Asymptotically Fast Solution of Toeplitz and Related Systems of Linear Equations*

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ABSTRACT

We present an inversion algorithm for the solution of a *generic* $N\times N$ Toeplitz system of linear equations with computational complexity $O(N\log^2 N)$ and storage requirements $O(N)$. The algorithm relies upon the known structure of Toeplitz matrices and their inverses and achieves speed through a doubling method. All the results are derived and stated in terms of the recent concept of displacement rank, and this is used to extend the scope of the algorithm to include a wider class of matrices than just Toeplitz and also to include block Toeplitz matrices.

1. INTRODUCTION

A Toeplitz matrix T is a matrix whose elements satisfy $T_{ij}=t(i-j)$, $0\leqslant i,j\leqslant n$, so that it has a "striped" appearance with stripes running parallel to the main diagonal. Toeplitz systems of linear equations arise in many physical data-processing applications (see e.g. [1]). In particular the solution of such systems of equations is seen as a key step in the solution of truncated Wiener-Hopf equations [2, 3] which arise in least-squares estimation problems.

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Because of this wide application, the desire for fast on-line solution, and the frequent large size of the system of equations, there is a considerable demand for solution methods utilizing the structure of the Toeplitz matrix to achieve solutions with a low number of computations and small storage. For an $N\times N$ Toeplitz system of equations solution may be effected without use of the particular structure by any standard solution package which will require $O(N^3)$ computations (additions and multiplications). Levinson [4] gave an algorithm which allowed solution of Toeplitz systems with computational complexity $O(N^2)$. His algorithm has been the basis of several other recursive schemes for the $O(N^2)$ solution of these systems and for the inversion of Toeplitz matrices. Among these schemes are the works of Trench [5], Zohar [6], and Akaike [7].

Recently Brent, Gustavson, and Yun [8] have developed new algorithms for the fast computation of solutions to large Toeplitz systems of equations having computational complexity $O(N\log N\log N)$ and linear storage [i.e. $O(N)$]. For an introduction to these results, see [24]. In this paper we shall also present a solution algorithm which has complexity and storage requirements of this same order, but which seems more flexible in its applicability and extension to classes of non-Toeplitz matrices and to block Toeplitz matrices.

The approach of [8] is via the computation of Padé approximants using fast doubling Euclidean algorithms akin to, but slightly faster than, the HGCD algorithm of [9] for calculating greatest common divisors of pairs of polynomials. Our method makes no contact with either of these ideas, and is therefore conceptually simpler. It draws upon the known structure of the Toeplitz inverse as the sum of two products of two triangular Toeplitz matrices and uses a doubling technique to achieve speed. As our algorithm and that of [8] have the same complexity and storage, many of the advantageous properties of the approach of [8] carry over without modification to the approach of this paper. These include the option of fast iterative refinement of the solution and certain faster methods for banded Toeplitz systems.

However, because our methods relate directly to the inverse and its description by sums of products of triangular matrices, the approach is easily modified to apply to systems of any given displacement rank [11–12] and the application of the method to a wide class of discrete-time linear estimation problems is apparent; see [12, 13]. This concept of displacement rank and the decomposition to triangular matrices developed in [10–13] have been used in other applications, e.g. [14, 15, 16]. Further, the extension from the inversion of Toeplitz matrices with scalar entries to the inversion of block Toeplitz matrices with matrix entries is relatively painless, as the structured formulae for scalar-entried Toeplitz inversion carry over directly to block Toeplitz inversion [17]. This latter extension could be difficult for the algorithm of [8], as currently available faster techniques for the computation of greatest common left and right divisors of matrix polynomial pairs may be numerically unstable [18] and in any case appear slower than HGCD.

One advantage of the methods of [8] over our algorithm is that they are able to present an algorithm which always yields the solution if it exists, while our scheme relies on a generic property of the system of equations. This is akin to the requirement of "normality" of the Padé table for the MD algorithm of [8] and is a familiar but very mild restriction, as will be pointed out later, in Secs. 2, 3, and 5.

We have learnt, since initially submitting this paper, that M. Morf has independently derived a very similar algorithm to that of this paper and that this has recently been submitted for publication [25].

The paper is organized as follows. In Sec. 2 we present the concept of displacement rank and show how it relates to a class of inversion problems which includes Toeplitz inversion. Section 2 also contains a collection of lemmas which demonstrate the computational aspects of dealing with matrices of given displacement rank. Section 3 consists of the statement of the solution algorithm after an examination of the displacement rank properties of the inverse of a matrix of given displacement rank. Section 4 presents the simple extension from the earlier results for matrices with scalar elements to matrices with matrix entries. The conclusion, Sec. 5, sums up, and we suggest possible areas for improvement of the results.

2. DISPLACEMENT RANK AND PRELIMINARY RESULTS

Of great importance in the development and generalization of our method is the concept of displacement rank. It was first stated in the open literature in [10], but see [11, 12] for detailed proofs.

Definition [12]. The $(+)$-displacement rank of an $N\times N$ matrix R is the smallest integer $\alpha_+(R)$ such that one may write

$$R=\sum_{i=1}^{\alpha_+(R)} L_i U_i$$

for some lower-triangular Toeplitz matrices $\{L_i\}$ and some upper-triangular Toeplitz matrices $\{U_i\}$.

Definition [12]. The $(-)$-displacement rank of an $N\times N$ matrix R is the smallest integer $\alpha_-(R)$ such that we can write

$$R=\sum_{i=1}^{\alpha_-(R)} \overline{U}_i\overline{L}_i$$

for some lower-triangular Toeplitz matrices $\{\bar{L}_i\}$ and some upper-triangular Toeplitz matrices $\{\bar{U}_i\}$.

These displacement ranks were proposed as a measure of the closeness to being Toeplitz of an arbitrary matrix R. Indeed, for a Toeplitz matrix T, $\alpha_+(T)=\alpha_-(T)\leqslant 2$ (and generically $=2$), since $T=T_+I+IT_-$, where T_+ and T_- are the upper and lower triangular parts of T. For invertible matrices the following relationship is also established.

LEMMA 1 [12]. *For an invertible matrix R, one has $\alpha_+(R)=\alpha_-(R^{-1})$ and $\alpha_-(R)=\alpha_+(R^{-1})$.*

This lemma tells us that the displacement ranks of the inverse of an invertible Toeplitz matrix are both two, i.e., $T^{-1}=L_1U_1+L_2U_2=U_3L_3+U_4L_4$ for some triangular Toeplitz matrices L_i, U_i. This latter result is of course well known [2, 20]. However, the following lemma clarifies the origins of the triangular Toeplitz matrices and indicates the reason for the name "displacement rank."

LEMMA 2 [12]. *The displacement ranks of a matrix R may be computed as*

$$\alpha_+(R)=\operatorname{rank}\{R-ZRZ'\},\qquad \alpha_-(R)=\operatorname{rank}\{R-Z'RZ\},$$

where the prime denotes transpose and Z is the displacement matrix

$$Z=\begin{bmatrix} 0 & & & \\ 1 & \cdot & 0 & \\ & \cdot & \cdot & \\ & & \cdot & \\ 0 & & 1 & 0 \end{bmatrix}.$$

Further, given column vectors $\{x_i, y_i;\ i=1,\dots,\alpha\}$, the functional equations

$$R-ZRZ'=\sum_{i=1}^{\alpha} x_iy_i' \quad \text{and} \quad R-Z'RZ=\sum_{i=1}^{\alpha} x_iy_i'$$

have the unique solutions (respectively)

$$R=\sum_{i=1}^{\alpha} L(x_i)U(y_i) \quad \text{and} \quad R=\sum_{i=1}^{\alpha} U(\bar{x}_i)L(\bar{y}_i),$$

where $L(x)$ denotes a lower-triangular Toeplitz matrix whose first column is x, $U(y)=L(y)'$, and $\bar{x}'=[x_N\ \cdots\ x_1]$ when $x'=[x_1\ \cdots\ x_N]$.

In the derivation of our inversion algorithm we shall be repeatedly using the properties of the triangular Toeplitz matrices and the knowledge of displacement rank to numerical advantage, and so here we present the following results, which are fundamental to the inversion algorithm. Where proofs are straightforward they are omitted.

LEMMA 3. *The product of the two lower-triangular Toeplitz matrices* $L(x)$ *and* $L(y)$ *with* $x'=[x_1\ x_2\ \cdots\ x_n]y'=[y_1\ y_2\ \cdots\ y_n]$ *is the lower-triangular Toeplitz matrix* $L(x)L(y)=L(z)$, *where* $z'=[z_1\ z_2\ \cdots\ z_n]$ *with* $z_i=\sum_{j=1}^{n}x_j y_{i-j+1}$ (*taking* $y_i=0$ *for* $i\notin[1,n]$).

This lemma demonstrates the closure of lower-triangular Toeplitz matrices under multiplication and further illustrates that the first column of the product is composed simply of half the convolution of the first columns of the multiplicands. There is an obvious counterpart for the product of two upper-triangular Toeplitz matrices. (To see this simply transpose the product of two lower-triangular Toeplitz matrices.) The nature of these products will later be shown to facilitate low-complexity, low-storage multiplication and handling of these matrices. Convolution is also relevant in considering UL and LU Toeplitz products, which we now examine.

LEMMA 4. *The elements of the product of a lower-triangular Toeplitz matrix* $L(x)$ *and an upper-triangular Toeplitz matrix* $U(y)$ *are computable as convolutions of subsequences of* $[x_1\ x_2\ \cdots\ x_n]$ *and* $[y_n\ y_{n-1}\ \cdots\ y_1]$. *In particular, denoting convolution by* $*$,

$$[L(x)U(y)]_{ij}$$

$$=\begin{cases} i\text{th element of } (x_1\ x_2\ \cdots\ x_j)*(y_n\ y_{n+1}\ \cdots\ y_{n-j+1}), & i<j,\\ 2n-j\text{th element of } (x_1\ x_2\ \cdots\ x_i)*(y_n\ y_{n-1}\ \cdots\ y_{n-i+1}), & i>j.\end{cases}$$

The equivalent formulation for the product of $U(x)L(y)$ is derivable simply by noting that $[U(x)L(y)]_{ij}=[L(x)U(y)]_{n-i+1,n-j+1}$. The role of this result in the algorithm derivation is to exhibit the fact that the first rows and columns and the last rows and columns of a product of dissimilar triangular Toeplitz matrices may be calculated by methods which take advantage of the structure.

Having presented the above results on the multiplication of the triangular Toeplitz matrices, we next consider how to describe an LU product as a sum of UL products and vice versa. In stating our lemma we extend a result of [12] that $|\alpha_+(R)-\alpha_-(R)|\leq 2$.

LEMMA 5. *We have the following identity*:

$$L(x)U(y)=IL(z)+U(w)I-U(\hat{x})L(\hat{y}),$$

where

$\hat{x}'=[0\ x_n\ \cdots\ x_3\ x_2]$, $\hat{y}'=[0\ y_n\ \cdots\ y_3\ y_2]$,
z' = *final row of* $L(x)U(y)$ *with order reversed*,
w = *final column of* $L(x)U(y)$ *with order reversed*.

Proof. By evaluating, for Z defined as in Lemma 2,

$$L(x)U(y)-Z'L(x)U(y)Z$$

$$=-\bar{\hat{x}}\bar{\hat{y}}'+[0\ \cdots\ 0\ 1]'\bar{z}'+\bar{w}[0\ \cdots\ 0\ 1].$$

We then apply Lemma 2 to obtain the result. ■

Having presented these last few lemmas concerned with the properties of combinations of triangular Toeplitz matrices, we return to the constructive method indicated in Lemma 2 for the generation of this sum of products of Toeplitz matrices. We ask the following question. Suppose that we know R to have (+)-displacement rank two and we are given R-ZRZ'. How do we calculate a dyadic decomposition of R-ZRZ' in the generic case? We present the following easily verified generic solution.

LEMMA 6. *Let S be a rank m matrix of size* $n\times n$ *with a generically nonsingular* $m\times m$ *leading principal submatrix A. Then* $S=\sum_{i=1}^{m}x_i y_i'$ *for n-vectors* $\{x_i\}$, $\{y_i\}$, *where* x_i *is the ith column of S and* y_i' *is the ith row of* A^{-1} *times the matrix consisting of the first m rows of S*.

This lemma will have application to the dyadic description of displaced matrices to yield a summed Toeplitz product decomposition. For LU sums we shall consider a leading principal submatrix as in Lemma 6, while for UL we shall look at a trailing principal submatrix, employing a minor variation on Lemma 6.

In the next section we shall apply the previous lemmas to derive an inversion procedure.

3. INVERSION AND SOLUTION ALGORITHM

Here we develop a fast $O(N\log N\log N)$-complexity algorithm for the solution of Toeplitz systems of equations. The method relies substantially on the results of the previous section to achieve speed and low storage, and proceeds in outline as follows. Firstly we express the inverse of a $2N\times 2N$ matrix in terms of operations on $N\times N$ submatrices and show that displacement-rank properties of the larger matrix are reflected in the properties of the submatrices. (There is no compulsion for even dimension, but we find that notation is simpler if we assume the dimension is a power of 2.) For matrices with low displacement rank, including the Toeplitz case, this then leads us to apply the earlier results to develop fast methods of computing the triangular Toeplitz matrices of a representation of the $2N\times 2N$ inverse by operations on the triangular Toeplitz matrices of the representation of the $N\times N$ submatrices. Since operations on triangular Toeplitz matrices basically involve calculation of convolutions, we employ fast-Fourier-transform (FFT) methods, which have complexity $O(N\log N)$, for the calculation of the convolution of two N-length sequences [9]. (It may be necessary to augment the sequences with zeros prior to FFT so that the sequence length is a power of 2.) Finally, having calculated the triangular Toeplitz decomposition of the inverse, we show how Trench's [5] method can be implemented via FFT to yield the solution to the Toeplitz system of equations. This last step has complexity $O(N\log N)$ and has also been discussed by Chin and Steiglitz [21] and by Brent, Gustavson, and Yun [8].

We begin with the following easily established elementary result.

Lemma 7. *Let T be an $m\times m$ invertible matrix subdivided into $k\times k$, $k\times(m-k)$, $(m-k)\times k$, and $(m-k)\times(m-k)$ submatrices T_{11}, T_{12}, T_{21}, T_{22} as shown below. Then $S=T^{-1}$ is partitioned similarly into S_{11}, S_{12}, S_{21}, S_{22}:*

$$T=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},\qquad S=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where, assuming T_{11} is invertible,

$$S_{11}=T_{11}^{-1}+T_{11}^{-1}T_{12}\left(T_{22}-T_{21}T_{11}^{-1}T_{12}\right)^{-1}T_{21}T_{11}^{-1},$$

$$S_{12}=-T_{11}^{-1}T_{12}\left(T_{22}-T_{21}T_{11}^{-1}T_{12}\right)^{-1},$$

$$S_{21}=-\left(T_{22}-T_{21}T_{11}^{-1}T_{12}\right)^{-1}T_{21}T_{11}^{-1},$$

$$S_{22}=\left(T_{22}-T_{21}T_{11}^{-1}T_{12}\right)^{-1}.$$

In our application of this result we shall be considering matrices T with given displacement rank, which for practical purposes should be significantly less than $\dim T$. Further, we shall generally be examining the subdivision of a

$2n\times 2n$ matrix into $n\times n$ submatrices, although the theory carries over for different subdivisions and there is no compulsion for the matrix to have even dimension for the algorithm to work. This latter operation allows us to achieve a doubling algorithm.

We now consider the hereditary nature of the displacement rank from T to its submatrices and the submatrices of its inverse.

LEMMA 8. *With T as in Lemma 7, suppose that each T_{ij} is $n\times n$ and that T has $(+)$-displacement rank α_0. Then: $\alpha_+(T_{11}), \alpha_-(T_{11}^{-1}), \alpha_+(T_{22}-T_{21}T_{11}^{-1}T_{12}), \alpha_-[(T_{22}-T_{21}T_{11}^{-1}T_{12})^{-1}]$ have maximum value α_0; $\alpha_+(T_{12})$ and $\alpha_+(T_{21})$ have maximum value α_0+1; $\alpha_+(T_{22})$ has maximum value α_0+2.*

Proof. To evaluate $\alpha_+(T_{11})$, $\alpha_+(T_{12})$, $\alpha_+(T_{21})$, and $\alpha_+(T_{22})$ we compare the submatrices of $T-Z_{2n}TZ'_{2n}$, each of which has maximum rank α_0, with the displaced submatrices such as $T_{11}-Z_nT_{11}Z'_n$, etc.

To evaluate $\alpha_-[(T_{22}-T_{21}T_{11}^{-1}T_{12})^{-1}]$ we first repeat the previous operation to compare the (2,2) block element of $S-Z'_{2n}SZ_{2n}$ with $S_{22}-Z'_nS_{22}Z_n$, and find that they are identical. Hence $\alpha_-[(T_{22}-T_{21}T_{11}^{-1}T_{12})^{-1}]\leqslant\alpha_-(S)$ which, by Lemma 1, equals $\alpha_+(S^{-1})=\alpha_+(T)=\alpha_0$. The remaining two properties follow from the above with Lemma 1. ■

We are now in a position to present the algorithm for expressing the inverse of a $2n\times 2n$ matrix T with $\alpha_+(T)=\alpha_0$ as a sum of α_0 products of pairs of upper-triangular Toeplitz matrices U_i and lower-triangular Toeplitz matrices L_i. For clarity we present the algorithm in stepwise form rather than in some high-level computer language which permits recursive calling of subroutines.

ALGORITHM.

Step 1. Call algorithm to provide $T_{11}^{-1}=\sum_{j=1}^{\alpha_0}U_jL_j$ for $n\times n$ upper- and lower-triangular Toeplitz matrices U_j, L_j.

Step 2. Use method of Lemma 2 and Lemma 6 to express $T_{21}=\sum_{k=1}^{\alpha_0+1}L_kU_k$ and $T_{12}=\sum_{l=1}^{\alpha_0+1}L_lU_l$.

Step 3. Multiply $T_{21}T_{11}^{-1}T_{12}$ by: combining adjacent L_iL_j and U_iU_j pairs as in Lemma 3; converting some L_iU_j pairs to U_kL_l as per Lemma 5; repeating until we have $T_{21}T_{11}^{-1}T_{12}=\sum_{m=1}^{\beta}L_mU_m$ for some (possibly non-minimal) β. Use FFT for multiplication and convolutions.

Step 4. Compute first α_0 rows and α_0 columns of $T_{22}-T_{21}T_{11}^{-1}T_{12}$ using Lemma 4.

Step 5. Factor $T_{22}-T_{21}T_{11}^{-1}T_{12}=\sum_{n=1}^{\alpha_0}L_nU_n$ using the method of Lemma 2 and Lemma 6.

Step 6. Call algorithm to provide $S_{22}=(T_{22}-T_{21}T_{11}^{-1}T_{12})^{-1}=\sum_{p=1}^{\alpha_0}U_pL_p$.

Step 7. Use FFT methods as in Step 3 to calculate $S_{12} = -T_{11}^{-1}T_{12}S_{22}$ and $S_{21} = -S_{22}T_{21}T_{11}^{-1}$ as possibly nonminimal sums of products of triangular Toeplitz matrices.

Step 8. Compute the last α_0+1 rows and α_0+1 columns of $S=T^{-1}$ from S_{12}, S_{21}, and S_{22} using Lemma 4.

Step 9. Compute the last α_0 rows and α_0 columns of $S-Z'SZ$ from information at step 8.

Step 10. Factor $S=\Sigma_{q=1}^{\alpha_0} U_q L_q$ using the method of Lemma 2 and Lemma 6.

The correctness of the algorithm for generic invertible matrices follows from the preceding remarks.[1] Here we now examine the computational complexity and storage requirements of the scheme.

THEOREM 1. *With fast Fourier transforms used to evaluate all convolutions, the above algorithm requires $O(N\log^2 N)$ computations (additions and multiplications) and $O(N)$ storage for the expression of the inverse of an $N\times N$ matrix of $(+)$-displacement* rank α_0 *as a sum of α_0 products of upper- and lower-triangular Toeplitz matrices.*

Proof. Denote by $f(N)$ the number of computations required by the algorithm to find the expression for the inverse of an $N\times N$ matrix as in the theorem statement. Then step 1 takes $f(N/2)$ computations, with storage requirements $O(N)$, since only the first row of each U_i and the first column of each L_i need be stored. Step 2 requires $O(N)$ computations and $O(N)$ storage. Step 3 requires only the calculation of a fixed number of $N\times N$ convolutions and hence $O(N\log N)$ computations and $O(N)$ storage. Similarly for the other steps: step 4, $O(N\log N)$ and $O(N)$; step 5, $O(N)$ and $O(N)$; step 6, $f(N/2)$ and $O(N)$; step 7, $O(N\log N)$ and $O(N)$; step 8, $O(N\log N)$ and $O(N)$; step 9, $O(N)$ and $O(N)$; step 10, $O(N)$ and $O(N)$.

From this evaluation we have for computational complexity

$$\begin{aligned} f(N) &= 2f\left(\frac{N}{2}\right) + O(N\log N) \\ &= O\left(N\log N + 2\cdot\frac{N}{2}\log\frac{N}{2} + 4\cdot\frac{N}{4}\log\frac{N}{4} + \cdots\right) \\ &= O(N\log^2 N), \end{aligned}$$

[1] If the principal minor chosen in the application of Lemma 6 includes elements in the first or last row and column of the displaced matrix, according as + or − displacement is involved, which row and column also appear in the original matrix, then we are generically assured that this minor is nonzero.

since there are $\log N$ terms in the sum.[2] Similarly for storage $g(N)$, since only one inversion is performed at a time and the storage space of the inverse is $O(N)$,

$$g(N) = g\left(\frac{N}{2}\right) + O(N)$$

$$= O(N). \qquad \blacksquare$$

We next apply the above result to the solution of the system of linear equations

$$Tx = y$$

for x, given a vector y and matrix T having $\alpha_+(T) = \alpha$. With $O(N\log^2 N)$ complexity we calculate, via the algorithm, $T^{-1} = \sum_{i=1}^{\alpha_0} U_i L_i$. We know then that

$$x = \sum_{i=1}^{\alpha_0} U_i L_i y,$$

and it becomes apparent by observation (of Lemma 3) that the elements of $z = L_i y$ are simply computable as part of the convolution of the first column of L_i and the elements of y. Similarly $x_i = U_i z$ is a part of the convolution of the first row of U_i and the elements of z. These $2\alpha_0$ convolutions to find x are performable by FFT.

COROLLARY 1 [21]. *Given the description of T^{-1} as $\sum_{i=1}^{\alpha} U_i L_i$, the solution of $Tx = y$ may be achieved via* FFT *with complexity $O(N\log N)$. Hence $Tx = y$ is solvable by the above methods in $O(N\log^2 N)$ computations.*

4. BLOCK TOEPLITZ AND RELATED SYSTEMS

Just as Toeplitz matrices with scalar entries arise in many applications, often through the solution of a truncated Wiener-Hopf equation, so do Toeplitz matrices with square matrix entries of equal dimension arise as solutions to truncated multivariable Wiener-Hopf equations. These equations

[2] A more exact derivation yields $f(N) = 48N\log N\log N - 36N\log N + 71N$ for Toeplitz matrices and approximately $f(N) = (4\alpha_0^3 + \alpha_0^4)N\log N\log N + (5\alpha_0 - 2\alpha_0^3 - \alpha_0^4)N\log N + (4\alpha_0^2 + 1)N$ in general. It would be desirable to calculate the crossover value of N for this method and for that of [11], but first it would be necessary to perform an accurate operation count on the algorithm of [11].

in turn arise in applications such as multiple time series, geophysical data processing, etc.

Gohberg and Heinig [17] have shown that many of the standard theorems concerned with the inversion of scalar Toeplitz matrices carry over with slight modification to block Toeplitz matrices (or actually any Toeplitz matrix with elements from a noncommutative normed algebra). In particular the expression for the inverse of the Toeplitz matrix as the sum of two products of pairs of triangular Toeplitz matrices goes over. Indeed, all of the lemmas and definitions of previous sections may be given independently of the commutativity of the elements of the Toeplitz matrices or the matrices of given displacement rank.

With this observation, all that remains to extend the method of the algorithm to include block Toeplitz matrices and maintain its speed and low storage is to demonstrate that we may still carry out matrix convolutions using the FFT with the same order of computational complexity. We have

LEMMA 9. *Let $\{A_k\},\{B_k\}$, $k=1,2,\dots,n$, be sequences of $p\times q$ and $q\times r$ matrices respectively. Define the convolution of these sequences as*

$$(A*B)_k=\sum_{\tau=1}^{k}A_\tau B_{k-\tau+1},\qquad k=1,2,\dots,2n-1.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be the element-by-element discrete Fourier transform (DFT) matrices of the sequences $\{A_k\}$, $\{B_k\}$. Then

$$\mathcal{A}\mathcal{B}=\mathrm{DFT}(A*B)$$

Proof. Elementary. ∎

This result shows that the convolution of the matrix sequences given in the lemma statement may be performed in $O(p^2N\log N)$ operations assuming $p=q=r$. If $p\ll N$, then this allows the straightforward extension of Theorem 1 and Corollary 1 to the block-element case.

THEOREM 2. *The algorithm of Sec. 3 may be used to evaluate the (block) solution x to the block $N\times N$ Toeplitz system of equations $Tx=y$ with computational complexity $O(N\log^2 N)\times p^2$, where $p\times p$ is the dimension of the blocks.*

This ease of extension from scalar-entried matrices to matrix-entried matrices is a key feature of our proposed scheme and demonstrates the value of its direct approach in passing from ordinary Toeplitz to related problems.

5. CONCLUSION

We have presented a recursive algorithm allowing the asymptotically fast solution of an $N\times N$ system of linear equations with a given displacement rank (Toeplitz systems have displacement rank 2). The computational complexity of the scheme is $O(N\log^2 N)$, while the storage is $O(N)$. The algorithm utilizes the structural and displacement-rank properties of the matrices and their inverses, and while the chief application is to Toeplitz systems, the scheme may also be used in other, non-Toeplitz situations such as some nonstationary estimation problems [10–16]. Further, the extension to block Toeplitz systems is straightforward.

As with any algorithm, there is the question of numerical stability and propagation of roundoff errors during calculations. We have not yet fully investigated this area, but recent results of Cybenko [22] on the propagation of roundoff errors for the $O(N^2)$ algorithms of Durbin, Levinson, and Trench are encouraging.

For banded Toeplitz systems there exists the possibility of tracing through the propagation of the banded property from the original matrix to its triangular Toeplitz components. This could yield a faster method of banded Toeplitz solution, but would probably be no faster than that of Jain [19] nor as fast as that of Dickinson [14] and considerably less elegant. Methods for sparse (but not banded) Toeplitz systems would also be valuable, and it would be interesting to see whether the method of [19] can be generalized.

It might be that such methods could be based on the fact that the LU decomposition of a sparse matrix has sparse triangular matrices [23]; this suggests that the triangular Toeplitz decomposition of a matrix with low displacement rank may involve sparse triangular Toeplitz matrices.

Perhaps the major area where new variations on the results given here would be useful is in connection with the genericity requirement and its appearance in Lemma 6. In particular, for the most important case of Toeplitz matrices which are positive definite symmetric covariance matrices, one can foresee that a search of the N 2×2 principal minors of the displaced matrix which include the fixed corner element would allow inversion whenever possible. This avenue remains to be explored, but it should be noted that a requirement in one of the standard formulas for the inverse of a Toeplitz matrix (see [2]) is that a certain $(N-1)\times(N-1)$ principal minor be nonzero. In the event that this condition on the minor does not hold, the problem is obviated by expanding the given $N\times N$ Toeplitz matrix to an $(N+1)\times(N+1)$ Toeplitz matrix, finding its inverse using the formula, and then recovering the inverse of the smaller matrix from that of the larger. This same device, which we stress is generically unnecessary, appears applicable to Lemma 6.

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