Time-Limited Time-Invariant Wiener Filtering*

J. M. POTTER

and

B. D. O. ANDERSON†

Department of Electrical and Computer Engineering, University of Newcastle, New South Wales, 2308, Australia

ABSTRACT

For the estimation of a signal observed with additive white noise, it is shown that the optimum linear least-squares filter constrained to have its impulse response time-limited to the interval $[0, T]$ satisfies a truncated version of the Wiener-Hopf equation. To solve this equation the covariance for the observed process need only be known for time lags less than T. There is a unique extension of the covariance for lags greater than T, for which the time-limited filter is the optimum Wiener filter; furthermore this same extension is that extension of the covariance for which the optimum Wiener filter gives maximum mean square error, i.e., given limited covariance information we have found the "worst possible" extension of the known information. Parallels are drawn with discrete-time maximum-entropy spectral analysis.

I. INTRODUCTION

Given observations $y(\cdot)$ of a signal process $z(\cdot)$ in additive white noise $n(\cdot)$,

$$y(\cdot)=z(\cdot)+n(\cdot),$$

an important filtering problem is to find the optimum linear least-squares estimate $\hat{z}(\cdot)$ of the signal process. A good survey of the problem is available in [1]. We assume that all processes involved are zero-mean and wide-sense stationary, with the following covariances:

$$En(t)n'(s)=I\delta(t-s),$$

$$Ey(t)y'(s)=I\delta(t-s)+K(t-s),$$

*Work supported by the Australian Research Grants Committee.

†Part of this work was done while the second author was on leave at Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106.

52 Vanderbilt Ave., New York, NY 10017 0020-0255/81/03153-06$02.50

and that $z(t)$ and $n(s)$ are uncorrelated for $t<s$. If $y(\cdot)$ is observed over the time interval $(-\infty, t]$, then the optimum estimate is given by

$$\hat{z}(t)=\int_0^\infty h(\tau)y(t-\tau)\,d\tau, \tag{1}$$

where $h(\cdot)$ satisfies the Wiener-Hopf equation

$$h(t)+\int_0^\infty h(\tau)K(t-\tau)\,d\tau=K(t), \qquad t>0. \tag{2}$$

If $y(\cdot)$ is observed over the time interval $[0,t]$, then the optimum estimate is given by

$$\hat{z}(t)=\int_0^t h(t,\tau)y(\tau)\,d\tau, \tag{3}$$

where $h(\cdot,\cdot)$ satisfies the nonstationary Wiener-Hopf-type equation

$$h(t,s)+\int_0^t h(t,\tau)K(\tau-s)\,d\tau=K(t-s), \qquad t>s\geqslant 0. \tag{4}$$

In the case of (2) we say that *h is the optimum Wiener filter for K*. A key advantage of the filter defined by (2) is that it is time-invariant; definition and implementation of the filter requires knowledge of both the observations and the covariance over an infinite time interval, which may be a disadvantage. On the other hand the filter defined by (4) only needs the observations and covariance over a finite time interval (increasing with t); its disadvantage is that it is time-varying.

This communication is devoted to achieving a compromise between the two approaches. The filter is time-invariant and the observations and covariance are only needed over a finite time interval of fixed length. The filter is seen to be obtainable directly from that found in (4), but more simply it is also obtainable from a time-limited version of (2). In Sec. II the filter is derived and various properties are expounded. Outline proofs are given in Sec. III, and Sec. IV looks at the relationships with previous work, especially with regard to discrete-time analogies.

II. THE TIME-LIMITED TIME-INVARIANT WIENER FILTER

For fixed $T>0$, define the (time-invariant) impulse response

$$\begin{aligned} h_T(t)&=h(T,T-t), && 0\leqslant t\leqslant T,\\ &=0 && \text{otherwise,} \end{aligned} \tag{5}$$

where $h(\cdot,\cdot)$ satisfies (4). Using (4), it is easy to verify that

$$h_T(t)+\int_0^T h_T(\tau)K(t-\tau)\,d\tau=K(t), \qquad 0<t\leqslant T, \tag{6}$$

which is simply a version of (2) limited to the time interval $[0,T]$; also $h_T(\cdot)$ may be evaluated from (6) with knowledge of $K(t)$ for $|t|\leqslant T$.

Suppose that we do only know $K(t)$ for $|t|\leqslant T$. Let us define a time invariant impulse response $k_T(\cdot)$ by (in operator terms)

$$(\delta I+k_T)(\delta I-h_T)=\delta, \tag{7}$$

or more precisely

$$k_T(t)-\int_0^{\min(t,T)} k_T(t-\tau)h_T(\tau)\,d\tau-h_T(t)=0. \tag{8}$$

Then let us further define a stationary covariance $\delta I+K_T$ by

$$(\delta I+K_T)=(\delta I+k_T)(\delta I+k_T^a), \tag{9}$$

or

$$K_T(t)=k_T(t)+k_T'(-t)+\int_{-\infty}^{\min(0,t)} k_T(t-\tau)k_T'(-\tau)\,d\tau. \tag{10}$$

We are now in a position to state the following properties for the filter h_T and the covariance $\delta I+K_T$; the properties will be verified in the next section.

P1: h_T is the optimum Wiener filter for K_T.

P2: Given $K(t)$, at least for $|t|<T$, h_T is the optimum linear estimator for $z(\cdot)$ under the constraint that support is restricted to the time interval $[0,T]$.[1]

P3: Given $K(t)$ for $|t|\leqslant T$, $K_T(t)$ is that extension of $K(\cdot)$ for which the optimum Wiener filter gives maximum mean square error (MSE).

[1]By support restricted to $[0,T]$ we mean, for an arbitrary time-varying filter, that the impulse response $g(t,s)$ has support restricted to the strip $t-s\in[0,T]$.

III. VERIFICATION OF PROPERTIES

Property P1 is an immediate consequence of Eqs. (7) and (9) above. We note that this implies that $K_T(t)$ can also be defined in the following way:

$$\begin{aligned} K_T(t) &= K(t) \qquad |t| \leqslant T, \\ &= \int_0^T h_T(\tau) K_T(t-\tau)\, d\tau, \qquad t > T, \\ &= K_T'(-t) \qquad t < -T. \end{aligned} \tag{11}$$

To verify P2 and P3 we first need to derive the MSE for an arbitrary linear filter $g(\cdot,\cdot)$ with support over $[0,T]$. Using the independence of $z(t)$ and $n(s)$ for $t<s$, it may be shown that

$$\begin{aligned} \text{MSE} &= \operatorname{tr} E\{[z(t)-\hat{z}(t)][z(t)-\hat{z}(t)]'\} \\ &= \operatorname{tr}\Big\{K(0) - \int_{t-T}^{t} g(t,\tau)K(\tau-t)\, d\tau - \int_{t-T}^{t} K(t-\sigma) g'(t,\sigma)\, d\sigma \\ &\quad + \int_{t-T}^{t}\int_{t-T}^{t} g(t,\tau)[I\delta(\tau-\sigma)+K(\tau-\sigma)] g'(t,\sigma)\, d\sigma\, d\tau \Big\}. \end{aligned} \tag{12}$$

Observe for later reference that (12) implies that the MSE for a filter with support over $[0,T]$ is independent of $K(t)$ for $|t|>T$. To prove P2 let us perturb the filter h_T by Δh, i.e., consider the filter $g(t,s)=h_T(t-s)+\Delta h(t,s)$, where Δh also has support restricted to $[0,T]$; using (12), we find that the increase in mean square error is

$$\begin{aligned} \Delta\text{MSE} &= \text{MSE}(h_T+\Delta h) - \text{MSE}(h_T) \\ &= \operatorname{tr}\int_{t-T}^{t} \Delta h(t,\tau)\Big[h_T'(t-\tau) + \int_{t-T}^{t} K(\tau-\sigma) h_T'(t-\sigma)\, d\sigma - K(\tau-t)\Big]\, d\tau \\ &\quad + \operatorname{tr}\int_{t-T}^{t}\Big[h_T(t-\sigma) + \int_{t-T}^{t} h_T(t-\tau)K(\tau-\sigma)\, d\tau - K(t-\sigma)\Big] \\ &\quad \times \Delta h'(t,\sigma)\, d\sigma + \operatorname{tr}\int_{t-T}^{t}\int_{t-T}^{t} \Delta h(t,\tau)[I\delta(\tau-\sigma)+K(\tau-\sigma)]\Delta h'(t,\sigma)\, d\sigma\, d\tau \\ &= \operatorname{tr}\int_{t-T}^{t}\int_{t-T}^{t} \Delta h(t,\tau)[I\delta(\tau-\sigma)+K(\tau-\sigma)]\, \Delta h'(t,\sigma)\, d\sigma\, d\tau, \quad \text{by (11)} \\ &\geqslant 0, \end{aligned}$$

since the covariance $\delta I + K$ is nonnegative definite. Therefore P2 holds.

To check the last property, let $K_0(t)$ be any extension for $K(\cdot)$ for $|t|>T$ such that $\delta I+K_0$ is a covariance. Since h_T is not necessarily the optimum Wiener filter for K_0,

$$
\begin{aligned}
&\text{optimum MSE with } K_0 \\
&\qquad \leqslant \text{MSE with } K_0, h_T \\
&\qquad = \text{MSE with } K_T, h_T, \qquad \text{by (12) and remark following (12)} \\
&\qquad = \text{optimum MSE with } K_T, \qquad \text{by P1,}
\end{aligned}
$$

whence P3 holds.

IV. COMMENTS

The discrete-time analogy of our time-limited time-invariant Wiener filter is an estimator using a moving average of the observations with a fixed and finite set of weighting coefficients. Equivalently we may think of this as fitting an autoregressive model to the observations (Box and Jenkins [2]). However the most significant connection with known discrete-time results is in the area of maximum-entropy spectral analysis; there the problem is to extend a covariance beyond a finite interval by taking that process which has maximum entropy. The maximum-entropy method uses an autoregressive model to estimate the covariance (van den Bos [3], Davis [4]); the method is a precise analogy of our extension of the covariance given by (11). In fact our property P3 can be shown to be equivalent to a maximum-entropy property.

Our results were inspired by a transmission-line model for the filtering equations developed by Anderson [5] and in further joint work by the present authors. Restricting the Wiener filter to have support over $[0, T]$ corresponds to a model where the transmission line has finite length and has a matched termination; the discrete-time analogy is a lattice filter model where the reflection coefficients are zero after a finite number of sections (see, for example, Robinson and Treitel [6]).

REFERENCES

1. T. Kailath, A view of three decades of linear filtering theory, *IEEE Trans. Information Theory* IT-20 (2): 146–181 (Mar. 1974).
2. G. E. Box and G. H. Jenkins, *Time Series Analysis: Forecasting and Control*, Holden-Day, San Francisco, 1970.
3. A. van den Bos, Alternative interpretation of maximum entropy spectral analysis, *IEEE Trans. Information Theory* IT-17 (4): 493–494 (July 1971).

4. D. A. Gray, Maximum entropy spectrum analysis technique—A review of its theoretical properties, Tech. Report 1812(W), Weapons Research Establishment, Adelaide, Australia, 1980.
5. B. D. O. Anderson, A physical basis for Krein's prediction formula, submitted for publication.
6. E. A. Robinson and S. Treitel, Maximum entropy and the relationship of the partial autocorrelation to the reflection coefficients of a layered system, *IEEE Trans. Acoustics, Speech and Signal Proc.* ASSP-28, No. 2 (Apr. 1980).

Received October 1980