. The simplification of [9] versus [7], [8] in 4) supports use in tracking a time-varying system, but loses a possible convergence speed improvement.

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The Robustness of Controllability and Observability of Linear Time-Varying Systems

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***Abstract*—Fixed point methods from nonlinear analysis are used to establish conditions under which the uniform complete controllability of linear time-varying systems is preserved under nonlinear perturbations in the state dynamics and the zero-input uniform complete observability of linear time-varying systems is preserved under nonlinear perturbation in the state dynamics and output read-out map. Robustness of partial controllability, observability, and a specific kind of nonzero input observability are also proven.**

I. Introduction

Controllability and observability are key issues in system theory. To be specific, consider a class of physical dynamical systems which are adequately modeled by ordinary differential equations with inputs u and a static read-out map h: more precisely,

$$\dot{x} = f(x, u, t) \tag{I.1}$$

$$y = h(x, t) \tag{I.2}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_i}$, $y \in \mathbb{R}^{n_o}$, $t \in \mathbb{R}_+$, and f, h are C^0 functions; f satisfies Lipschitz and growth conditions for existence and uniqueness of solutions on $\mathbb{R}_+$.

The definition of uniform complete controllability for (I.1) is as follows: $\exists T \in \mathbb{R}_+$ such that given any t_0, initial time, and any two states x_0, the initial state, and x_1, the final state, there exists a control $u \in L_2^{n_i}([t_0, t_0 + T])$ which will steer the system (I.1) from x_0 at t_0 to x_1 at $t_0 + T$. Zero-input uniform complete observability is defined as: $\exists T \in \mathbb{R}_+$ such that given any t_0 and the output of the system with zero input on $[t_0, t_0 + T]$, we can determine (uniquely) the state of the system at t_0.

In this paper, we prove that if a system of the form (I.1), (I.2) is "close" to a linear system which is uniformly completely controllable and zero-input observable, then the original nonlinear system is also uniformly completely controllable and zero-input observable. Further, if the linear system is only partially controllable or observable, then the original nonlinear system at least retains these partial controllability and observability properties (see Section V). These results are termed robustness results for the controllability and observability of linear time-varying systems because of the perturbational nature of the proofs and estimates about a nominal linear system.

The major mathematical tool is a solvability theorem for operator equations with a quasibounded nonlinearity due to Granas [4] which is reminiscent of the familiar small gain theorem (see [2], for example). The heart of the theorem lies in the Rothe fixed point theorem. The use of fixed point theorems in proving global nonlinear controllability results is not new—the Arzela–Ascoli theorem was used in Lukes [5], the contraction mapping theorem by Mirza and Womack [8], and the Schauder fixed point theorem by Vidyasagar [14] and by Dauer [16], [17] and Aronsson [18]. For global nonlinear observability, we have the work of Yamamoto and Sugiura [15]. Our estimates by virtue of our new technique are different from those reported in the literature (with some overlap, Theorem V.1 is also proven by Lukes).[1] Detailed comments on how the results reported above are special cases of our theorems appear in the text of the paper.

The results of Dauer [17] and Aronsson [18] start from rather different assumptions from ours on the nonlinear perturbations and are not compared. We have illustrated the use of our results in this paper in the derivation of control laws during a very specific emergency, for interconnected power systems, by posing the alert state control problem as a steering problem in [11].

II. Notation

The dynamical systems that we study are *differential dynamical systems* (DDS) with finite dimensional *vector spaces* as input, output, and state space, respectively $\mathbb{R}^{n_i}$, $\mathbb{R}^{n_o}$, and $\mathbb{R}^n$, with the representation

$$\dot{x} = f(x, u, t) \tag{II.1}$$

$$y = h(x, t) \tag{II.2}$$

where $t \in \mathbb{R}_+$, f is a C^0 function from $\mathbb{R}^n \times \mathbb{R}^{n_i} \times \mathbb{R}_+ \to \mathbb{R}^n$ which is globally Lipschitz continuous in its first argument (to guarantee uniqueness of solution to (2.1) when the initial condition is given), and h is a C^0 function from $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n_o}$. *Finite dimensional linear dynamical systems* (FDLS) with a *bounded realization* are defined as differential dynamical systems of the form (II.3), (II.4):

$$\dot{x} = A(t)x + B(t)u \tag{II.3}$$

$$y = C(t)x \tag{II.4}$$

$$\text{with } \|A(\cdot)\|, \|B(\cdot)\|, \|C(\cdot)\| \text{ bounded on } \mathbb{R}_+. \tag{II.5}$$

III. Characterization of Controllability for Finite Dimensional Linear Systems

The definitions and propositions of this section are well known, although not standardized. We restate them here to establish the terminology and notation. The definitions are drawn from Silverman [12], and the proofs may be found in standard books (see, e.g., [1]). To obtain the desired characterization define, for fixed $t_0 \in \mathbb{R}_+$, the linear map $\mathcal{L}_R$ (called the *reachability map*) from $L_2^n([t_0, t_0 + T])$ to $\mathbb{R}^n$ by

$$\mathcal{L}_R u = \int_{t_0}^{t_0+T} \Phi(t_0 + T, \tau) B(\tau) u(\tau)\, d\tau \tag{III.1}$$

where $\Phi(t, \tau)$ is the state-transition function for the linear system. Then at $t = t_0 + T$, (III.1) leads to

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[1] Since this paper was written, the work of Dacka [3], which is similar in spirit, has appeared. His main result is a generalization of our Theorem V.1. This generalization can be derived from our methods as well (see Section V-A).

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