3) the resultant $R(x)$ of $D_1(x, z_2)$ and $D_2(x, z_2)$ has no real roots in the interval $-1 \leqslant x \leqslant 1$.

Noting that the conditions 1) and 2) are obtained as a direct consequence of applying the Schussler's result for (10) at $x = 0$, and from the discussion in the previous section, it can be seen that these three conditions only guarantee the following.

$$D(x, z_2) = 0, \quad -1 \leqslant x \leqslant 1, \quad |z_2| < 1 \text{ or } |z_2| > 1. \tag{11}$$

To make sure that the above is true only for $|z_2| < 1$ the following additional condition should also be included.

$$\left| \frac{d_0(x)}{d_{2n_2}(x)} \right| < 1, \quad -1 \leqslant x \leqslant 1. \tag{12}$$

However, following the argument given by Bose for the "if part" of his theorem, it can be seen that if $D(x, z_2)$ is guaranteed to be stable at $x = 0$, then satisfaction of his condition 3) assures that $D(x, z_2)$ is stable for $-1 \leqslant x \leqslant 1$ as well. It then follows that the additional condition to be included to Bose's theorem to make it necessary and sufficient is

$$\left| \frac{d_0(0)}{d_{2n_2}(0)} \right| < 1.$$

References

[1] H. W. Schussler, "A stability theorem for discrete systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-24, pp. 87-89, Feb. 1976.

[2] N. K. Bose, "Implementation of a new stability test for two-dimensional filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-25, pp. 117-120, Apr. 1977.

Corrections to "Fortran Subroutines for the Solution of Toeplitz Sets of Linear Equations"

T. R. HOPKINS

Abstract—**Corrections and improvements to previously published subroutines are presented.**

Recently in [1] routines were presented for the solution of both symmetric and nonsymmetric Toeplitz systems of M linear equations. Both the routines were tested on ICL 2960 and CDC 7600 machines. The following faults were found.

1) Both routines failed to solve the trivial case of a single equation correctly. In fact, they returned $x = b/a^2$ as the solution to $ax = b$. The symmetric routine presented by Farden [2] also failed in this case. This may be avoided by treating a single equation as a special case.

2) As already pointed out in [3], the nonsymmetric routine accesses array elements which do not have to be specifically set by the user, namely $A(M)$ and $R(M)$. This caused the program to halt on both the machines on which it was tested. The correction given in [3] is to assign zero to these elements. A cleaner solution is to eliminate the argument RZERO and for the user to provide the whole of the first row and column of the matrix.

3) Neither routine tests that all the principal minors of the matrix are nonzero (within the bounds of machine arithmetic), although a description of how this may be achieved is given in [1]. Without these tests it is possible to obtain results of dubious value without numerical overflow.

Versions of these routines correcting the deficiencies given above and incorporating other minor improvements are available from the author.

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References

[1] S. Zohar, "Fortran subroutines for the solution of Toeplitz sets of linear equations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-27, pp. 656-658, Dec. 1979.

[2] D. C. Farden, "Solution of a Toeplitz set of linear equations," *IEEE Trans. Antennas Propagat.*, vol. AP-24, pp. 906-907, Nov. 1976.

[3] S. Zohar, "Correction to Fortran subroutines for the solution of Toeplitz sets of linear equations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, p. 601, Oct. 1980.

Sufficient Excitation and Stable Reduced-Order Adaptive IIR Filtering

C. RICHARD JOHNSON, JR. AND B. D. O. ANDERSON

Abstract—**A perturbed error system is used to describe the parameter and output error behaviors of reduced-order application of adaptive identifier/filters such as the hyperstable adaptive recursive filter (HARF). Given satisfaction of a sufficient excitation condition, this error system is shown to exhibit a bounded-input, bounded-state (BIBS) property. This implies that, despite order insufficiency, the output and parameter estimates of HARF (and similar adaptive identifier/filter algorithms) remain bounded.**

I. Introduction

In [1] the equivalence of the adaptive infinite impulse response (IIR) filtering problem and the adaptive output error identification problem solved in [2] and [3] is exploited. For asymptotic stability the algorithms of [1]-[3] require that the order of the autoregressive moving average (ARMA) described adaptive identifier/filter (AIF) match or exceed that of the minimal ARMA process generating the desired output. This paper addresses the reduced-order AIF question. That is, if the order of the AIF is lower than that necessary to yield no output error but for some bounded parameter values could yield a bounded output error, will the output and parameters of the AIF remain bounded?

Resolution of this robustness question is based on study of the perturbed error system arising from reduced-order application of [4] and [5]. A limited affirmative answer to the boundedness question is provided in the next section by Lyapunov function analysis. The third section comments on the sufficient excitation requirement arising in this approach to reduced-order adaptive IIR filtering analysis.

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II. Reduced-Order Adaptive Identifier/Filter Error System BIBS Stability

Consider the reduced-order use of the adaptive IIR filter HARF of [1] and [5]:

$$\hat{y}(k) = \sum_{i=1}^{n} \hat{a}_i(k)\, z(k-i) + \sum_{j=1}^{m} \hat{b}_j(k)\, u(k-j) \tag{1}$$

$$\hat{a}_i(k+1) = \hat{a}_i(k) + \mu_i z(k-i)\, v(k), \qquad \mu_i > 0 \tag{2}$$

$$\hat{b}_j(k+1) = \hat{b}_j(k) + \rho_j u(k-j)\, v(k), \qquad \rho_j > 0 \tag{3}$$

$$v(k) = y(k) - z(k) + \sum_{i=1}^{n} c_i [y(k-i) - z(k-i)] = \frac{y(k) - \hat{y}(k) + \sum_{i=1}^{n} c_i [y(k-i) - z(k-i)]}{1 + \sum_{i=1}^{n} \mu_i z^2(k-i) + \sum_{j=1}^{m} \rho_j u^2(k-j)} \tag{4}$$

$$z(k) = \sum_{i=1}^{n} \hat{a}_i(k+1)\, z(k-i) + \sum_{j=1}^{m} \hat{b}_j(k+1)\, u(k-j) \tag{5}$$

(or the output error identifier in [2] and [3]) in matching the desired output

$$y(k) = y_M(k) + y_U(k) \tag{6}$$

where y_M is the modeled portion of y and has the same order as the identifier filter

$$y_M(k) = \sum_{i=1}^{n} a_i y_M(k-i) + \sum_{j=1}^{m} b_j u(k-j) \tag{7}$$

and $y_U(k)$ is the unmodeled portion of y. Note that the extraction of y_M from y is, in this development, arbitrary. (Note also that $y_U(k)$ could represent a noise signal.) A practical concern is whether or not $z(k)$ and, subsequently, $\hat{y}(k)$ in (1)–(5) will remain bounded if $y_U(k)$ remains bounded. Sought, using (6) in (1)–(5), will be the implication that

$$\|y_U(k)\|^2 \leq \delta^2 < \infty, \quad \forall k \;\Rightarrow\; \|y(k) - z(k)\|^2 \leq \epsilon^2 < \infty, \quad \forall k > \bar{k}. \tag{8}$$

In [1]–[3] it was proven that $\delta^2 = 0 \Rightarrow \epsilon^2 \to 0$. With $\delta^2 \neq 0$ it is clear that, in general, $\epsilon^2 \neq 0$. Using [5], as shown in [6], the perturbed composite output and parameter error system can be written as

$$\boldsymbol{e}(k+1) = \boldsymbol{A}\boldsymbol{e}(k) + \boldsymbol{b}w(k) \tag{9}$$

$$v(k) = \boldsymbol{h}^T \boldsymbol{e}(k) + dw(k) + p(k) \tag{10}$$

$$w(k) = \boldsymbol{\phi}^T(k)\boldsymbol{x}(k) - \alpha \boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)\, v(k) \tag{11}$$

$$\boldsymbol{\phi}(k+1) = \boldsymbol{\phi}(k) - \boldsymbol{\Gamma} v(k)\, \boldsymbol{x}(k) \tag{12}$$

where $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > 0$, $\alpha > \frac{1}{2}$, and all other variables are as defined in [5] (e.g., $\boldsymbol{x}$ is the composite vector of n past *a posteriori* output estimates z and m past inputs u, and $\boldsymbol{\phi}$ is the vector of parameter estimate errors) except $\boldsymbol{e}$, which is now

$$\boldsymbol{e}^T(k) \triangleq [\, y_M(k-1) - z(k-1) \cdots y_M(k-n) - z(k-n)] \tag{13}$$

and

$$p(k) \triangleq y_U(k) + \sum_{i=1}^{n} c_i y_U(k-i) \tag{14}$$

which did not appear in the free error system of [4] and [5].

Evaluating the Lyapunov function of [5]

$$V(k) = \boldsymbol{e}^T(k)\boldsymbol{P}\boldsymbol{e}(k) + \tfrac{1}{2}\boldsymbol{\phi}^T(k)\boldsymbol{\Gamma}^{-1}\boldsymbol{\phi}(k) \tag{15}$$

which, due to the required [1]–[3] strict position reality (SPR) of $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{h}, d)$, yields [6]

$$\begin{aligned} \Delta V(k) &\triangleq V(k+1) - V(k) \\ &= -[\boldsymbol{e}^T(k)\boldsymbol{q} - \nu w(k)]^2 - \epsilon \boldsymbol{e}^T(k)\boldsymbol{L}\boldsymbol{e}(k) \\ &\quad - (\alpha - \tfrac{1}{2})\, v^2(k)\, \boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k) - p(k)\, w(k). \end{aligned} \tag{16}$$

Due to the positive definiteness of ϵ, $\boldsymbol{\Gamma}$, and $\boldsymbol{L}$ and the requirement that $\alpha > \frac{1}{2}$, $\Delta V(k) \leq 0$ when $p(k) = 0$. To evaluate $\Delta V(k)$ for nonzero $p(k)$, substitute (10) into (11) and solve for $w(k)$ as

$$\begin{aligned} w(k) = [1 + \alpha d \boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)]^{-1} [\boldsymbol{\phi}^T(k)\boldsymbol{x}(k) \\ - \alpha \boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)(\boldsymbol{h}^T\boldsymbol{e}(k) + p(k))]. \end{aligned} \tag{17}$$

Substitution of (17) into (16) yields

$$\begin{aligned} V(k) = &-\left[\boldsymbol{e}^T(k)\boldsymbol{q} - \frac{\nu\boldsymbol{\phi}^T(k)\boldsymbol{x}(k)}{1 + \alpha d\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)} + \frac{\nu\alpha\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)}{1 + \alpha d\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)}(\boldsymbol{h}^T\boldsymbol{e}(k) + p(k))\right]^2 \\ &- \epsilon\boldsymbol{e}^T(k)\boldsymbol{L}\boldsymbol{e}(k) - \left(\alpha - \frac{1}{2}\right)v^2(k)\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k) \\ &- \frac{\boldsymbol{\phi}^T(k)\boldsymbol{x}(k)\,p(k)}{1 + \alpha d\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)} + \frac{\alpha\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)}{1 + \alpha d\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)} \\ &\cdot (\boldsymbol{h}^T\boldsymbol{e}(k) + p(k))\, p(k). \end{aligned} \tag{18}$$

Observation 1: Assuming that $p(k)$ remains bounded, i.e.

$$0 < |p(k)| < N < \infty \qquad \forall k \tag{19}$$

and that large $\|\boldsymbol{\phi}(k)\|$ implies large $\|\boldsymbol{x}^T(k)\,\boldsymbol{\phi}(k)\|$, i.e., sufficient excitation occurs, there exists some $0 < \beta < \infty$, such that

$$\|\boldsymbol{e}(k)\|^2 + \|\boldsymbol{\phi}(k)\|^2 > \beta^2 \tag{20}$$

for which $\Delta V(k) \leq 0$.

Justification of Observation 1: Three possibilities must be considered: 1) $\|\boldsymbol{e}\| > 1$ and large relative to $\|\boldsymbol{\phi}\|$ (and $\|\boldsymbol{\phi}^T\boldsymbol{x}\|$) and $|p|$, 2) $\|\boldsymbol{\phi}\|$ (and $\|\boldsymbol{\phi}^T\boldsymbol{x}\|$) > 1 and large relative to $\|\boldsymbol{e}\|$ and $|p|$, and 3) both $\|\boldsymbol{e}\|$ and $\|\boldsymbol{\phi}\|$ (and $\|\boldsymbol{\phi}^T\boldsymbol{x}\|$) are greater than one and large relative to $|p|$. In each case negative quadratics of the large terms will dominate any sum leaving ΔV in (18) negative. □

Therefore, given the respective output and parameter error interpretations of $\boldsymbol{e}$ and $\boldsymbol{\phi}$, for output and parameter estimate errors of summed squared norm greater than β^2, the algorithm causes their decay since $\Delta V(k)$ is negative.

To establish that once (20) is dissatisfied the increase in $\|\boldsymbol{e}\| + \|\boldsymbol{\phi}\|$ is bounded, form

$$\begin{bmatrix} \boldsymbol{e}(k+1) \\ \boldsymbol{\phi}(k+1) \end{bmatrix} = \boldsymbol{F}(k) \begin{bmatrix} \boldsymbol{e}(k) \\ \boldsymbol{\phi}(k) \end{bmatrix} + \boldsymbol{G}(k)\, p(k). \tag{21}$$

Such a forced, time-varying system description is derived by solution of (10) and (11) for

$$\begin{aligned} v(k) = [1 + \alpha d\boldsymbol{x}^T(k)\boldsymbol{\Gamma}\boldsymbol{x}(k)]^{-1} [\boldsymbol{h}^T\boldsymbol{e}(k) \\ + d\boldsymbol{\phi}^T(k)\boldsymbol{x}(k) + p(k)] \end{aligned} \tag{22}$$

its substitution into (12), and the substitution of (17) into (9), yielding

$$\boldsymbol{F}(k)=\begin{bmatrix} \boldsymbol{A}-\dfrac{\alpha \boldsymbol{x}^T(k)\,\boldsymbol{\Gamma x}(k)\,\boldsymbol{b h}^T}{1+\alpha d\boldsymbol{x}^T(k)\,\boldsymbol{\Gamma x}(k)} & \dfrac{\boldsymbol{b x}^T(k)}{1+\alpha d\boldsymbol{x}^T(k)\,\boldsymbol{\Gamma x}(k)} \\ \dfrac{-\boldsymbol{\Gamma x}(k)\,\boldsymbol{h}^T}{1+\alpha d\boldsymbol{x}^T(k)\,\boldsymbol{\Gamma x}(k)} & \boldsymbol{I}-\dfrac{d\boldsymbol{\Gamma x}(k)\,\boldsymbol{x}^T(k)}{1+\alpha d\boldsymbol{x}^T(k)\,\boldsymbol{\Gamma x}(k)} \end{bmatrix} \tag{23}$$

and

$$\boldsymbol{G}(k)=\begin{bmatrix} \dfrac{-\boldsymbol{b x}^T(k)\,\boldsymbol{\Gamma x}(k)}{1+\alpha d\boldsymbol{x}^T(k)\,\boldsymbol{\Gamma x}(k)} \\ \dfrac{-\boldsymbol{\Gamma x}(k)}{1+\alpha d\boldsymbol{x}^T(k)\,\boldsymbol{\Gamma x}(k)} \end{bmatrix}. \tag{24}$$

Formation of (21) validates the approach proposed in [7] for interpretation of the reduced-order modeling problem with the algorithms of [1]-[3] such that the unmodeled output serves as a forcing function into the homogeneous equations of the adequately ordered case. Due to the boundedness of $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{h}$, and d and the normalization of $\boldsymbol{x}(k)$, $\boldsymbol{F}(k)$ and $\boldsymbol{G}(k)$ are bounded, i.e.,

$$0<\|\boldsymbol{F}(k)\|<\gamma_1<\infty \tag{25}$$

$$0<\|\boldsymbol{G}(k)\|<\gamma_2<\infty. \tag{26}$$

Observation 2: If $p(k)$ is bounded, as in (19), and $\|\boldsymbol{e}(k)\|+\|\boldsymbol{\phi}(k)\|$ is bounded by β^2, then the succeeding state norm $\|\boldsymbol{e}(k+1)\|+\|\boldsymbol{\phi}(k+1)\|$ is bounded.

Justification of Observation 2: Reformulate (21) using properties of matrix norms [8] as

$$\left\|\begin{matrix}\boldsymbol{e}(k+1)\\ \boldsymbol{\phi}(k+1)\end{matrix}\right\| \leqslant \|\boldsymbol{F}(k)\| \left\|\begin{matrix}\boldsymbol{e}(k)\\ \boldsymbol{\phi}(k)\end{matrix}\right\| + \|\boldsymbol{G}(k)\|\;|p(k)|. \tag{27}$$

Given (19), (25), (26), and dissatisfaction of (20) such that

$$\left\|\begin{matrix}\boldsymbol{e}(k)\\ \boldsymbol{\phi}(k)\end{matrix}\right\|^2 = \|\boldsymbol{e}(k)\|^2+\|\boldsymbol{\phi}(k)\|^2 \leqslant \beta^2 \tag{28}$$

(27) becomes

$$\left\|\begin{matrix}\boldsymbol{e}(k+1)\\ \boldsymbol{\phi}(k+1)\end{matrix}\right\| < \gamma_1\beta+\gamma_2 N. \qquad \square \tag{29}$$

Combining Observations 1 and 2 leads to Observation 3.

Observation 3: If $p(k)$ is bounded, as in (19), and large $\|\boldsymbol{\phi}(k)\|$ implies large $\|\boldsymbol{x}^T(k)\boldsymbol{\phi}(k)\|$, i.e., sufficient excitation occurs, then $\|\boldsymbol{e}(k)\|+\|\boldsymbol{\phi}(k)\|$ in (9)-(12) is asymptotically bounded and the objective (8) is achieved.

Justification of Observation 3: If (20) is satisfied, then from Observation 1, $\Delta V(k)\leqslant 0$, implying a decay in $\|\boldsymbol{e}(k+1)\|^2+\|\boldsymbol{\phi}(k+1)\|^2$ until (20) is dissatisfied. Assume, without loss of generality, that

$$(\gamma_1\beta+\gamma_2 N)^2>\beta^2. \tag{30}$$

Then from Observation 2, once any initially bounded $\|\boldsymbol{e}(0)\|^2+\|\boldsymbol{\phi}(0)\|^2$ decays inside β^2 it will never exceed $(\gamma_1\beta+\gamma_2 N)^2$. $\square$

These observations are summarized in Fig. 1. Trajectory ① is impossible due to Observation 1 and Observation 2 prohibits trajectory ②. This leaves only eventually bounded trajectories similar to ③ as possibilities as noted in Observation 3. Note that the trajectories can occur only in the first quadrant due to the positivity of $\|\boldsymbol{e}\|$ and $\|\boldsymbol{\phi}\|$.

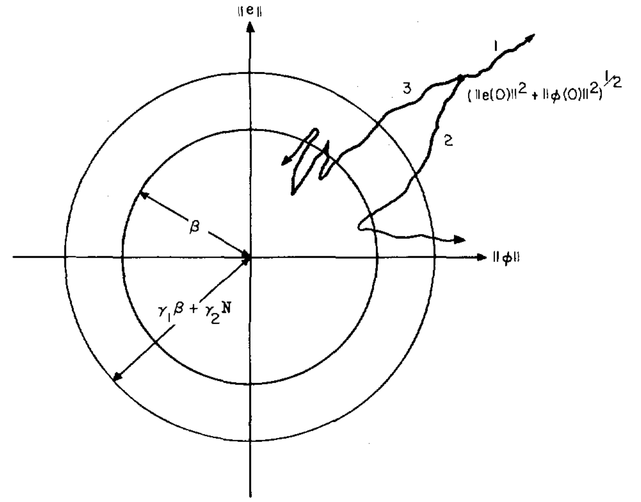

Fig. 1. Error trajectories.

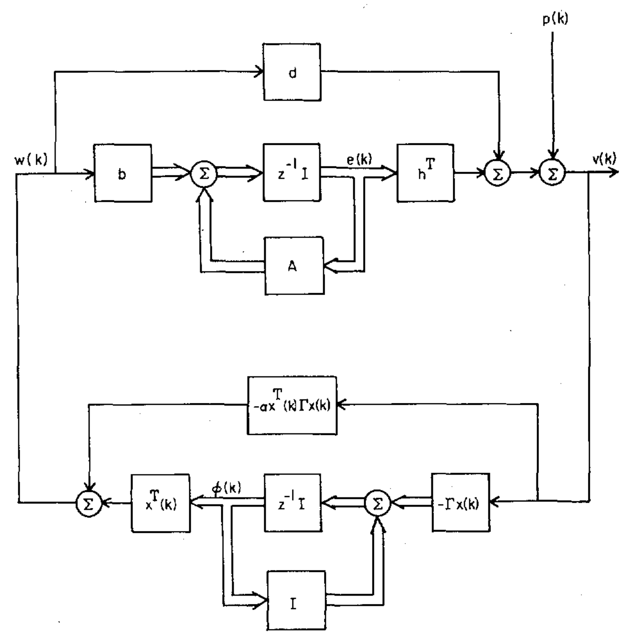

Fig. 2. Perturbed error system.

III. Sufficient Excitation Requirement

The validity of the insights of the last section rest on the satisfaction of the condition embodied in Observation 1 that for nonzero $p(k)$, large $\|\boldsymbol{\phi}(k)\|$ implies large $\|\boldsymbol{x}^T(k)\boldsymbol{\phi}(k)\|$. This is similar to the nonorthogonality sufficient excitation condition in equation error estimation [9] required for consistent parameter estimation. However, this $\boldsymbol{x}$ includes both estimator inputs and outputs, unlike equation error.

The need for this sufficient excitation requirement on the error system in (9)-(12) is possibly best illustrated by composition of Fig. 2 from (9)-(12). The perturbed error system in Fig. 2 clearly demonstrates the state vector nature of $[\boldsymbol{e}^T(k)\,\boldsymbol{\phi}^T(k)]^T$. The single loop system is also clearly divided into the two passive components, one linear and time-invariant and the other time-varying, isolated for hyperstability analysis [1]-[3]. Consider a nonzero $p(k)$ increasing $\boldsymbol{\phi}(k)$. However, if $\boldsymbol{x}^T(k)\;\boldsymbol{\phi}(k)$ is very small, then the energy in the time-varying component appreciably dissipates through the linear, time-invariant SPR component. Assume, for example, that $\boldsymbol{x}^T(k)\boldsymbol{\phi}(k)=0$ for $k>0$. From (12) $\|\boldsymbol{\phi}\|$ could clearly become incessantly larger. Such a possibility is noted in [6]. The question still remains if, for any bounded u and y_U (or p), $\boldsymbol{\phi}$ can grow without bound due to the boundedness of $\boldsymbol{\phi}^T\boldsymbol{x}$.

The exponential stability of (21) with $p \equiv 0$, proven in [7] for sufficiently rich $\boldsymbol{x}$, would be sufficient to invoke the BIBS property demonstrated in the last section if nonzero p influenced (21) only as a forcing function as explicitly shown. However, $\boldsymbol{F}(k)$ in (21) changes with $p(k) \neq 0$, i.e., the linear appearance of (21) masks its nonlinear nature. This influence of p on $\boldsymbol{F}$ arises since $\boldsymbol{F}$ includes $\boldsymbol{x}$, as shown in (23), some entries of $\boldsymbol{x}$ are $z(k-i)$, $i = 1, \cdots, n$, and z depends via (2), (3), and (5) on v, which depends on p in (10). Therefore, though this characteristic of exponential stability can be extended to the *nonlinear*, time-varying (21), as is done in [10], it is not immediate and in fact is much less transparent than the arguments of the preceding section.

IV. Conclusions

This correspondence established the bounded-input, bounded-state (BIBS) character of the error system associated with HARF and similar adaptive recursive filters and output error identifiers given a sufficient excitation condition. This BIBS property supports the conclusions that, since a stable low-order model of a higher order system always exists, the output estimates and parameter estimates of an AIF such as HARF will remain bounded depite reduced-order application, given sufficient excitation. This sufficient excitation condition is peculiar in that it excludes the near-orthogonality of the vector of autoregressive and moving-average parameter estimate errors with the vector of past outputs and inputs of the AIF when the parameter error vector is large. Whether or not this undesirable condition can occur for an AIF such as HARF has not been ascertained. It was shown in [6] that it is possible for the general underlying perturbed error system derived from [4] and [5]. Quantitative description of this BIBS robustness in reduced-order application can be formally derived [10] from the exponential convergence character of HARF in adequate-order application proven in [11] under a sufficient excitation condition on the filter input alone. As noted in the preceding section, this extension is complicated by the "hidden" nonlinearity of (21).

References

[1] C. R. Johnson, Jr., "A convergence proof for a hyperstable adaptive recursive filter," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 745–749, Nov. 1979.

[2] I. D. Landau, "Unbiased recursive identification using model reference adaptive techniques," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 194–202, Apr. 1976.

[3] ——, "An addendum to 'Unbiased recursive identification using model reference adaptive techniques,'" *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 97–99, Feb. 1978.

[4] Y.-H. Lin and K. S. Narendra, "A new error model for adaptive systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 585–587, June 1980.

[5] C. R. Johnson, Jr., "Another use of the Lin–Narendra error model: HARF," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 985–988, Oct. 1980.

[6] C. R. Johnson, Jr. and B. D. O. Anderson, "Sufficient excitation and stable reduced-order adaptive IIR filtering," in *Proc. 1981 IEEE Int. Conf. Acoust., Speech, Signal Processing*, Atlanta, GA, Mar. 1981, pp. 534–537.

[7] R. R. Bitmead and B. D. O. Anderson, "Performance of adaptive estimation algorithms in dependent random environments," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 788–794. Aug. 1980.

[8] H. H. Rosenbrock and C. Storey, *Mathematics of Dynamical Systems*. New York: Wiley, 1970, p. 118.

[9] C. R. Johnson, Jr., "Adaptive parameter matrix and output vector estimation via an equation error formulation," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-9, pp. 392–397, July 1979.

[10] B. D. O. Anderson and C. R. Johnson, Jr., "On reduced order adaptive output error identification and adaptive IIR filtering," in *Proc. 1981 Joint Automat. Contr. Conf.*, Charlottesville, VA, June 1981, paper FA-4A.

[11] ——, "Exponential convergence of adaptive identification and control algorithms," *Automatica*, vol. 18, Jan. 1982.

Maximum Entropy Power Spectrum Estimation of Signals with Missing Correlation Points

JAE S. LIM AND NAVEED A. MALIK

Abstract—**A computationally simple algorithm has been recently proposed by Lim and Malik [1] to solve the two-dimensional (2-D) maximum entropy (ME) power spectrum estimation (PSE) problem. In this note, we illustrate that this algorithm also solves the ME PSE problem for both 1-D and 2-D signals when the region in which the correlation function is known has any arbitrary shape that includes the origin.**

The problem of power spectrum estimation (PSE) arises in a number of application areas [2]–[6] and a variety of techniques for PSE have been developed in the literature. One technique which has been studied extensively due to its high resolution characteristics is the maximum entropy (ME) method. For one-dimensional (1-D) signals whose known correlation points are connected and include the origin, the ME method is equivalent to autoregressive signal modeling, and thus it leads to a linear problem formulation that is theoretically tractable and computationally attractive [7]. When the known correlation points are not connected, which could arise in situations such as when the correlation points are obtained from nonuniformly sampled data or by an interferometer [8], a computationally simple algorithm has not yet been found, despite some recent attempts [9]. The purpose of this correspondence is to illustrate that the computationally simple algorithm recently proposed by Lim and Malik [1] to solve the two-dimensional (2-D) ME PSE problem also solves the ME PSE problem when the region in which the correlation function is known has any arbitrary shape that includes the origin.

Using 2-D signal notation, the ME PSE problem can be stated [1] as follows:

Given $R_x(n_1, n_2)$ for $(n_1, n_2) \in A$, determine $\hat{P}_x(\omega_1, \omega_2)$ such that $\hat{P}_x(\omega_1, \omega_2)$ has the form

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