# A Note on the Reduced Schur–Cohn Criterion

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***Abstract*—This correspondence simplifies the reduced Schur–Cohn matrices of an earlier published paper [1]. In particular, the symmetric matrices $B$ for $n$-even, and the $A$-matrix for $n$-odd in connection with theorems $I_e$ and $I_0$ of [1] are simplified.**

## Schur–Cohn and Reduced Criteria

In an earlier published paper [1], the authors obtained several stability criteria for the real polynomial

$$F(z)=\sum_{i=0}^{n} a_i z^i \qquad a_n>0 \tag{1}$$

to have all its roots inside the unit circle, by modifying the standard Schur–Cohn criterion for stability. This criterion is that the $n\times n$ matrix $C=(\gamma_{ij})$ be positive definite where

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$$\gamma_{ij}=\sum_{p=1}^{\min(i,j)}(a_{n-i+p}a_{n-j+p}-a_{i-p}a_{j-p}). \quad (2)$$

The modification involved matrices $A=(\alpha_{ij})$, $B=(\beta_{ij})$ where

$$\alpha_{ij}=\gamma_{ij}+\gamma_{i,n+1-j} \qquad \beta_{ij}=\gamma_{ij}-\gamma_{i,n+1-j}(n \text{ even}) \quad (3)$$

with a minor adjustment to the $\alpha_{ij}$ equation only arising when $n$ is odd. For $n=2m$, $A$ and $B$ are $m\times m$ and for $n=2m-1$, $A$ is $m\times m$ and $B$ is $(m-1)\times(m-1)$.

The various alternative stability criteria discussed in [1] are provided by

$$A>0 \text{ and } B>0 \quad (4)$$

and

one of $A$ and $B$ is positive definite and certain linear combination of the $a_i$ are positive. (5)

(The precise linear combinations are discussed in [1], [2] and do not concern us here.)

## Simplification of Schur–Cohn and Related Criteria

We shall now illustrate how a simplification in one of $A$ and $B$ can be achieved. Define $\tilde{F}(z)=zF(z)$, and let the tilde denote quantities associated with $\tilde{F}(z)$. Without loss of generality, suppose $a_n=1$. Then $\tilde{a}_{n+1}=1$, $\tilde{a}_0=0$. The formula for $\tilde{\gamma}_{ij}$ shows immediately that the first and last column and row of $\tilde{C}$ are linear in the $\tilde{a}_i$ and, thus, $a_i$, rather than (as normal) quadratic. By (3), the same is true of the first row and column of $\tilde{A}$ and $\tilde{B}$. Obviously, $F(z)$ is stable if and only if $\tilde{F}(z)$ is stable, i.e., if and only if

$$\tilde{A}>0 \quad \text{and} \quad \tilde{B}>0 \quad (6)$$

or

one of $\tilde{A}$ and $\tilde{B}$ is positive definite and certain linear combinations of the $a_i$ are positive. (7)

In the case $n=2m$, $B$ and $\tilde{B}$ are $m\times m$, and $\tilde{B}$ is simpler than $B$, in that its first row and column are simpler. If $n=2m-1$, $A$ and $\tilde{A}$ are both $m\times m$ and $\tilde{A}$ is simpler than $A$.

The question can be considered as what is the simplest test of the various possibilities. In the case $n=2m$, $\tilde{B}>0$ together the positivity of certain linear combinations of the $a_i$ is the easiest condition to check; if $n=2m-1$, $B$ (together with linear inequalities) is simplest to check for positive definiteness, being $(m-1)\times(m-1)$, even though $\tilde{A}$ is simpler than $A$. Later results, however, will indicate yet other possibilities.

*Example:* Let $F(z)=z^4+a_3z^3+a_2z^2+a_1z+a_0$. Then one obtains

$$B=\begin{bmatrix} 1-a_0^2-a_1+a_0a_3 & a_3-a_0a_1-a_2+a_0a_2 \\ a_3-a_0a_1-a_2+a_0a_2 & 1-a_1^2+a_3^2-a_0^2-a_3-a_2a_3+a_0a_1+a_1a_2 \end{bmatrix}$$

and

$$\tilde{B}=\begin{bmatrix} 1-a_0 & a_3-a_1 \\ a_3-a_1 & 1+a_3^2-a_0^2-a_2+a_0a_2-a_1a_3 \end{bmatrix}.$$

Besides illustrating what happens in the first row and column, this example also illustrates that fewer summands appear in all terms in $\tilde{B}$, as compared with $B$. This phenomenon is actually general, and can be established by use of (2) and (3). (The same is true for $\tilde{A}$.)

## Interpretation via Positive Innerwise Matrices

Define

$$X_{n+1}=\begin{bmatrix} a_n & a_{n-1} & \cdots & a_0 \\ 0 & a_n & \cdots & a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \quad Y_{n+1}=\begin{bmatrix} 0 & \cdots & 0 & a_0 \\ 0 & \cdots & a_0 & a_1 \\ \vdots & ⋰ & & \vdots \\ a_0 & \cdots & a_{n-1} & -a_n \end{bmatrix}. \quad (8)$$

Also define $X_n$, $[X_{n-1}]$ to be $X_{n+1}$ without its first (first and second) row and column and $Y_n$, $[Y_{n-1}]$ to be $Y_{n+1}$ without its last (last and second last) row and column. Finally, define

$$\Delta_i^{\pm}=X_i\pm Y_i \quad (9)$$

These quantities appear in various stability criteria discussed in [2] and [3]. Furthermore, various relations are described of these matrices to symmetric matrices appearing in stability criteria, like $A$, $B$, $\tilde{B}$; usually the symmetric matrices are symmetrized forms of the Schur complement of some $\Delta_i^{\pm}$ [3], [4]. More precisely, one can argue for $n=2m$, see [2], [3],

$$B>0\Leftrightarrow\Delta_n^- \text{ is positive innerwise } (n=2m) \quad (10)$$

$$A>0\Leftrightarrow\Delta_n^+ \text{ is positive innerwise } (n=2m) \quad (11)$$

and for $n=2m-1$

$$B>0\Leftrightarrow\Delta_{n-1}^- \text{ is positive innerwise } (n=2m-1) \quad (12)$$

$$A>0\Leftrightarrow\Delta_{n+1}^+ \text{ is positive innerwise } (n=2m-1). \quad (13)$$

In fact, the signs of the leading principal minors of the matrices on the left of (10)–(13) are in one-to-one correspondence with the signs of the inner determinants of the matrices on the right of (10)–(13), with small dimension minors corresponding to small dimension inner determinants.

Let $n=2m$. Then $\tilde{B}>0\Leftrightarrow\tilde{\Delta}_{2m}$ is positive innerwise and a calculation in [2, see p. 79] relating $\tilde{\Delta}_n^-$ and $\Delta_{n-1}^-$ provides

$$\tilde{B}>0\Leftrightarrow\Delta_{n-1}^- \text{ is positive innerwise } (n=2m). \quad (14)$$

If $n=2m-1$, $\tilde{A}>0\Leftrightarrow\Delta_{2m}^+$ is positive innerwise, and one can conclude similarly that

$$\tilde{A}>0\Leftrightarrow\Delta_n^+ \text{ is positive innerwise } (n=2m-1). \quad (15)$$

Equations (10) and (14) should be compared as should (13) and (15); they also suggest the simplification.

## Example Continued

$$\Delta_4^-=\left[\begin{array}{cc|cc} 1 & a_3 & a_2 & a_1-a_0 \\ 0 & 1 & a_3-a_0 & a_2-a_1 \\ \hline 0 & -a_0 & 1-a_1 & a_3-a_2 \\ -a_0 & -a_1 & -a_2 & 1-a_3 \end{array}\right]=\left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right]. \quad (16)$$

When one computes the Schur complement $S-RP^{-1}Q$ and symmetrizes through premultiplication by

$$\begin{bmatrix} 1 & 0 \\ a_3 & 1 \end{bmatrix}$$

one obtains $B$. The same operations applied to

$$\Delta_3^-=\left[\begin{array}{c|cc} 1 & a_3 & a_2-a_0 \\ \hline 0 & 1-a_0 & -a_1 \\ -a_0 & -a_1 & 1-a_2 \end{array}\right] \quad (17)$$

yield $\tilde{B}$.

## A Further Simplification

In case any stability test is used which requires examination of certain linear inequalities in the $a_i$, it should be noted that such inequality sets always include

$$F(1)>0 \qquad (-1)^nF(-1)>0. \quad (18)$$

The inequality sets for $\tilde{F}(\cdot)$ include $\tilde{F}(1)>0$ and $(-1)^{n+1}\tilde{F}(-1)>0$, which are the same as (18). Now it is established in [2, see p. 135] that

$$|\Delta_{n+1}^+|=2(-1)^nF(1)F(-1)|\Delta_{n-1}^-|. \quad (19)$$

By virtue of the connection between the $\Delta_i^{\mp}$ and the $A$, $B$, $\tilde{A}$, $\tilde{B}$, this means that for $n=2m-1$

$$|A|=\tfrac{1}{2}(-1)^n F(1)F(-1)|B| \tag{20a}$$

and for $n=2m$

$$|\tilde{A}|=\tfrac{1}{2}(-1)^n F(1)F(-1)|\tilde{B}|. \tag{20b}$$

This means that checking that $A$ is positive definite and that certain linear inequalities hold when $n=2m-1$ can proceed by checking the positive definiteness of the matrix $A_1$ formed from $A$ by deleting the last row and column, checking the linear inequalities, and checking that $|B|>0$. The checking of the last inequality is satisfied by checking for $B>0$, in (4). For $n=2m$, a similar statement holds for $\tilde{A}$.

## Conclusion

In this correspondence we have shown that the reduced Schur–Cohn matrices of [1], see especially Theorem 1, can be replaced by simpler matrices. Positive definiteness of the various matrices is equivalent to the positive innerwise character of certain other matrices $\Delta_i^{\mp}$ and the simplification of the reduced Schur-Cohn matrices can be paralleled in terms of the $\Delta_i^{\mp}$.

## References


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