Automatica, Vol. 17, No. 5, pp. 703–712
Printed in Great Britain

0005-1098/81/050703-10 $02.00/0
Pergamon Press Ltd

Algebraic Characterization of Fixed Modes in Decentralized Control*

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Algebraic characterizations for the existence of closed-loop fixed modes with decentralized control extend the class of controllers for which fixed modes are present.

Key Words—Controllers; large-scale systems; feedback; multivariable control systems; stabilizers; decentralized control.

Abstract—Algebraic characterizations are presented for the existence of fixed modes of a linear closed-loop system with decentralized feedback control. The class of controllers for which fixed modes are present is extended beyond that currently known.

1. INTRODUCTION

IT IS now becoming recognized that the design of feedback controllers for some applications requires restrictions on the particular system output-input pairs that the controller may connect. As a very simple example, it may be that for a two-input, two-output system, control of input one may only use output one and control of input two may only use output two. Decentralization restrictions of this type often translate into restrictions on the ability of the controller to achieve certain purposes. For example, Wang and Davison (1973) discuss the use for linear, time-invariant, finite-dimensional plants of linear, time-invariant, finite-dimensional and decentralized controllers, and concludes that the decentralization requirement may force the closed-loop system to have certain natural frequencies ('fixed modes') which are independent of the particular controller used.

Fixed modes are defined later in the paper using matrix fraction descriptions; for completeness, we state the definition given by Wang and Davison (1973). Consider the system

$$\dot{x}=Fx+\sum_{i=1}^{m} G_i u_i \quad y_i=H_i' x \qquad (i=1,\ldots,m). \tag{1}$$

Here, $G_i \in R^{n\times \gamma_i}$, $H_i \in R^{n\times \nu_i}$. Let $\mathscr{K}$ be the set of block diagonal matrices

$$\mathscr{K}=\{K|K=\operatorname{diag}(K_1,K_2,\ldots,K_m),\quad K_i\in R^{\gamma_i\times \nu_i}\}. \tag{2}$$

Then the set of fixed modes of $\{F, G_i, H_i, i=1,2,\ldots,m\}$ with respect to $\mathscr{K}$ is defined as

$$\wedge(F,G_i,H_i,\mathscr{K})=\bigcap_{K\in\mathscr{K}} \sigma\left(F+\sum_{i=1}^{m} G_i K_i H_i'\right) \tag{3}$$

where $\sigma(\cdot)$ denotes the set of eigenvalues of $(\cdot)$.

In this paper, we present algebraic tests for the existence of fixed modes. To be sure, 'computational' tests are known such as that of Davison (1977). (For several controllers chosen randomly, one computes the resulting closed-loop natural frequencies; if certain frequencies are common to all controllers, with probability one, these are fixed modes.) Such tools at the very least deny the theoretical insight that our tests appear to give, particularly in relation to identifying the parts of the overall system which may be held to be responsible for the occurrence of the fixed modes. In particular, our tests clarify the origins of fixed modes in a decentralized system examined in some recent correspondence (Wang, 1978; Fessas, 1979a; Ikeda and Siljak, 1979). Further arguments for the potential worth of algebraic tests are given below.

Actually, an algebraic test has been available for a restricted situation—that where the plant has two inputs and two outputs and input i can only be controlled from output i for $i=1,2$. This test has been developed in Clements (1979) and Fessas (1979b).

A further advantage of our procedures is that we have been able to extend a result of Wang and Davison (1973) on the range of controller

*Received 26 November 1979; revised 24 November 1980. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor E. J. Davison.

Work supported by the Australian Research Grants Committee.

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i.e. if and only if $A(s)$, $B(s)$ are not coprime. This result (albeit a state-variable version) is contained in Wang and Davison (1973) and further discussed in Davison, Gesing and Wang (1978).

(2) Suppose that $m=2$, $\gamma_1=\nu_1=\gamma_2=\nu_2=1$ and that $\operatorname{rank}[A(s_0)\ B(s_0)]=2$, i.e. $A(s)$, $B(s)$ are coprime. Then either

$$A_1(s_0)=B_1(s_0)=0$$

or

$$A_2(s_0)=B_2(s_0)=0$$

This result can be recovered from Clements (1979) and Fessas (1979b).

By exploiting our freedom to change MFDs using unimodular transformations, we can more graphically illustrate the effect of equation (12). To clarify matters, suppose that $\nu_m=\gamma_m=0$, and reorder the input and output blocks so that $i_1=1,\ldots,i_j=j$. Let U_{s_0} be a constant possibly complex matrix with determinant unity such that

$$U_{s_0}[A_1(s_0)\ B_1(s_0)\ldots A_j(s_0)\ B_j(s_0)] = \begin{bmatrix} \hat{A}_1(s_0) & & \hat{B}_j(s) \\ 0 & \ldots & 0 \end{bmatrix}$$

where $[\hat{A}_1(s_0)\ldots\hat{B}_j(s_0)]$ has full row rank. Setting

$$\hat{\mathscr{A}}_1(s_0)=[\hat{A}_1(s_0)\ldots\hat{A}_j(s_0)],$$
$$\mathscr{A}_2(s_0)=U_{s_0}[A_{j+1}(s_0)\ldots A_{m-1}(s_0)]$$

and $\hat{\mathscr{B}}_1(s_0)$, $\mathscr{B}_2(s_0)$ similarly, we have

$$U_{s_0}A(s_0)=\begin{bmatrix} \hat{\mathscr{A}}_1(s_0) & \mathscr{A}_2(s_0) \\ 0 & \end{bmatrix}$$

$$U_{s_0}B(s_0)=\begin{bmatrix} \hat{\mathscr{B}}_1(s_0) & \mathscr{B}_2(s_0) \\ 0 & \end{bmatrix}$$

Equation (12) guarantees that the zero block in $U_{s_0}A(s_0)$ has more columns than rows. Because $U_{s_0}\bar{A}(s_0)$ must have the same zero block (as a trivial calculation shows), we see immediately that $U_{s_0}\bar{A}(s_0)$ is singular. It is easy to construct a unimodular matrix of real polynomials $U(s)$ such that $U(s_0)=U_{s_0}$. Thus we have shown:

Corollary 3.2 With the same hypotheses as Theorem 3.1, but with $\nu_m=\gamma_m=0$, there exists a (real) unimodular transformation $U(s)$ with $U(s_0)=U_{s_0}$ of unity determinant such that the matrices $U(s_0)A(s_0)$, $U(s_0)B(s_0)$ have a zero block; in $U(s_0)A(s_0)$, this block has more rows than columns, and its columns are those corresponding to a selection $\{i_1,i_2,\ldots,i_j\}$ of the m blocks of outputs. The rows of the zero block of $U(s_0)B(s_0)$ are the same as those of $U(s_0)A(s_0)$, and the columns correspond to a selection $\{i_1,i_2,\ldots,i_j\}$ of m blocks of inputs.

It is possible, though we do not prove it here, to find a real unimodular matrix $U(s)$ which interpolates unity determinant constant matrices $U_{s_0}, U_{s_1},\ldots$ at arbitrary points $s_0,s_1,\ldots$ subject to $s_i=s_j^*$ implying $U_{s_i}=U_{s_j}^*$. This allows extension of the corollary to encompass all fixed modes; the case of $\gamma_1=\gamma_2=\nu_1=\nu_2=1$, $m=2$ is treated in Clements (1979).

So far, we have considered only constant feedback gains K_i. We now turn to dynamic feedback.

Theorem 3.3 Consider the MFD $A^{-1}(s)B(s)$ with the feedback pattern (9) defined by the positive integers $\nu_1,\ldots,\nu_{m-1}$, $\gamma_1,\ldots,\gamma_{m-1}$ and non-negative ν_m, γ_m and suppose it has a fixed mode at s_0. Consider the use of controllers of the form, in Laplace transform notation

$$U_i(s)=-K_i(s)Y_i(s)+V_i(s) \tag{13}$$

in lieu of equation (9), where $K_i(s)$ is any realizable† transform function matrix of appropriate dimensions, arbitrary save that $K_i(s)$ can be written as $D_i(s)C_i^{-1}(s)$ in which $C_i(s_0)$ and $D_i(s_0)$ are finite.‡ Then the closed-loop system associated with (13) has a mode at s_0.

Proof. It is easily verified that the closed-loop system has transfer function matrix

$$\begin{bmatrix} C_1(s) \\ C_2(s) \\ \vdots \\ C_{m-1}(s) \\ I \end{bmatrix} [A_1(s)C_1(s)+B_1(s)D_1(s)\ldots A_{m-1}(s)$$

$$\times C_{m-1}(s)+B_{m-1}(s)D_{m-1}(s)\quad A_m(s)]^{-1}B(s)$$

†To fix ideas, we might require $K_i(s)$ to be the Laplace transform of a

$$L_a(t)+\sum_{j=0}^{\infty} L_j\delta(t-t_j)$$

with

$$\int_0^{\infty} e^{\sigma t}\|L_a(t)\|dt$$

and

$$\sum_0^{\infty} e^{\sigma t_j}\|L_j\|$$

finite for some real σ.

‡Note that $C_i(s)$ and $D_i(s)$ are *not* necessarily polynomial.

and it has a mode at s_0 if and only if

$$[A_1(s_0)C_1(s_0)+B_1(s_0)D_1(s_0)\dots A_{m-1}(s_0) \times C_{m-1}(s_0)+B_{m-1}(s_0)D_{m-1}(s_0) \quad A_m(s_0)]$$

has singular determinant. This is an easy consequence of equation (12).

A more limited version of this result, proved using state-variable descriptions, and requiring the controllers to be finite-dimensional, can be found in Wang and Davison (1973).

The relevance of the graph of a system to the development of conditions for decentralized controllability has been stressed in Corfmat and Morse (1976b) and to a lesser extent Fessas (1979a); as these references explain, decentralized controllability is closely linked to the absence of fixed modes. We explore the idea briefly here. Instead of introducing graph theoretic ideas, we shall simply postulate that, after reordering of input and output blocks if necessary, the system transfer function matrix has the form

$$W(s)=\begin{bmatrix} W^{11}(s) & W^{12}(s) \\ 0 & W^{22}(s) \end{bmatrix}. \qquad (14)$$

[The connection of such an assumption with a graph theoretic property is covered in Corfmat and Morse (1976b).] In equation (14), $W^{11}(s)$ is the transfer function from $[u_1' \dots u_k']'$ for some $k < m$ to $[y_1' \dots y_k']'$, and $W^{12}(s)$, $W^{22}(s)$ have obvious similar interpretations; the main point illustrated by (14) is that there is no path in the system from u_i, $1 \leqq i \leqq k$ to y_j, $k < j \leqq m$.

Suppose that $A^{-1}(s)B(s)$ is a left coprime MFD for $W(s)$. Without loss of generality, we may assume $A(s)$ is upper triangular [unimodular transformation to Hermite form can ensure this, (Wolovich, 1974)]. Then $B(s)$ has the same block triangular structure as $W(s)$. Clearly, MFDs for $W^{11}(s)$ and $W^{22}(s)$ are induced [via submatrices of $A(s)$ and $B(s)$], and paralleling ideas of Cormat and Morse (1976b), we assert that the set of fixed modes associated with the feedback pattern of equation (9) is the union of the sets of fixed modes of the MFDs of $W^{11}(s)$ and $W^{22}(s)$ associated with the feedback pattern of (9), $i = 1,\dots,k$ being relevant for $W^{11}(s)$ and $i > k$ for $W_{22}(s)$. This assertion follows easily through application of Theorem 3.1, and also extends easily to block triangular $W(s)$ with more blocks than that of equation (14).

4. FIXED MODES USING STATE-VARIABLE DESCRIPTIONS

State-variable descriptions have been used for some time in discussing fixed modes, and it is pertinent to try to characterize algebraically fixed modes in state-variable terms. We shall therefore recall the fixed mode description in state-variable terms, and then show how the basic theorem of Section 2 can be applied.

The open loop system is

$$\dot{\mathbf{x}} = F\mathbf{x} + G_1\mathbf{u}_1 + \cdots + G_m\mathbf{u}_m \qquad (15a)$$

$$\mathbf{y}_i = H_i\mathbf{x} \quad i = 1,\dots,m \qquad (15b)$$

and we allow feedback of the form of equation (9), i.e.

$$\mathbf{u}_i = -N_i\mathbf{y}_i + \mathbf{v}_i \qquad (16)$$

for $i = 1,\dots,m-1$, allowing the possibilities of some inputs or outputs not being involved in feedback. The fixed modes associated with the feedback pattern (16) defined by positive integers $p_1,\dots, p_{m-1}, q_1,\dots, q_{m-1}$ and non-negative p_m, q_m(p_j and q_j being the dimension of u_j and y_j, respectively) are those eigenvalues of

$$F + \sum_{i=1}^{m-1} G_iN_iH_i$$

which are independent of N_i, if any.

To provide N_i-independent conditions for fixed modes, we shall prove a lemma which enables application of the main result of Section 2. This lemma proves easier to state if we temporarily cause $\dim u_i = \dim y_i$. This is done as follows. Let $\pi_i = \max(p_i, q_i)$. By adding columns of zeros to G_i when $p_i < q_i$ and rows of zeros to H_i when $q_1 < p_1$ we can create matrices $\bar{G}_i$, $\bar{H}_i$ with π_i columns and rows respectively. With $\bar{N}_i$ denoting an arbitrary $\pi_i \times \pi_i$ matrix, it follows that the set

$$\sum_{i=1}^{m-1} G_iN_iH_i$$

achievable with arbitrary N_i is the same as the set

$$\sum_{i=1}^{m-1} \bar{G}_i\bar{N}_i\bar{H}_i$$

achievable with arbitrary $\bar{N}_i$. We now state the following.

Lemma 4.1 With quantities as defined above and with λ any complex number

$$\lambda I - F - \sum_{i=1}^{m-1} G_iN_iH_i$$

has rank $n - \alpha$ ($n = \dim F, \alpha \geqq 0$) for all real N_i if and only if

$$\operatorname{rank}\begin{bmatrix} \lambda I-F & \bar{G}_1 & \dots & \dots & \bar{G}_{m-1} \\ \bar{H}_1 & L_1 & 0 & \dots & 0 \\ \vdots & 0 & L_2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \bar{H}_{m-1} & 0 & 0 & \dots & L_{m-1} \end{bmatrix} < (n-\alpha)+\sum_{i=1}^{m-1}\pi_i \tag{17}$$

for all $\pi_i \times \pi_i$ real matrices L_i.

Proof. Suppose temporarily that the L_i all have full rank; with M the matrix on the left-hand side of (17), we have

$$\begin{bmatrix} I & -\bar{G}_1 L_1^{-1} & \dots & -\bar{G}_{m-1}L_{m-1}^{-1} \\ 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{bmatrix} M$$

$$= \begin{bmatrix} \lambda I - F - \sum_{i=1}^{m-1} \bar{G}_i L_i^{-1}\bar{H}_i & 0 & \dots & 0 \\ \bar{H}_1 & L_1 & & \cdot \\ \vdots & & \ddots & \vdots \\ \bar{H}_{m-1} & & & L_{m-1} \end{bmatrix}$$

and it is evident that

$$\operatorname{rank} M = \operatorname{rank}\left[\lambda I - F - \sum_{i=1}^{m-1}\bar{G}_i L_i^{-1}\bar{H}_i\right] + \sum_{i=1}^{m-1}\pi_i.$$

Now if

$$\operatorname{rank}\left[\lambda I - F - \sum_{i=1}^{m-1}\bar{G}_i\bar{N}_i\bar{H}_i\right] < n-\alpha \textit{ for all } \bar{N}_i,$$

then in particular this is true for nonsingular $\bar{N}_i$, and accordingly (17) holds for full rank L_i. But then it must also hold for all L_i. The argument is easily reversible to also yield the reverse conclusion. □

To tie this to the main result of Section 2, set

$$A_1 = \begin{bmatrix} \lambda I\text{-}F \\ \bar{H}_1 \\ \vdots \\ \bar{H}_{m-1} \end{bmatrix} \quad A_2 = \begin{bmatrix} \bar{G} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad A_m = \begin{bmatrix} \bar{G}_{m-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ \cdot \\ \vdots \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ I \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad B_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I \end{bmatrix} \tag{18}$$

With the K_i in equation (4) corresponding to the L_i in (17), the matrix on the left of (17) is $[A_1 + B_1K_1 \dots A_m + B_mK_m]$. The number of rows is the same as the number of columns, viz.

$$n + \sum_{i=1}^{m-1} \pi_i$$

and the rank is less than

$$n - \alpha + \sum_{i=1}^{m-1}\pi_i.$$

So the quantities δ and ε in (4) are both equal to α. We shall now consider several choices for the set $\mathscr{I} = \{i_1, i_2, \dots, i_j\}$ arising in the statement of the main theorem.

Case 1 $\mathscr{I} = \{1\}$. The main theorem yields

$$\operatorname{rank} A_1 = \operatorname{rank}\begin{bmatrix} \lambda I - F \\ \bar{H}_1 \\ \vdots \\ \bar{H}_{m-1} \end{bmatrix} < n - \alpha. \tag{19}$$

Recognize that this is equivalent to the same condition with $\bar{H}_i$ replaced by H_i, and is simply an unobservability condition. Unobservability always produces fixed modes (Wang and Davison, 1973; Davison, Gesing and Wang, 1978).

Case 2 $\mathscr{I} = \{1, 2, \dots, m\}$. Then

$$\operatorname{rank}\begin{bmatrix} \lambda I - F & \bar{G}_1 & \bar{G}_2 & \dots & \bar{G}_m & 0 & \dots & 0 \\ \bar{H}_1 & & & & & I & & \cdot \\ \vdots & & & & & & \ddots & \vdots \\ \bar{H}_{m-1} & & & & & & & I \end{bmatrix} < n - \alpha + \sum_{i=1}^{m-1}\pi_i.$$

Since the identity matrices are of rank $\pi_1, \dots, \pi_{m-1}$, an obvious simplification yields

$$\operatorname{rank}[\lambda I - F \quad \bar{G}_1 \quad \bar{G}_2 \dots \bar{G}_m] = \operatorname{rank}[\lambda I - F \quad G_1 \quad G_2 \dots G_m] < n - \alpha. \tag{20}$$

This corresponds to an uncontrollable system.

Case 3 $1 \notin \mathscr{I}$, but $\mathscr{I}$ is otherwise arbitrary. We can show this is impossible. On the one hand, if (5) holds

$$\operatorname{rank}[B_{i_1} \dots B_{i_j}] < \text{number of columns in } [B_{i_1} \dots B_{i_j}].$$

On the other hand, as (18) shows, each B_i for $i \geq 2$ has a nonzero identity submatrix occupying different rows to those occupied in B_j for $j \neq 1$, $j \geq 2$. So we must have

$$\operatorname{rank}[B_{i_1} \dots B_{i_j}] = \text{number of columns in } [B_{i_1} \dots B_{i_j}]$$

Case 4 $\mathscr{I} = \{1, 2, \dots, l\}$, $1 < l < m-1$. To within inessential reordering, this is the remaining case to be considered. The rank condition Theorem 2.1 becomes

$$\operatorname{rank}\begin{bmatrix} \lambda I - F & \bar{G}_1 & \dots & \bar{G}_l & 0 & \dots & 0 \\ \bar{H}_i & 0 & \dots & 0 & I & & 0 \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ \bar{H}_l & & & & & & I \\ \bar{H}_{l+1} & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ \bar{H}_{m-1} & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} < n - \alpha + \sum_{j=1}^{l} \pi_j$$

which immediately simplifies to

$$\operatorname{rank}\begin{bmatrix} \lambda I - F & \bar{G}_1 & \dots & \bar{G}_l \\ \bar{H}_{l+1} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \bar{H}_{m-1} & 0 & \dots & 0 \end{bmatrix} < n - \alpha$$

and then

$$\operatorname{rank}\begin{bmatrix} \lambda I - F & G_1 & \dots & G_l \\ H_{l+1} & 0 & \dots & 0 \\ \vdots & \vdots & & \\ H_{m-1} & 0 & \dots & 0 \end{bmatrix} < n - \alpha. \quad (21)$$

The preceding analysis has established the necessity part of the following result.

Theorem 4.1 Consider the system (15) with control law (16) for $i = 1, 2, \dots, m-1$. Then a necessary and sufficient condition for

$$\lambda I - F - \sum_{i=1}^{m-1} G_i N_i H_i$$

to have rank $< n - \alpha$ for all N_i, some fixed complex λ and non-negative α is that for some partition of the set $\{1, \dots, m-1\}$ into disjoint subsets $\{i_1, \dots, i_k\}$ and $\{i_{k+1}, \dots, i_{m-1}\}$, there holds

$$\operatorname{rank}\begin{bmatrix} \lambda I - F & G_{i_1} & \dots & G_{i_k} \\ H_{i_{k+1}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{i_{m-1}} & 0 & \dots & 0 \end{bmatrix} < n - \alpha. \quad (22)$$

As noted above, necessity has been established. Sufficiency follows easily by reversing the necessity argument.

Naturally for equation (22) to hold, λ must be an eigenvalue of F. So if the eigenvalues are known, (22) can be checked. Alternatively, the Smith form of the matrix on the left of (22) could be found, thus allowing checking of (22) without eigenvalue evaluation. This would naturally not be so computationally appealing.

Example. (Wang, 1978; Fessas, 1979a; Ikeda and Siljak, 1979): Take

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$H_1 = [I \ 0 \ 0] \quad H_2 = [0 \ I \ 0] \quad H_3 = [0 \ 0 \ I].$$

Observe that the following matrix has rank 5

$$\begin{bmatrix} -F & G_2 & G_3 \\ H_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is a special case of (22), corresponding to $\{i_1, i_2\} = \{2, 3\}$, $\{i_1\} = 1$, $n = 6$, $\alpha = 0$ and $\lambda = 0$.

Accordingly, the system has a fixed mode at $s=0$. As (22) shows in general, and this example in particular, the presence of fixed modes means that a certain subsystem of the overall system is both uncontrollable and unobservable. The behaviour of the examples of Ikeda and Siljak (1979) can also be explained by these sorts of considerations.

There is an obvious parallel of Theorem 3.3 and of the remarks of the last section on triangular transfer function matrices in terms of state variable ideas.

5. CONCLUSIONS

The two main results of the paper are those of Theorems 3.1 and 4.1, presenting algebraic conditions for fixed modes in terms of matrix fraction and state-variable plant descriptions. The results have indicated that finite-dimensionality of a controller is not essential for fixed modes to be retained under dynamic control, but, as further work based on this paper (Anderson and Moore, 1980) has shown, time-invariance of the controller is relevant.

Let us now list a number of issues which can or might be resolvable using the ideas of this paper. (Recent work of one of the co-authors with A. S. Morse has suggested some of the following.)

(1) Development of the concepts of decentralized controllability and observability (Morse, 1973; Corfmat and Morse, 1976a, b) using matrix fraction descriptions.

(2) Analysis of the 'fine structure' of fixed modes, i.e. their multiplicity, and their detailed location within the system.

(3) Definition of structures which have no fixed modes.

(4) Elimination of fixed modes using nonlinear controllers.

(5) Statement of a condition independent of eigenvalues or Smith forms on $\{F, G_i, H_i, i=1,\ldots, m\}$ for the existence of fixed modes.

(6) Analysis of fixed modes using transfer function matrix ideas. Fixed modes of a (centralized) controllable and observable system are quantities associated with the input/output properties of the system, and thus should be capable of study via the system transfer function matrix.

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APPENDIX: NECESSITY PROOF OF BASIC THEOREM

We proceed with the aid of several lemmas. We shall use the abbreviation g.r. to denote generic rank, sometimes placing a symbol underneath to indicate the variable.

Lemma A.1 Let A be a $\rho\times\gamma$ matrix, B a $\rho\times\nu$ matrix and suppose that for all $\nu\times\gamma$ matrices K

$$\underset{K}{\text{g.r.}}\,[A+BK]<\rho,\gamma \tag{A1}$$

Then

$$\text{rank}\,[A\quad B]=\underset{K}{\text{g.r.}}\,[A+BK]. \tag{A2}$$

Proof. Choose $K=\bar{K}$ to maximize rank$(A+BK)$. Choose a nonsingular E such that

$$E(A+B\bar{K})=\begin{bmatrix}F\\0\end{bmatrix}=\begin{bmatrix}I & G\\0 & 0\end{bmatrix}P.$$

Here, F is a full row rank matrix and P is a permutation matrix. Without loss of generality, $B\neq 0$. The zeros in

$$\begin{bmatrix}I & G\\0 & 0\end{bmatrix}$$

both have positive dimensions as a result of (A1). Suppose that

$$EB=\begin{bmatrix}\hat{B}\\B_2\end{bmatrix} \tag{A3}$$

and consider for arbitrary K_2 the following choice of $E(A+BK)$:

$$E[A+B\bar{K}+B[0 \quad K_2]P]=\begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}P+\begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}[0 \quad K_2]P$$

$$=\begin{bmatrix} I & G+\hat{B}K_2 \\ 0 & B_2K_2 \end{bmatrix}P.$$

Because $\bar{K}$ was chosen to give maximum row rank to $A+BK$, $B_2K_2=0$ and since K_2 is arbitrary, $B_2=0$ in (A3). Further

$$EA=E(A+BK)-EBK=\begin{bmatrix} F \\ 0 \end{bmatrix}-\begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}K=\begin{bmatrix} \hat{A} \\ 0 \end{bmatrix}$$

for some A. Taken with (A3), where $B_2=0$, this establishes that rank $[A \quad B]=\text{rank}[\hat{A} \quad \hat{B}]$. Since

$$[\hat{A} \quad \hat{B}]\begin{bmatrix} I \\ \bar{K} \end{bmatrix}=F$$

has full row rank, so does $[\hat{A} \quad \hat{B}]$; the row rank of $[\hat{A} \quad \hat{B}]$ is equal to the maximum rank of $(A+BK)$, which is the generic rank of this matrix. □

Now, using Lemma A.1, we shall study a slightly more complicated matrix than that considered in Lemma A1.

Lemma A2 Let A, B, K be as defined in Lemma A.1, and let C be a $\rho\times\delta$ matrix. Suppose that for some non-negative α

$$\underset{K}{\text{g.r.}}[C \quad A+BK]<\rho-\alpha. \tag{A4}$$

Then either

$$\text{rank}[C]+\gamma=\underset{K}{\text{g.r.}}[C \quad A+BK] \tag{A5}$$

or

$$\text{rank}[C \quad A \quad B]=\underset{K}{\text{g.r.}}[C \quad A+BK]. \tag{A6}$$

Proof. Let E be a nonsingular $\rho\times\rho$ matrix such that

$$E[C \quad A+BK]=\begin{bmatrix} \hat{C} & A^1+B^1K \\ 0 & A^2+B^2K \end{bmatrix} \tag{A7}$$

with $\hat{C}$ possessing full row rank. Suppose (A5) fails. Since obviously

$$\underset{K}{\text{g.r.}}[C \quad A+BK]>\text{rank}[C]+\gamma$$

is impossible (γ is the number of columns of $A+BK$), one has

$$\underset{K}{\text{g.r.}}[C \quad A+BK]<\text{rank}[C]+\gamma.$$

From (A7), we have

$$\underset{K}{\text{g.r.}}[C \quad A+BK]=\text{rank}\,\hat{C}+\underset{K}{\text{g.r.}}(A^2+B^2K)$$

$$=\text{rank}\,C+\underset{K}{\text{g.r.}}(A^2+B^2K)$$

so that

$$\underset{K}{\text{g.r.}}(A^2+B^2K)<\gamma,$$

the number of columns not only of A but of A^2. Also, since $\hat{C}$ has full row rank, while $[C \quad A+BK]$ does not,

$$\underset{K}{\text{g.r.}}(A^2+B^2K)<\text{number of rows of } A^2.$$

Using Lemma A.1 on the pair A^2, B^2 yields

$$\underset{K}{\text{g.r.}}[C \quad A+BK]=\text{rank}[\hat{C}]+\text{rank}[A^2 \quad B^2]$$

$$=\text{rank}[C \quad A \quad B]. \qquad \square$$

Now we can complicate the situation further again, introducing a collection of variable gain matrices $K_1,\ldots,K_k$. This lemma depends on Lemma A.2.

Lemma A.3 Let $A_1,\ldots A_k$, $B_1,\ldots B_k$ be prescribed matrices with ρ rows, and $\gamma_1,\ldots\gamma_k$, $\nu_1,\ldots\nu_k$ columns. Let K_i be variable $\nu_i\times\gamma_i$ matrices and let $A_j(K_j)$ denote $A_j+B_jK_j$. Suppose that for some nonnegative α

$$\underset{K_1,\ldots,K_i}{\text{g.r.}}[A_1(K_1)\ldots A_i(K_i) \quad A_{i+1} \quad B_{i+1}\ldots A_k \quad B_k]<\rho-\alpha \tag{A8}$$

where $1\leqq i\leqq k$ and $k\geqq 2$. Then either

$$\underset{K_1,\ldots,K_{i-1}}{\text{g.r.}}[A_1(K_1)\ldots A_{i-1}(K_{i-1}) \quad A_i \quad B_i\ldots A_k \quad B_k]<\rho-\alpha \tag{A9}$$

or

$$\underset{\substack{K_1,\ldots,K_{i-1}\\K_{i+1},\ldots,K_k}}{\text{g.r.}}[A_1(K_1)\ldots A_{i-1}(K_{i-1}) \quad A_{i+1}(K_{i+1}) \times\ldots A_k(K_k)]<\rho-\alpha-\gamma_i \tag{A10}$$

Proof. Fix $K_1=\bar{K}_1,\ldots,K_{i-1}=\bar{K}_{i-1}$ and set

$$C=[A_1(\bar{K}_1)\ldots A_{i-1}(\bar{K}_{i-1}) \quad A_{i+1} \quad B_{i+1}\ldots A_k \quad B_k].$$

Then

$$\underset{K_i}{\text{g.r.}}[C \quad A_i(K_i)]\leqq \underset{K_1,\ldots,K_i}{\text{g.r.}}[A_1(K_1)\ldots A_i(K_i) \times A_{i+1} \quad B_{i+1}\ldots A_k \quad B_k] <\rho-\alpha.$$

Thus, from Lemma A.2, either: (a)

$$\text{rank}[C \quad A_i \quad B_i]=\underset{K_i}{\text{g.r.}}[C \quad A_i(K_i)];$$

or (b)

$$\text{rank}[C]+\gamma_i=\underset{K_i}{\text{g.r.}}[C \quad A_i(K_i)].$$

If (a) holds, then clearly

$$\begin{aligned}\text{rank}[A_1(\bar{K}_1)\ldots A_{i-1}(\bar{K}_{i-1}) \quad A_i \quad B_i\ldots A_k \quad B_k] &=\text{rank}[C \quad A_i \quad B_i]\\ &=\underset{K_i}{\text{g.r.}}[C \quad A_i(K_i)]\\ &<\rho-\alpha \end{aligned} \tag{A11}$$

However, if (b) holds, then

$$\begin{aligned}&\text{rank}[A_1(\bar{K}_1)\ldots A_{i-1}(\bar{K}_{i-1}) \quad A_{i+1}(K_{i+1})\ldots A_k(K_k)]\\ &\leqq\text{rank}[A_1(\bar{K}_1)\ldots A_{i-1}(\bar{K}_{i-1}) \quad A_{i+1} \quad B_{i+1}\ldots A_k \quad B_k] \quad \text{for all } K_{i+1},\ldots,K_k\\ &=\text{rank}[C]\\ &=\underset{K_i}{\text{g.r.}}[C \quad A_i(K_i)]-\gamma_i\\ &<\rho-\alpha-\gamma_i \qquad \text{for all } K_{i+1},\ldots,K_k. \end{aligned} \tag{A12}$$

The first inequality above comes from the fact that the matrix on the left-hand side is the product of that on the right-hand side with

$$I\oplus\cdots\oplus I\oplus\begin{bmatrix} I \\ K_{i+1}\end{bmatrix}\cdots\oplus\begin{bmatrix} I \\ K_k\end{bmatrix}.$$

Thus, we have shown that either (A11) or (A12) holds for each choice of $K_1,\ldots,K_{i-1},K_{i+1},\ldots,K_k$. It then easily follows that (A9) or (A10) hold. □

Finally in the chain, we have the following Lemma.

Lemma A.4 Assume the same hypotheses as Lemma A.3, save that (A8) is strengthened to

$$\underset{K_1,\ldots,K_i}{\text{g.r.}}\ [A_1(K_1)\ldots A_i(K_i)A_{i+1}\quad B_{i+1}\ldots A_k\quad B_k] < \rho-\alpha,\ \sum_{j=1}^{k}\gamma_j-\beta \tag{A13}$$

for some non-negative β, with $1\leq i\leq k$, $k\geq 2$. Then either

$$\underset{K_1,\ldots,K_{i-1}}{\text{g.r.}}\ [A_1(K_1)\ldots A_{i-1}(K_{i-1})\quad A_i\quad B_i\ldots A_k\quad B_k] < \rho-\alpha,\ \sum_{j=1}^{k}\gamma_j-\beta \tag{A14}$$

or

$$\underset{\substack{K_1,\ldots,K_{i-1},\\ K_{i+1},\ldots,K_k}}{\text{g.r.}}\ [A_1(K_1)\ldots A_{i-1}(K_{i-1})\quad A_{i+1}(K_{i+1})\ldots A_k(K_k)] < \rho-\alpha-\gamma_i,\ \sum_{\substack{j=1\\ j\neq i}}^{k}\gamma_j-\beta. \tag{A15}$$

Proof. If

$$\rho\leq\sum_{j=1}^{k}\gamma_j-\beta$$

the result is a trivial consequence of Lemma A.3 with $\alpha=0$. If

$$\rho>\sum_{j=1}^{k}\gamma_j-\beta,\quad \text{set}\quad \alpha=\rho-\sum_{j=1}^{k}\gamma_j+\beta$$

and apply Lemma A.3.

Necessity proof of the theorem. The proof of the theorem is based on successive applications of Lemma A.4. In particular, we make the following observations.

(1) Condition (4) of the theorem is a special case of (A13) (take $i=k=m$ and $\alpha=\delta$, $\beta=\varepsilon$) and accordingly, the lemma allows replacement of equation (4) by one of two alternatives.

(2) The lemma is capable of reapplication. More precisely, if (A14) holds with $i-1\geq 1$, the lemma may be reapplied, i.e. with inessential adjustment of subscripts (A14) has the same form as (A13). Further, if (A15) holds, the lemma may also be reapplied i.e. again with inessential readjustment of subscripts, (A15) has the form of (A13).

(3) Successive reapplications of the lemma must ultimately cease to be possible. This is because each application either reduces the number of K_j involved while maintaining the set of subscripts which occur in the matrix whose rank is computed [alternative (A14)], or reduces the number of subscripts which occur, and simultaneously the ranks, though increasing the number of K_j involved [alternative (A15)]. Ultimately, one of three possibilities is encountered: one obtains a matrix in (A14) with no K_i involved, i.e. a matrix as on the left of (5); or one obtains a matrix in (A15) with just one subscript, (A15) is of the form $\text{rank}[A_l+B_lK_l]<\ldots$, and then Lemma A.1 will provide a condition of the form $\text{rank}[A_l\ B_l]<\ldots$, assuming the rank bounds are satisfactory; or one obtains a matrix in (A15) with zero rank, in which case all A_j, B_j with subscripts j appearing as a subscript in (A15) must be zero. This yields a special case of (5).

(4) In case the chain of applications of the Lemma is initiated by the inequality (4), and any one of (A13), (A14) or (A15) result at some point in the chain of applications (perhaps with renumbering of subscripts), then, as may be readily verified, the value of the constant α in (A13), (A14) and (A15) will be

$$\alpha=\delta+\sum_{j>k}\gamma_j.$$

In effect, every time a subscript l disappears in the matrix whose rank is being examined, α increases by the number of columns γ_l in the associated A_l. This can be checked by examining (A13) and (A14) where there is no change of α and no removal of a subscript, and (A13) and (A15) where in effect α increases by γ_i, subscript i disappearing. This observation guarantees the first rank bound of (5).

(5) The second rank bound of (5) is consistent with and follows from the fact that the second rank bound in (A13) through (A15) is the sum of the number of columns γ_l in the A_l over the subscripts l involved in the relevant matrix, less a quantity β which remains constant throughout the calculation.