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DETECTABILITY AND STABILIZABILITY OF TIME-VARYING DISCRETE-TIME LINEAR SYSTEMS*

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Abstract. The concepts of detectability and stabilizability are explored for time-varying systems. We study duality, invariance under feedback, an extended version of the lemma of Lyapunov, existence of stabilizing feedback laws, linear quadratic filtering and control, and the existence of approximate canonical forms.

1. Introduction. The dual concepts of observability and reachability for linear finite-dimensional time-invariant systems have found application in a variety of filtering and control problems. For example, signal model observability is a sufficient condition for an optimal minimum variance filter (Kalman filter) to exist, and signal model controllability ensures its asymptotic stability—at least in the case of linear time-invariant continuous time, finite dimensional signal models. See [1] for a leisurely exposition. Dual results hold for the linear-quadratic optimal control problem. Moreover, there have been at least two significant generalizations of these results. First, retaining the time-invariance assumption, the reachability and observability hypotheses have been weakened to stabilizability and detectability, and the results are still valid, [2], [3]. Second, time-variable problems have been considered, and with the imposition of a uniformity constraint in the now time-varying reachability and observability hypotheses, the results extend to the time-varying situation [4]. For time-varying systems one is tempted to avoid the weaker conditions of detectability and stabilizability since the technical issues raised involve nontrivial generalizations of the time-invariant results and on first glance appear formidable. However, the desirability of extending known control and filtering results to important classes of time-varying filtering and control problems is clear.

A lead has been taken in [5] with the introduction of definitions of detectability and stabilizability for discrete-time, time-varying linear systems and some applications to control and filtering problems. However an exploration of equivalent definitions and properties such as duality and invariance under feedback is not attempted in [5].

In this paper, motivated by the lead given in [5], we explore a generalization of the dual concepts of observability and controllability to linear, time-varying, discrete-time, finite-dimensional systems.

Following definitions of the concepts of detectability and stabilizability in § 2, we indicate their formal duality, and establish two simple consequences. In § 3, we show the invariance of the properties under appropriate feedback, and in § 4, we prove a significant generalization of the lemma of Lyapunov, which is useful for studying the stability of linear systems using quadratic Lyapunov functions. Section 5 considers linear quadratic problems, and we prove one of the main results of the paper: detectability and stabilizability are the key properties required to guarantee an exponentially stable Kalman filter. We also show the equivalence of the definitions of § 2 with the existence of stabilizing feedback laws of an appropriate form. Section 6 contains the concuding remarks.

Two general points can be noted. First, almost all the results are stated just for detectable pairs, rather than detectable and stabilizable pairs. Given the duality

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established in the paper, there is no loss of generality. Second, the results are all stated for discrete-time systems. We elected to work with discrete-time systems rather than continuous-time systems because we were aware that, as illustrated by ideas in [6]–[9], it is often harder to get the discrete-time result than the continuous-time result. Particularly is this so when the discrete-time transition matrix can be singular. Of course, for the continuous-time case, regularity conditions must be imposed and techniques as in, for example, [12] exploited. In some cases, the continuous-time proofs are likely to be harder.

2. Detectability definitions and some implications. Consider the linear finite-dimensional state space system in discrete time

$$x_{k+1} = F_k x_k + G_k u_k, \tag{2.1a}$$

$$y_k = H'_k x_k, \tag{2.1b}$$

where x_k is the n-vector state, u_k is an input m-vector, y_k is an output p-vector, and F_k, G_k, H_k are matrices of appropriate dimension. The state transition matrix is denoted $\phi_{k,l}$ for $k \geqq l$ where $\phi_{k+1,k} = F_k$ and $\phi_{k,l} = \phi_{k,k-1}\phi_{k-1,l}$.

The detectability definition we work with is a specialization of one in [5] for finite-dimensional systems.

DEFINITION 2.1. The pair $[F_k, H_k]$ is *uniformly detectable* if there exist integers s, $t \geqq 0$ and constants d, b with $0 \leqq d < 1$, $0 < b < \infty$, such that whenever

$$\|\phi_{k+t,k}\xi\| \geqq d\|\xi\| \tag{2.2}$$

for some ξ and k, then

$$\xi' M_{k+s,k}\xi \geqq b\xi'\xi, \tag{2.3}$$

where

$$M_{k+s,k} = \sum_{i=k}^{k+s} \phi'_{i,k} H_i H'_i \phi_{i,k}. \tag{2.4}$$

Remarks. 1. For time invariant systems, detectability definitions have been given (see [3]) which require that the unstable modes of a system be observable. The above definition is a time-varying version of this notion. In fact, the definition says roughly that when a state trajectory is not fast decaying, i.e., (2.2) is satisfied, then that trajectory must be observable, i.e., (2.3) holds. Conversely, trajectories which are not observed with much output energy, i.e., those for which (2.3) fails, must be trajectories which decay, i.e., (2.2) fails. Further justification of this remark is provided by Lemma 2.2.

2. Recall (see, e.g., [4]), that if $[F_k, H_k]$ is uniform with respect to observability, the observability Gramian $M_{k+s,k}$ satisfies (for some integer s, and constants β_1, β_2)

$$0 < \beta_1 I \leqq M_{k+s,k} \leqq \beta_2 I.$$

This is clearly a sufficient condition for detectability as above. Notice, however that there are no upper bounds in the detectability definition. It is not surprising therefore that in most of the results to follow we impose upper bounds in F_k and H_k.

3. Without loss of generality, $s \geqq t$ can be assumed in the above definition, since if $s < t$, s can be replaced by t. Henceforth, we shall assume that $s \geqq t$.

The second definition we give is that of uniform stabilizability. As argued following the definition, the definition is related to that of detectability via a certain duality.

DEFINITION 2.2. The pair $[\hat{F}_k, \hat{G}_k]$ is *uniformly stabilizable* if there exist integers s, $t \geqq 0$ and constants d, b with $0 \leqq d < 1$, $0 < b < \infty$, such that whenever

$$\|\hat{\phi}_{k+1,k+1-t}\xi\| \geqq d\|\xi\| \tag{2.5}$$

for some ξ, k, then

$$\xi'\hat{Y}_{k,k-s}\xi \geqq b\xi'\xi, \tag{2.6}$$

where $\hat{\phi}_{k,l}$ is the transition matrix associated with $\hat{F}_k$ and

$$\hat{Y}_{k,k-s} = \sum_{i=k-s}^{k} \hat{\phi}_{k+1,i+1}\hat{G}_i\hat{G}_i'\hat{\phi}_{k+1,i+1}'. \tag{2.7}$$

Without loss of generality, $s \geqq t$ can be assumed.

Remark. In the time-invariant case, stabilizability is equivalent to the requirement that any uncontrollable mode be asymptotically stable (see [3]). But as the name suggests, stabilizability is also equivalent (but this must be proved) to the property that there exists a stabilizing state feedback law. The first idea is reflected in the definition above. The second will be taken up later.

The duality between detectability and stabilizability is taken up in the following lemma.

LEMMA 2.1. *Make the definitions*

$$F_k = \hat{F}_{-k}', \qquad H_k = \hat{G}_{-k}. \tag{2.8}$$

Then

(a) $\phi_{i,k} = \hat{\phi}_{-k+1,-i+1}'$ *and* $M_{k+s,k} = Y_{-k,-k-s}$.
(b) $[\hat{F}_k, \hat{G}_k]$ *is uniformly stabilizable if and only if* $[F_k, H_k]$ *is uniformly detectable.*
(c) $x_{k+1} = F_k x_k$ *is exponentially stable if and only if* $\hat{x}_{k+1} = \hat{F}_k\hat{x}$ *is exponentially stable.*

Proof is via direct calculation. Notice that use of a dual relationship of the form $F_k = (\hat{F}_k')^{-1}$ is not suitable, requiring as it does the existence of the inverse. Use of this second, rather unsatisfactory dual, appears to be behind many ideas of [8], [9].

We conclude the section by noting two simple consequences of the detectability definitions. The first confirms the first remark following the definition. The second will be used in a later section.

LEMMA 2.2. *With* $[F_k, H_k]$ *detectable and* F_k *bounded above, then for the system* (2.1) *with a zero input,*

$$H_k'x_k \to 0 \text{ as } k \to \infty \Rightarrow \|x_k\| \to 0 \text{ as } k \to \infty.$$

Proof (by contradiction). Assume there exists an x_0 with $H_k'x_k \to 0$ as $k \to \infty$, but with $\|x_k\| \nrightarrow 0$ as $k \to \infty$. Now if for all $k \geqq k_1$ for some k_1, $\|\phi_{k+t,k}x_k\| < d\|x_k\|$, then $\|x_k\| \to 0$ as $k \to \infty$. Thus there exists a sequence $k_i \to \infty$ with $\|\phi_{k_i+t,k_i}x_{k_i}\| \geqq d\|x_{k_i}\|$ defining the k_i. We can further assume that $\|x_{k_i}\| \nrightarrow 0$, for if $\|x_{k_i}\| \to 0$, as $k_i \to \infty$, then as we now show, $\|x_k\| \to 0$ as $k \to \infty$ which is a contradiction.

Let $k \in (k_i, k_{i+1})$ but be otherwise arbitrary, and set $k = k_i + t\alpha + \beta$, with α, β integers with $\beta < t$. Then

$$x_k = \phi_{k,k_i+t\alpha}\phi_{k_i+t\alpha,k_i+t(\alpha-1)}\cdots\phi_{k_i+2t,k_i+t}x_{k_i+t}.$$

The first matrix in the product has bounded norm because F_k is bounded and $k-(k_i+t\alpha)$ is bounded.

From the definition then of the k_i,

$$\|x_k\| \leqq \gamma_1 d^{\alpha-1}\|x_{k_i+l}\| \\ \leqq \gamma_1\gamma_2 d^{\alpha-1}\|x_{k_i}\|,$$

with γ_2 existing because of the bound on F_k. It readily follows that $\|x_{k_i}\| \to 0$ implies $\|x_k\| \to 0$ for all k. So we return to the assumption that $\|x_{k_i}\| \nrightarrow 0$.

Define a subsequence $\{l_i\}$ of the $\{k_i\}$ such that $\|x_{l_i}\| > \gamma_3$ for all i and some $\gamma_3 > 0$. Now

$$\|\phi_{l_i+t,l_i}x_{l_i}\| \geqq d\|x_{l_i}\|,$$

and by detectability

$$\|x'_{l_i}M_{l_i+s,l_i}x_{l_i}\| \geqq b\|x_{l_i}\|^2 > b\gamma_3^2.$$

However, from the definitions of M_{l_i+s,l_i} and assumptions on H'_kx_k and u_k,

$$\|x'_{l_i}M_{l_i+s,l_i}x_{l_i}\| = \sum_{l=l_i}^{l_i+s}\|H'_lx_l\|^2 \to 0 \quad \text{as } l_i \to \infty$$

So we have a contradiction and the lemma is established. □

LEMMA 2.3. *Let* $[F_k, H_k]$ *be detectable. Then* $[\rho F_k, H_k]$ *is detectable for* $1+\varepsilon \geqq \rho > 1$ *and* $\varepsilon > 0$ *sufficiently small.*

Proof. Let a tilde denote quantities associated with $\tilde{F}_k = \rho F_k$ and H_k. Now

$$\|\tilde{\phi}_{k+t,k}\xi\| = \rho^t\|\phi_{k+t,k}\xi\|.$$

Now choose $\rho > 1$ such that $\tilde{d} = d\rho^t < 1$, where d appears in the detectability definitions for $[F_k, H_k]$. Then $\|\tilde{\phi}_{k+t,k}\xi\| \geqq \tilde{d}\|\xi\|$ if and only if $\|\phi_{k+t,k}\xi\| \geqq d\|\xi\|$. Also

$$\tilde{M}_{k+s,k} = \sum_{i=k}^{k+s}\rho^{2(i-k)}\phi'_{i,k}H_iH'_i\phi_{i,k} \geqq M_{k+s,k}.$$

Consequently, if $\xi' M_{k+s,k}\xi \geqq b\xi'\xi$, then also $\xi'\tilde{M}_{k+s,k}\xi \geqq b\xi'\xi$. Thus tying the above results together we have that whenever $\|\tilde{\phi}_{k+t,k}\xi\| \geqq \tilde{d}\|\xi\|$, then $\xi'\tilde{M}_{k+s,k}\xi \geqq \xi'\xi$ as required by the detectability definition. This establishes the lemma. □

Remarks. 1. As the proof shows, ρ can be taken as any number for which $\rho^t d < 1$, d being the quantity appearing in the uniform detectability definition associated with $[F_k, H_k]$.

2. The above lemma is a special case of a more general result which is almost as easily proved: if $[F_k, H_k]$ is detectable, there exists ε, depending on $[F_k, H_k]$, such that $[\tilde{F}_k, \tilde{H}_k]$ is detectable for all $\tilde{F}_k$, $\tilde{H}_k$ with $\|F_k - \tilde{F}_k\| < \varepsilon$, $\|H_k - \tilde{H}_k\| < \varepsilon$.

3. Invariance under feedback. With $[F_k, H_k]$ denoting an open-loop system matrix pair, there is interest in a closed-loop system matrix pair $[\bar{F}_k, H_k]$ where $\bar{F}_k = F_k - K_kH'_k$.

LEMMA 3.1. *The observability Gramians* $M_{l,k}$ *for the open-loop pair* $[F_k, H_k]$ *and* $\bar{M}_{l,k}$ *for the closed-loop pair* $[\bar{F}_k, H_k]$, *where* $\bar{F}_k = F_k - K_kH'_k$, *bear the following relationship*:

$$\bar{M}_{l,k} = H_{l,k}C'_kC_kH'_{l,k}, \qquad M_{l,k} = H_{l,k}H'_{l,k}, \tag{3.1}$$

where

$$H_{l,k} = [H_k \quad \phi'_{k+1,k}H_{k+1} \quad \cdots \quad \phi'_{l,k}H_l], \tag{3.2}$$

$$C_k = \begin{bmatrix} I & 0 & \cdots & 0 \\ * & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & I \end{bmatrix} \tag{3.3}$$

and $*$ *denotes terms involving* F_i, K_i, H_i, *for* $i = k, k+1 \cdots l$. *Moreover, with* F_i, K_i, H_i *bounded, then for some positive constants* α_1, α_2,

$$\alpha_1 M_{l,k} \leqq \bar{M}_{l,k} \leqq \alpha_2 M_{l,k}. \tag{3.4}$$

Proof. The relationship (3.1) follows by inductive arguments, the definitions of $M_{l,k}$, and straightforward manipulations. The bounds (3.4) follow from a premultiplication by $H_{l,k}$ and postmultiplication by $H'_{l,k}$ of the inequalities,

$$0 < \alpha_1 I \leqq \lambda_{\min}(C'_k C_k) I \leqq C'_k C_k \leqq \lambda_{\max}(C'_k C_k) I \leqq \alpha_2 I < \infty.$$

The above bounds are verified as follows. First, $\lambda_{\max}(C'_k C_k) \leqq \operatorname{tr}(C'_k C_k) < \alpha_2 I$ for some $\alpha_2 < \infty$ under the boundedness assumptions, and

$$\lambda_{\min}(C'_k C_k) \geqq \frac{|C'_k C_k|}{[\lambda_{\max}(C'_k C_k)]^{n-1}} \geqq \frac{|C'_k||C_k|}{\alpha_2^{n-1}} = \frac{1}{\alpha_2^{n-1}} = \alpha_1,$$

for some $\alpha_1 > 0$. □

LEMMA 3.2. *In the notation of the previous lemma, and with* $\bar{\phi}_{l,k}$ *the transition matrix associated with the closed loop system matrix* $\bar{F}_k = F_k - K_k H'_k$, *then for all* $l \geqq k$,

$$\bar{\phi}_{l,k} = \phi_{l,k} + [*]H'_{l,k}, \tag{3.5}$$

where $[*]$ *denotes a matrix involving* F_i, K_i, H_i *for* $i = k, k+1, \cdots l-1$.

Proof. From straightforward manipulations. □

We now have the following main result.

THEOREM 3.3. *With* F_k, H_k, *and* K_k *bounded, and with* $\bar{F}_k = F_k - K_k H'_k$, *then* $[F_k, H_k]$ *is uniformly detectable if and only if* $[\bar{F}_k, H_k]$ *is uniformly detectable.*

Proof. It clearly suffices to prove that under the boundedness conditions, $[F_k, H_k]$ uniformly detectable implies $[F_k - K_k H'_k, H_k]$ uniformly detectable. Let s, t, d, b be as in the uniform detectability definition applied to $[F_k, H_k]$.

Under the boundedness assumptions, Lemma 3.2 yields

$$\begin{aligned} \|\bar{\phi}_{k+t,k}\xi\| &= \|\phi_{k+t,k}\xi + [*]H'_{k+t,k}\xi\| \\ &\leqq \|\phi_{k+t,k}\xi\| + \alpha_3\|H'_{k+t,k}\xi\|, \end{aligned}$$

for some α_3. Let α_1 be as in Lemma 3.1, with $l = k + s$, and define

$$\bar{b} = \min\left\{\alpha_1 b, \alpha_1\left(\frac{1-d}{2\alpha_3}\right)^2\right\}, \qquad \bar{d} = d + \alpha_3\sqrt{\frac{\bar{b}}{\alpha_1}}.$$

Notice that $1 - \bar{d} > 0$. We shall show that $s, t, \bar{d}, \bar{b}$ characterize the uniform detectability of $[\bar{F}_k, H_k]$.

Suppose that $\xi'\bar{M}_{k+s,k}\xi < \bar{b}\xi'\xi$. Using (3.4), we have

$$\xi' M_{k+s,k}\xi < \frac{\bar{b}}{\alpha_1}\xi'\xi \leqq b\xi'\xi,$$

and also

$$\|H'_{k+t,k}\xi\| = (\xi' M_{k+t,k}\xi)^{1/2} \leqq (\xi' M_{k+s,k}\xi)^{1/2} < \sqrt{\frac{b}{\alpha_1}}\|\xi\|.$$

Consequently,

$$\|\bar{\Phi}_{k+t,k}\xi\| < d\|\xi\| + \alpha_3\sqrt{\frac{b}{\alpha_1}}\|\xi\|,$$

or

$$\|\bar{\Phi}_{k+t,k}\xi\| < \bar{d}\|\xi\|.$$

Equivalently, $[\bar{F}_k, H_k]$ is uniformly detectable. □

There are two useful corollaries to this theorem.

COROLLARY 3.4. *A sufficient condition for the pair $[F_k, H_k]$ to be uniformly detectable is that there exist a bounded gain K_k such that the closed-loop system $\bar{x}_{k+1} = (F_k - K_k H'_k)\bar{x}_k$ is exponentially stable.*

Proof. If $\bar{F}_k$ defines an exponentially stable system, $[\bar{F}_k, H_k]$ is uniformly detectable for any H_k. □

Later in the paper, we shall show that the sufficiency condition just stated is in fact also a necessity condition.

COROLLARY 3.5. *With notation as above, the following quantity is feedback invariant*:

$$\tilde{d} = \inf [d \mid d \in [0, 1) \text{ and } \|\phi_{k+t,k}\xi\| \geqq d\|\xi\| \text{ implies } \|\xi' M_{k+s,k}\xi\| \geqq b\|\xi\|^2 \text{ for some } b > 0].$$

Proof. Let K_k be a gain sequence, and let α_1, α_2 be constants defined in the statement of Lemma 3.1 and the proof of Lemma 3.3. Take $\varepsilon > 0$, arbitrary save that $\tilde{d} + \varepsilon < 1$. Then for $d = \tilde{d} + \varepsilon/2$, there exists $b > 0$ such that $\|\phi_{k+t,k}\xi\| \geqq d\|\xi\|$ implies $\|\xi' M_{k+s,k}\xi\| \geqq b\|\xi\|^2$.

Without loss of generality, we may replace b by $\min(b, \varepsilon^2/(4\alpha_3^2\alpha_1))$. Then with $\bar{d}$ referring to the uniform detectability definition applied to $[F_k - K_k H'_k, H_k]$, the proof of the theorem shows that we can take $\bar{d}$ with

$$\begin{aligned}
\bar{d} &= d + \frac{\alpha_3}{\sqrt{\alpha_1}}\left[\min\left\{\alpha_1 b, \alpha_1\left(\frac{1-d}{2\alpha_3}\right)^2\right\}\right]^{1/2} \\
&\leqq d + \frac{\alpha_3}{\sqrt{\alpha_1}}\sqrt{\alpha_1 b} \\
&\leqq \tilde{d} + \varepsilon/2 + \frac{\alpha_3}{\sqrt{\alpha_1}}\frac{\varepsilon}{2\alpha_3} \\
&= \tilde{d} + \varepsilon.
\end{aligned}$$

Consequently, $\inf \bar{d} \leqq \inf d = \tilde{d}$. But also we can argue that $\inf d \leqq \inf \bar{d}$, whence the feedback invariance of $\tilde{d}$. □

In case $\tilde{d} = 0$, the result simply says that uniform observability is invariant under feedback. For $\tilde{d} > 0$, the quantity has the following interpretation. Consider trajectories which are exactly, or approximately unobserved. Then they all decay at least as fast as $(\tilde{d})^{1/t}$ but they do not all decay faster. Feedback does not vary the conclusion, precisely because the trajectories are unobserved.

Finally in this section, we note that there are obvious duals of these results tied to uniform stabilizability.

4. Lemma of Lyapunov. In this section, we attempt to parallel a result relevant to estabishing stability. The continuous-time lemma of Lyapunov [10] is concerned with the matrix equation $PA + A'P = -Q$ with $Q > 0$, linking positive definiteness of P with asymptotic stability of $\dot{x} = Ax$; relaxation of $Q > 0$ is accomplished in [11], and a time-varying version using uniform complete observability ideas can be found in [12] for continuous time. A discrete-time statement applicable to the time-invariant case can be found in, e.g., [13], while a time-varying version parallel to [12] is easy to find. Here, we aim to relax the observability assumption to detectability.

The following result is comparatively straightforward to obtain. We state it as a lead-in to the more difficult time-varying result.

PROPOSITION 4.1. *Let $[F, H]$ be a detectable pair of constant matrices. Then the equation $P - F'PF = HH'$ has a unique solution $P = P' \geqq 0$ if and only if $x_{k+1} = Fx_k$ is asymptotically stable.*

Proof. We use the characterization of detectability that $Fw = \lambda w$, $H'w = 0$ and $w \neq 0$ only if $|\lambda| < 1$, ([13]). Suppose there exists $P = P' \geqq 0$. Let $Fx = \lambda x$, $x \neq 0$. Then $0 \leqq x'^{*}HH'x = (1 - |\lambda|^2)x'^{*}Px$. So $|\lambda| < 1$, or $|\lambda| \geqq 1$ and also $H'x = 0$, contradicting detectability. Conversely, if $x_{k+1} = Fx_k$ is asymptotically stable,

$$P = \sum_{i=0}^{\infty} (F')^i HH'F^i$$

is well-defined, symmetric, and nonnegative definite, and satisfies the equation $P - F'PF = HH'$. Uniqueness is easily obtained. □

To obtain a time-varying generalization, consider the sequence $\Pi_{k,N}$ defined for $k = N, N-1, N-2, \cdots$, by

$$F'_k \Pi_{k+1,N} F_k - \Pi_{k,N} = -H_k H'_k \quad \Pi_{N,N} = 0. \tag{4.1}$$

Evidently, for $k \leqq N-1$,

$$\Pi_{k,N} = \sum_{j=k}^{N-1} \phi'_{j,k} H_j H'_j \phi_{j,k}, \tag{4.2}$$

where $\phi_{j,k}$ has the usual association with F_k. We now have a parallel to one half of Proposition 4.1.

LEMMA 4.2. *With notation as above, suppose that $x_{k+1} = F_k x_k$ is exponentially stable and F_k, H_k are bounded. Then*

$$P_k = \lim_{N \to \infty} \Pi_{k,N} \tag{4.3}$$

exists as a bounded nonnegative definite symmetric matrix, it satisfies

$$F'_k P_{k+1} F_k - P_k = -H_k H'_k, \tag{4.4}$$

for all $k \geqq 0$, and $\{P_k\}$ is the unique bounded sequence to do so.

Proof. All claims are clear, except perhaps for the last. Let $\{Q_k\}$ be a second bounded sequence satisfying (4.4). Set $R_k = P_k - Q_k$. Then

$$F'_k R_{k+1} F_k - R_k = 0,$$

whence

$$\phi'_{k+l,k} R_{k+l} \phi_{k+l,k} - R_k = 0.$$

Letting $l \to \infty$, and using the exponential decay of $\phi_{k+l,k}$ and boundedness of R_{k+l}, gives $R_k = 0$. □

We now seek the converse to this result; i.e., we seek to establish exponential stability, given (4.4). One might think that $V(x_k, k) = x_k'P_kx_k$ could serve as a Lyapunov function for $x_{k+1} = F_kx_k$. After all (4.4) would then imply $V(x_{k+1}, k) - V(x_k, k) \leqq 0$. One difficulty is that $V(x_k, k)$ is not necessarily positive definite, and in fact it is easy to construct examples where it fails to be positive definite; another difficulty is that the monotone decreasing property of V along trajectories is not strict. Nevertheless we have the following result:

THEOREM 4.2 (Extended lemma of Lyapunov). *Suppose that $[F_k, H_k]$ is uniformly detectable, that F_k and H_k are bounded, that there is a bounded nonnegative definite symmetric matrix sequence P_k satisfying (4.4) on $[k_0, \infty)$. Then $x_{k+1} = F_kx_k$ is exponentially stable.*

Proof. We shall associate with $x_{k+1} = F_kx_k$ a "Lyapunov-like" function,

$$V(x_k, k) = x_k'(P_k + \varepsilon I)x_k, \tag{4.5}$$

for some ε still to be determined. While V may not decrease at every step, we shall show that over a larger number of steps than 1, it must strictly decrease.

Setting $V_k = V(x_k, k)$, we observe that

$$V_k - V_{k+1} = x_k'H_kH_k'x_k + \varepsilon x_k'(I - F_k'F_k)x_k.$$

Two cases arise. Let d, b be the quantities of the uniform detectability definition.

Case 1. If $\|\phi_{k+t,k}x_k\| \geqq d\|x_k\|$, then under the detectability assumption $x_k'M_{k+s,k}x_kx_k \geqq bx_k'x_k$, so that

$$\begin{aligned} V_k - V_{k+s+1} &= x_k'(M_{k+s,k} + \varepsilon)x_k - \varepsilon x_k'\phi_{k+s+1,k}'\phi_{k+s+1,k}x_k \\ &\geqq [b + \varepsilon(1-\gamma)]x_k'x_k, \end{aligned}$$

where γ is an upper bound on $\|\phi_{k+s+1,k}'\phi_{k+s+1,k}\|$, which exists by virtue of the assumption of the boundedness of F_k.

Case 2. If $\|\phi_{k+t,k}x_k\| < d\|x_k\|$, then

$$\begin{aligned} V_k - V_{k+t} &= x_k'(M_{k+t-1,k} + \varepsilon)x_k - \varepsilon x_k'\phi_{k+t,k}'\phi_{k+t,k}x_k \\ &\geqq \varepsilon(1-d^2)x_k'x_k. \end{aligned}$$

Hence if ε is sufficiently small, there exists $\eta > 0$ such that

$$\max\{V_k - V_{k+t}, V_k - V_{k+s+1}\} \geqq \eta\|x_k\|^2 \geqq \delta V_k,$$

where the existence of δ follows from the bound on P_k.

This inequality shows that there is a subsequence $\{V_{k_i}\}$ of $\{V_k\}$, depending on x_0 and with ith member V_{k_i} where $k_i \leqq i(s+1)$ irrespective of x_0, such that the subsequence decays exponentially fast; i.e.,

$$V_{k_i} \leqq ab^i,$$

for some $a > 0$, $0 < b < 1$. Then because $\varepsilon x_k'x_k \leqq V_k$, a subsequence $\{\|x_{k_i}\|\}$ of $\{\|x_k\|\}$, again depending on x_0, decays at least at the same rate.

We must now show that as a result, $\{\|x_k\|\}$ also decays exponentially fast. For arbitrary k, there exists a greatest k_i with

$$k_i \leqq k < k_i + s + 1.$$

Thus any x_k can be written as $x_k = \phi_{k,k_i}x_{k_i}$, where $0 \leqq k - k_i \leqq s$. Consequently

$\|\phi_{k,k_i}\|$ is bounded, and we have

$$\|x_k\| \leqq a'b^i \leqq a'b^{k_i(s+1)^{-1}} \leqq a''(b')^k,$$

where $b' = b^{(s+1)^{-1}} < 1$ and $a'' = a'/(b')^s$. This bound holds irrespective of the sequence $\{k_i\}$ induced by the particular x_0, and exhibits the required exponential convergence. □

Remark. If (4.4) holds not over $[k_0, \infty)$ but over $[k_0, k_1]$, we can still conclude the existence of constants $\alpha > 0$, $\beta \in [0, 1)$, independent of k_0, k_1, such that

$$\|\phi_{l,k}\| \leqq \alpha\beta^{l-k},$$

for $k_0 \leqq k \leqq l \leqq k_1$ by minor variation on the above argument. This remark is crucial in establishing the dual of Lemma 4.2 and Theorem 4.3, the proof of which is otherwise straightforward and is omitted. The dual will be of use in the next section.

THEOREM 4.3. *Suppose that $[\hat{F}_k, \hat{G}_k]$ is uniformly stabilizable, that $\hat{F}_k$ and $\hat{G}_k$ are bounded, and that there is a bounded nonnegative definite matrix sequence $\hat{P}_k$ satisfying*

$$\hat{P}_{k+1} = \hat{F}_k\hat{P}_k\hat{F}'_k + \hat{G}_k\hat{G}'_k \tag{4.6}$$

on $[k_0, \infty)$. Then $x_{k+1} = \hat{F}_{kx}$ is exponentially stable. Conversely, if $x_{k+1} = \hat{F}_kx_k$ is exponentially stable, and $\hat{F}_k$, $\hat{G}_k$ are bounded, there exists a unique bounded nonnegative definite matrix sequence $\hat{P}_k$ satisfying (4.6) *on* $[k_0, \infty)$.

5. Detectability, stabilizability and state estimation. Consider the problem of state estimation for the signal process

$$\begin{aligned} x_{k+1} &= F_kx_k + G_kw_k, \\ y_k &= H'_kx_k + v_k. \end{aligned} \tag{5.1}$$

Here, $\{w_k\}$, $\{v_k\}$ are independent, zero mean, white processes with $E[w_kw'_k] = I$, $E[v_kv'_k] = I$. (A nonunit covariance for w_k is absorbed in G_k and a nonunit covariance for v_k is absorbed by scaling y_k and H'_k, so long as the covariance is nonsingular). We assume that $E[x_0x'_0] = P_0$, $E[x_0] = m$, and x_0, $\{w_k\}$, $\{v_k\}$ are independent. Finally, we assume F_k, G_k, and H_k are bounded.

The main results of the section are: $[F_k, H_k]$ uniformly detectable is sufficient for the optimal (Kalman filter) error covariance to be bounded. Furthermore, if $[F_k, G_k]$ is uniformly stabilizable, the Kalman filter is exponentially stable. Uniform detectability of $[F_k, H_k]$ is sufficient for there to exist a bounded sequence K_k such that $x_{k+1} = (F_k - K_kH'_k)x_k$ is exponentially stable.

LEMMA 5.1. *With notation and assumptions as above, and with $[F_k, H_k]$ uniformly detectable, there exists a state estimator producing a filtered state estimate for* (5.1) *with bounded error covariance.*

Proof. (By construction.) Let s, t, d, b have their usual meanings. By orthogonal transformation of the state coordinate basis at each time instant, we may assume that $M_{k+s,k} = M^1_{k+s,k} \dotplus M^2_{k+s,k}$ where $M^1_{k+s,k} \geqq bI$ and $M^2_{k+s,k} < bI$. (The symbol $\dotplus$ denotes direct sum).

Define the smoothed estimate

$$\bar{x}_{k+t|k+s} = \phi_{k+t|k}\left\{\begin{bmatrix} (M^1_{k+s,k})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \sum_k^{k+s} \phi'_{i,k}H_iy_i + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{x}_{k|k+s-t}\right\}, \tag{5.2}$$

with initialization $\hat{x}_{i|i+s-i}=0$ for $i=0,\cdots,t-1$. The partitioning in the matrix multiplying $\hat{x}_{k|k+s-i}$ is the same as that in the matrix multiplying $\sum_k^{k+s}\phi'_{i,k}H_i y_i$. Now $y_i=H'_i\phi_{i,k}x_k+[*]$, where $[*]$ is a bounded linear combination of w_j, v_j, $j\in[k,k+s]$. Consequently,

$$\hat{x}_{k+t|k+s}=\phi_{k+t,k}\left\{\begin{bmatrix}(M^1_{k+s,k})^{-1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}M^1_{k+s,k} & 0\\ 0 & M^2_{k+s,k}\end{bmatrix}x_k+\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\hat{x}_{k|k+s-t}\right\}+[*],$$

or

$$\hat{x}_{k+t|k+s}-x_{k+t}=\phi_{k+t,k}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}(\hat{x}_{k|k+s-t}-x_k)+[*].$$

Using the detectability definition and structure of $M_{k+s,k}$, it is easily seen that

$$\left\|\phi_{k+t,k}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\right\|\leqq d<1.$$

Consequently, $E[\|\hat{x}_{k+t|k+s}-x_{k+t}\|^2]$ is bounded.

Now define $\hat{x}_{k+s|k+s}=\phi_{k+s,k+t}\hat{x}_{k+t|k+s}$. Then

$$\hat{x}_{k+s|k+s}-x_{k+s}=\phi_{k+s,k+t}(\hat{x}_{k+t|k+s}-x_{k+t})+[*],$$

and clearly the error covariance bound associated with the smoothed estimate yields a bound for the filtered estimate error covariance. □

The following is immediate by optimality.

COROLLARY 5.2. *With notation and assumptions as above and with $[F_k, H_k]$ uniformly detectable, the (Kalman) filter error covariance $\Sigma_{k|k}$ and one-step predictor error covariance $\Sigma_{k+1|k}$ are bounded.*

Now we couple in a stabilizability constraint to obtain exponential stability of the Kalman filter.

THEOREM 5.3. *With notation and assumptions as above, and with $[F_k, H_k]$ detectable and $[F_k, G_k]$ stabilizable, the Kalman filter is exponentially stable.*

Proof. With K_k the Kalman filter gain, and with $\Sigma_{k|k}$ and $\Sigma_{k-1|k}$ the filtered and one-step prediction error covariances, we have from [13], quoting only the equations we need,

$$K_k=\Sigma_{k|k}H_k \tag{5.3}$$

and

$$\Sigma_{k+1|k}=(F_k-K_kH'_k)\Sigma_{k|k-1}(F_k-K_kH'_k)'+[G_kK_k][G_kK_k]'. \tag{5.4}$$

The homogeneous equation associated with the one-step predictor is $x_{k+1}=(F_k-K_kH'_k)x_k$, and it is exponentially stable if and only if the filter equation is exponentially stable. This is easily checked; a formal calculation can be found in [14].

By the assumptions and Corollary 5.2, K_k is bounded as is $\Sigma_{k+1|k}$. By Theorem 4.3, exponential stability follows if the pair $[F_k-K_kH'_k]$, $[G_kK_k]$ is uniformly stabilizable.

This is easily established as follows:

$[F_k, G_k]$ is uniformly stabilizable

$\Rightarrow$

$[F_k, [G_kK_k]]$ is uniformly stabilizable (by applying the definition)

$\Rightarrow$

$$\left[F_k + [G_kK_k]\begin{bmatrix} 0 \\ -H_k \end{bmatrix}, [G_kK_k]\right] \text{ is uniformly stabilizable}$$

(applying invariance under feedback). □

Now we have the converse to Corollary 3.4, which together with Corollary 3.4 shows the equivalence for bounded F_k, H_k of the detectability property and the existence of an output-to-state feedback law providing exponential stability.

COROLLARY 5.4. *If $[F_k, H_k]$ is uniformly detectable (and F_k, H_k are bounded), there exists a bounded sequence K_k such that $x_{k+1} = (F_k - K_kH'_k)x_k$ is exponentially stable.*

Proof. Consider the process (5.1) with $G_k = I$. Then $[F_k, G_k]$ is stabilizable, and Theorem 5.3 provides the result.

We can also consider the necessity of the detectability condition. Certainly, detectability is necessary for there to be an exponentially stable estimator of the type

$$\hat{x}_{k+1|k} = F_k\hat{x}_{k|k-1} + K_k(y_k - H'_k\hat{x}_{k|k-1});$$

(this is effectively the content of Corollary 3.4). However, we can get a slightly sharper result.

COROLLARY 5.5. *Consider the process (5.1) and associated assumptions, and with $[F_k, G_k]$ uniformly stabilizable. Suppose that the associated optimal filter error covariance is bounded. Then $[F_k, H_k]$ is uniformly detectable.*

Proof. In (5.4), $\Sigma_{k+1|k}$ is bounded and the pair $F_k - K_kH'_k$, $[G_kK_k]$ is uniformly stabilizable. By Theorem 4.3, $x_{k+1} = (F_k - K_kH'_k)x_k$ is exponentially stable, and Corollary 3.4 then yields the result. □

Remarks. 1. The main theorem of this section appeals to almost all the important results of the preceding section. As well, it appeals to the suboptimal estimator construction of Lemma 5.1, which is not trivial and considerably more complicated than constructions which have been used in studying observable processes; see, e.g., [4], [14]. In particular, we were not able here to define an exponentially stabilizing feedback law K_k simply in terms of the observability matrix M, as can be done in the observable case, [14].

2. The corresponding regulator result of course follows by duality, though some care has to be taken because of the fact that with the interval for which the filter is studied being $[0, \infty)$, its dual is $[-\infty, 0)$, while we wish to study the regulator over $(0, \infty)$. The considerations of the remark preceding Theorem 4.3 can be applied to overcome this difficulty. An alternative approach to the regulator problem is to show that complete stabilizability implies the existence of a control yielding a bounded performance index, to conclude then that the optimal index is bounded and achievable with a linear feedback law, and to show under a detectability assumption that the closed-loop is exponentially stable. The construction of the control yielding a bounded performance index is not straightforward; the construction procedure in some way has to parallel the construction of Lemma 5.1.

Finally, in this section, we illustrate that the feedback invariant $\tilde{d}$ defines an achievable bound on how stable we can make a closed-loop system via feedback.

THEOREM 5.6. *With F_k, H_k bounded and uniformly detectable, and with $\bar{d}$ as defined in Corollary 3.5, there exists a feedback law K_k such that all trajectories of $x_{k+1} = (F_k - K_k H'_k)x_k$ decay at least as fast as $(\bar{d}+\varepsilon)^k$ for arbitrary $\varepsilon > 0$.*

Proof. Define the detectability property of $[F_k, H_k]$ using $\bar{d}+\varepsilon/2$, b, t, and s, as we are entitled to do. Choose so that $\rho'(\bar{d}+\varepsilon) = 1$. Then $[\bar{\rho}F_k, H_k]$ is detectable, by Lemma 2.3 and the remark following the lemma. Find K_k so that $x_{k+1} = (\rho F_k - K_k^{\rho} H'_k)x_k$ is exponentially stable. Then choosing $K_k = \rho^{-1}K_k^{\rho}$ ensures that $x_{k+1} = (F_k - K_k H'_k)x_k$ has the desired property.

Remarks. 1. The discussion following Corollary 3.5 makes it clear that we could not obtain a feedback law K_k such that the closed-loop system trajectories decay as fast as $(\bar{d}-\varepsilon)^{k/t}$ for some ε with $0 \leqq \varepsilon \leqq \bar{d}$.

2. As was noted in Remark 1 following Corollary 5.5, we are unable to define a stabilizing gain sequence K_k simply in terms of the observability matrix associated with a detectable pair $[F_k, H_k]$. The construction given for the stabilizing gain sequence via Corollary 5.4 has the potential disadvantage that K_k depends on F_l, H_l for all $l \leqq k$. This is at least a "causal" dependence; when one considers the problem of constructing a stabilizing sequence $\hat{K}_k$ for a stabilizable pair $[\hat{F}_k, \hat{G}_k]$, the disadvantage is that $\hat{K}_k$ depends on $\hat{F}_l$, $\hat{G}_l$ for all $l \geqq k$. This leads one to consider whether or not there might be a sequence dependent on a finite "window" only of $[F_l, H_l]$ or $[\hat{F}_l, \hat{G}_l]$. Indeed there is. We describe the detectable situation only. Following the idea of the proof of Theorem 5.6, the Kalman filter equations are solved forward in time with F_k replaced by ρF_k, $\Sigma_{0|-1} = I$, $Q_k = I$, save that at some set of times $r_1, r_2, \cdots$, the choice of which is described below, we replace the value of $\Sigma_{r_i|r_i-1}$ predicted by the equations by $\Sigma_{r_i|r_i-1} = I$, amounting to a reinitialization. This has the effect of causing K_k^{ρ} and K_k for $k \in [r_i, r_{i+1}-1]$ to depend on F_l, H_l for $l \in [r_i, r_{i+1}-1]$, i.e., on no more than $r_{i+1} - r_i$ values of F_k, H_k. The integers r_i are chosen so that, with $\bar{\phi}_{k,l}$ the transition matrix associated with $F_k - K_k H'_k$, one has $\|\bar{\phi}_{r_{i+1},r_i}\| < 1$. Actually, a lengthy argument will show that r_i may be taken as ir for some appropriately large integer r. The upshot is the desired exponential decay property.

6. Coordinate basis choice to display detectability. If $[F, H]$ is a time-invariant detectable pair, it is well known (see [3]) that if the coordinate basis is chosen satisfactorily, then we can have

$$F = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}, \qquad H' = [H'_1 \quad 0], \tag{6.1}$$

with $[F_{11}, H_1]$ observable and $|\lambda_i(F_{22})| < 1$. We seek here a time-varying version of this result.

Assume $[F_k, H_k]$ is detectable, and d, b, s, and t have their usual meanings. We shall take b here to be very small and assume also that, for some arbitrary but fixed k,

(a) $M_{k+s+1,k}$, $M_{k+s,k}$ and $M_{k+s,k-1}$ have precisely p eigenvalues less than b, and the remaining eigenvalues much greater than b.

(b) By orthogonal changes of basis and without loss of generality or variation of the stability properties, $M_{k+s,k-1}$ and $M_{k+s+1,k}$ are diagonal, with diagonal elements taking decreasing values down the diagonal.[1]

Under these assumptions and with

$$H'_{k-1} = [H^{1'}_{k-1} \quad \overbrace{H^{2'}_{k-1}}^{p}], \qquad F_{k-1} = \begin{bmatrix} F^{11}_{k-1} & \overbrace{F^{12}_{k-1}}^{p} \\ F^{21}_{k-1} & F^{22}_{k-1} \end{bmatrix}\}p,$$

[1] Note that if $\bar{x}_k = T_k x_k$ defines the orthogonal basis change, $M_{k+s,k} \to \bar{M}_{k+s,k} = T_k M_{k+s,k} T'_k$.

we assert that $\|H^2_{k-1}\|^2$ and $\|F^{12}_{k-1}\|^2$ are $O(b)$. Since b is small, this means that F_{k-1} and H_{k-1} are approximately of the form of (6.1). Furthermore, we can show that $\phi^{22}_{k+i,k} < d + O(b^{1/2})$, and $\|F^{22}_{k+i-1}F^{22}_{k+i-2}\cdots F^{22}_k\| < d + O(b^{1/2})$. This mimics the requirement in the time-invariant case that $|\lambda_i(F_{22})| < 1$. Finally, an observability result can be obtained for the pair $[F^{11}_k, H^1_k]$. In case F_k, H_k are constant, the time-invariant results are evidently recovered. Because of length restrictions, proofs are omitted.

7. Conclusions. Given the now widespread knowledge of the linear-quadratic problem and its solutions, the results of this paper are not particularly surprising. Certainly, when the ideas of the paper were being developed many of the conjectures were clear. In hindsight, there is also no real surprise in the techniques required to obtain the results. However, we must admit that many of the specific techniques, especially that of Lemma 5.1, surfaced only after exploring a number of misleading approaches and conjectures. Perhaps this accounts for the comparatively long time between the intuitive grasp of the general nature of these results and their formal derivation.

It is clear that one of the main applications of the results is to the linear-quadratic problem. However, we feel it likely that the extended lemma of Lyapunov is a result of some power, which should also find significant application. We have, for example, recently been able to use this lemma to establish that if a linear, finite-dimensional, uniformly stabilizable and detectable system is bounded-input, bounded-output stable, then the system is necessarily exponentially stable.

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