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Theoretical Aspects of Optimal Autocorrelation Synthesis*

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ABSTRACT

In this paper we consider several theoretical aspects of signal design when the signal-matched filter output is required to possess certain optimality properties. The problem is formulated as a constrained minimax problem in function space. Necessary conditions for optimality are obtained, and a useful numerical algorithm is derived. Several uncertainty-principle-type results are also established.

1. INTRODUCTION

A particularly important problem in radar systems is the design of signals which allow both optimal detection and unambiguous resolution of targets. For the receiver configuration shown in Fig. 1 it is well known that passing the noise-corrupted received signal s through a *matched filter* maximizes the probability of detecting the presence of s, and that the detection probability is independent of the detailed signal shape, with the signal energy [1, 2]

$$\|s\|^2=\int s^2(\tau)\,d\tau=E$$

being the only relevant signal quantity when the noise is white. If the noise power spectrum is a constant amplitude of $N_0/2$ watts/Hz, the matched-filter output signal-to-noise ratio is given by E/N_0. The proof of this is a direct consequence of the Schwarz inequality. For any linear time-invariant filter

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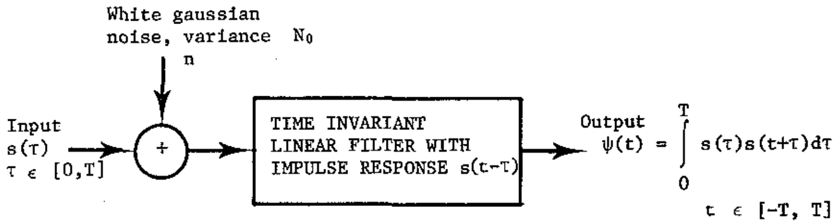

Fig. 1. Basic receiver configuration.

with impulse response u the output signal-to-noise ratio becomes

$$\left(\frac{S}{N}\right)=\frac{\max_t\left|\int u(\tau)s(t-\tau)\,d\tau\right|^2}{N_0\int u^2(\tau)\,d\tau}$$

$$\leqslant\frac{\int s^2(\tau)\,d\tau\int u^2(\tau)\,d\tau}{N_0\int u^2(\tau)\,d\tau}=\frac{E}{N_0},$$

and when u is a matched filter $[u(\tau)=s(t-\tau)]$, equality is attained. Clearly then, to achieve good detection we need to ensure that our receiver is a matched filter.

Now to achieve high resolution without detection ambiguities we also require that the matched-filter response,

$$\psi(t)=\int s(\tau)s(t+\tau)\,d\tau,$$

has a narrow mainlobe and small sidelobes (see Fig. 2). If ψ looks like a delta function, then the resolution is high and the detailed structure of radar targets will be retained. If ψ has a broad mainlobe and/or high sidelobes, these will blur and confuse the radar-target properties and cause resolution problems.

One obvious theoretically attractive method for overcoming these problems is to select the transmitted signal s to be a delta function; then ψ is a delta function. This is unrealistic in practice, however, because even modern radar transmitters have peak power and bandwidth constraints, and the transmitted pulse must contain sufficient energy to satisfy the detection requirements. Thus

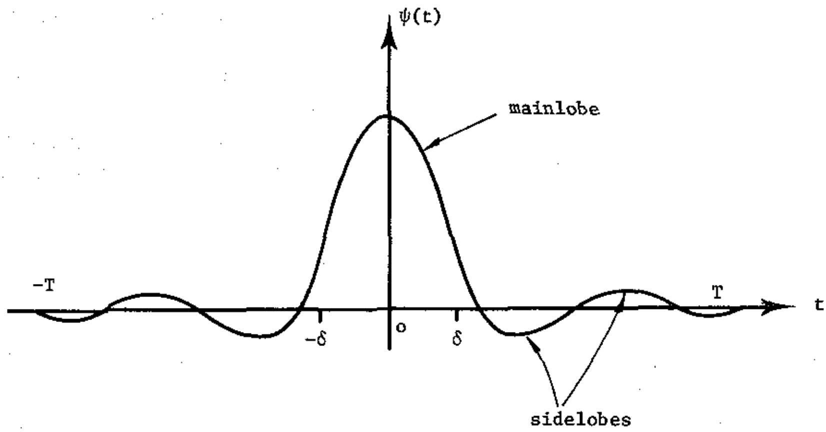

Fig. 2. Typical compressed pulse.

when searching for signals it is essential to impose certain constraints on the allowable set.

One possible approach for improving resolution is to minimize the matched-filter sidelobe energy while constraining the bandwidth of the allowable signals. This can be posed as an optimization problem:

$$\text{minimize} \qquad \left\| \int s(\tau)s(t+\tau)\,d\tau \right\|^2 = \int_{-T}^{T} \psi^2(t)\,dt$$

$$\text{subject to} \qquad \int s^2(\tau)\,d\tau = 1,$$

$$|S(\omega)| = 0, \quad \omega \notin \left[-\frac{B}{2\pi}, \frac{B}{2\pi} \right]$$

where $S(\omega)$ is the Fourier transform of the signal s and B is its bandwidth. This problem can be solved by transforming the cost functional into the frequency domain, setting its Frechet derivative to zero, and only considering real signals. The optimal signal satisfies

$$S(\omega) = \begin{cases} \text{constant} & \text{on } \left[-\frac{B}{2\pi}, \frac{B}{2\pi} \right], \\ 0 & \text{elsewhere.} \end{cases}$$

This result is well known in radar, and the derived clutter-resistant signal has an unsatisfactory matched-filter sidelobe structure [8].

An alternative approach, and the one we shall investigate in this paper, is the minimax point of view. The idea is either to minimize the maximum sidelobe level subject to certain signal constraints, or to constrain the maximum sidelobe level and optimize some signal property. For example the minimax problem can be written as

$$\underset{s\in\mathbb{S}}{\text{minimize}}\ \max_{t\in\mathfrak{T}}|\psi(t)-d(t)|$$

where $\mathbb{S}$ is a suitably defined set of allowable signals, $\mathfrak{T}$ is a time set, and $d(\cdot)$ is the desired output shape. This problem is difficult to attack directly because of the nondifferentiable cost functional; however, in Sec. III necessary conditions for an optimum are obtained via the theory of directional Gateaux differentials. In Sec. II below we shall present a precise mathematical formulation of the minimax problem and discuss several aspects relating to the sets $\mathbb{S}$ and $\mathfrak{T}$, and also consider various alternative problem formulations. Before doing this, however, we shall briefly consider relevant portions of the history of the optimal-signal-synthesis problem.

The problem of optimal autocorrelation synthesis has attracted considerable interest over the past two or three decades [33, 34]. The first approach required specification of ψ, checking that the Fourier transform Ψ was positive and realizable, and if so taking any signal spectrum which satisfied $|S(\omega)|^2=\Psi(\omega) = \int_{-\infty}^{\infty}\psi(t)e^{-j\omega t}\,dt$. This approach is clearly unsatisfactory because of the difficulty in specifying a ψ which is in fact an autocorrelation. Various techniques for approximating the chosen ψ by a $\tilde{\psi}$ which was known to be an autocorrelation were also tried, but the whole approach turned out to be very cumbersome [3]. A more successful line of attack, associated with the names of Dolph and Van der Maas (see [9, 10]), was to determine the narrowest autocorrelation function for a given sidelobe level (or vice versa). These autocorrelations are related to Chebyshev polynomials and have infinite unattenuated sidelobes. Moreover the optimal $\Psi(\omega)$ is "Gaussian-like" with infinite spikes at the edges of the allowable passband; thus the resulting signals are unrealizable. Several approximations to the optimal autocorrelations have been developed so that the optimal performance can be closely approached with realizable signals. The spectrum $\Psi(\omega)$ of the approximation is generally Gaussian out to the band edge, and it becomes more uniform as the sidelobe level increases [10, 12]. A second important development was provided by the energy-optimal signals of Slepian, Landau, and Pollock [11], which maximize the mainlobe energy subject to a constraint on the total energy. Using spheroidal wave functions, it can be shown that the autocorrelations are also

"Gaussian-like," becoming uniform as the sidelobe level increases. One signal which has used this uniform-spectrum idea is the popular linear FM waveform.

A different approach is to force $\Psi(\omega)$ to look like the power spectrum of a single narrow pulse autocorrelation. The optimal signal in this case is a phase-coded waveform called a Barker code. The autocorrelation function for a Barker code of length N has $2N$ sidelobes each of height $1/N$, but unfortunately no codes have been found for $N>13$ [1, 2], and such codes might not even exist. A variety of more complex codes are available, including polyphase codes and the impulse equivalent waveforms of Huffman [1, 2]. The Huffman codes have both amplitude and phase modulation, and the autocorrelation has the maximum possible clear area on either side of the mainlobe peak. Unfortunately, however, there is no useful method for designing these codes, although some work is being done on this problem [4–7].

As far as algorithms for designing optimal waveforms go, not much work has been done. There are some methods for producing better codes [4], but there does not appear to have been any previous consideration along the lines developed in this paper.

One of the major differences between the results presented here and previous work is that we work directly on the signal and develop results and algorithms for ensuring good autocorrelation functions, whereas most of the treatments referred to above have worked with the autocorrelation function itself, subject to evenness and positivity constraints on its Fourier spectrum, and then found the signal by factorization.

2. SIGNAL ANALYSIS AND PROBLEM FORMULATION

In this section we shall formulate in precise mathematical terms a number of useful autocorrelation synthesis problems. In order to proceed with this task it is first necessary to define certain signal spaces and establish pertinent properties of signals and autocorrelation functions on these spaces. These properties can be then used to ensure that the problem formulations are mathematically meaningful.

We shall assume throughout that the signal $s(\cdot)$ has the following properties:

(1) s is differentiable, with piecewise continuous derivatives.
(2) $s(0)=s(T)=0$, $s(t)=0$, $t\notin[0,T]$.

Denote the set of such s by $\mathbb{S}[0,T]$.

As we have already noted, the signal autocorrelation function is given by

$$\psi(t) \triangleq \int_0^T s(\tau)s(t+\tau)\,d\tau=\langle s,s_t\rangle$$

where $s_t(\tau) \triangleq s(t+\tau)$. Note (i) that $\psi \in C[-T,T]$, the Banach space of continuous functions on $[-T,T]$, (ii) that ψ is symmetric in t,

$$\langle s, s_t \rangle = \langle s, s_{-t} \rangle,$$

and finally (iii) that $\psi(0) = \langle s, s \rangle = \|s\|^2$. (We shall subsequently have occasion to use a sup norm. When distinction between this norm and the L_2 norm is important, we shall use a subscript ∞ or 2 as appropriate. An unsubscripted norm will always denote an L_2 norm.)

It will often be convenient to define a time set $\mathfrak{T}$ associated with the sidelobes of ψ—for example

$$\mathfrak{T} \triangleq \{t : t \in [-T, -\delta] \cup [\delta, T]\}.$$

Thus the mainlobe is supported on $(-\delta, \delta)$, and the sidelobes are defined as any part of ψ which is outside this interval. The set of allowable signals $\mathbb{S}$ will depend upon the problem formulation, but some typical examples are

$$\mathbb{S} = \mathbb{S}_E \triangleq \{s \in \mathbb{S}[0,T] : \|\dot{s}\|^2 \leqslant B^2, \ \|s\|^2 = 1\},$$

which constrains the signal energy and bandwidth, and

$$\mathbb{S} = \mathbb{S}_P \triangleq \Big\{s \in \mathbb{S}[0,T] : \max_{\tau \in [0,T]} |s(\tau)| \leqslant P, \\ \|\dot{s}\|^2 \leqslant B^2 \|s\|^2\Big\},$$

which constrains the signal bandwidth and peak power.

The signal bandwidth constraint we are using ($\|\dot{s}\|^2 \leqslant B^2 \|s\|^2$) is a second-moment constraint which can be related to the signal Fourier transform by Parseval's theorem, as we now show. It forces the signal to stay "smooth" and is quite satisfactory for our problem.

If we define the signal bandwidth β via

$$\beta^2 \triangleq \frac{\int_{-\infty}^{\infty} f^2 |S(f)|^2 \, df}{\int_{-\infty}^{\infty} |S(f)|^2 \, df},$$

where $S(f)$ is the Fourier transform of the signal s, then direct application of

Parseval's theorem gives

$$\beta^2=\left(\frac{1}{2\pi}\right)^2\frac{\|\dot{s}\|^2}{\|s\|^2}.$$

Thus, a constraint of the form

$$\|\dot{s}\|^2<B^2\|s\|^2$$

constrains the signal bandwidth to

$$\beta<B/2\pi.$$

The peak power constraint is of practical importance, since real radar systems are always peak power limited. Further, the signal only has support on a finite time interval $[0,T]$; thus the peak power constraint also implies a maximum energy constraint. Some useful relationships between these constraints are contained in the following lemma.

LEMMA 2.1. *For a signal* $s\in\mathbb{S}[0,T]$, *suppose*

$$\|s\|^2=E, \qquad \|s\|_\infty=P^{1/2}, \qquad \|\dot{s}\|^2\leqslant B^2\|s\|^2$$

Then

$$E\leqslant TP\leqslant\tfrac{1}{4}T^2B^2E.$$

Proof. The proof of the first inequality is trivial. We prove the second. Suppose that $\|s(t)\|$ attains its maximum value in [0,T] at $t=t_1$. Define $v(\cdot)$ and $w(\cdot)$ by

$$\dot{s}(t)=\frac{P^{1/2}}{t_1}+v(t), \qquad 0\leqslant t\leqslant t_1,$$

$$\dot{s}(t)=-\frac{P^{1/2}}{T-t_1}+w(t), \qquad t_1\leqslant t\leqslant T.$$

Notice that

$$\int_0^{t_1}v(t)\,dt=\int_{t_1}^{T}w(t)\,dt=0.$$

Consequently,

$$\int_0^T \dot{s}^2\,dt = \int_0^{t_1} \frac{P}{t_1^2}\,dt + \int_0^{t_1} v^2(t)\,dt + \int_{t_1}^T \frac{P}{(T-t_1)^2}\,dt + \int_{t_1}^T w^2(t)\,dt$$

$$\geqslant \frac{P}{t_1} + \frac{P}{T-t_1} \geqslant \frac{4P}{T},$$

which yields the result. Equality results if and only if $\dot{s}(t)=2P^{1/2}/T$ for $0 \leqslant t < T/2$, $\dot{s}(t) = -2P^{1/2}/T$ for $T/2 \leqslant t \leqslant T$ (or the negative). ∎

Some implications of this result are:

(1) if the peak power and signal duration are constrained, then the signal energy is automatically constrained to $E \leqslant PT$;

(2) if the second moment bandwidth, signal duration, and signal energy are constrained, then the peak power is automatically constrained to $P \leqslant \frac{1}{4} B^2 ET$.

Also it is important to observe that an arbitrary choice of B and δ will not always force a sensible constraint on the allowable signal set. For example, if $B\delta$ is chosen too large, then it may be possible to find a signal of bandwidth B such that its autocorrelation function ψ has support only on $(-\delta, \delta)$, i.e., ψ has no sidelobes. The following theorem makes this notion precise.

THEOREM 2.2. *With β defined as $(1/2\pi)\|\dot{s}\|_2/\|s\|_2$ and $\psi(t)=\langle s, s_t\rangle$, for any $\beta\delta \geqslant \frac{1}{2}$, there exists a signal $s \in \mathbb{S}[0,T]$ such that $|\psi(t)|=0\ \forall |t| \geqslant \delta$.*

Proof. Consider the variational problem:

$$\begin{aligned} &\text{minimize} && \|\dot{s}\|^2\delta^2, \\ &\text{subject to} && \|s\|^2 = 1, \\ &&& \langle s, s_t\rangle = 0, \quad |t| \geqslant \delta. \end{aligned}$$

The Lagrangian functional[1] for this problem becomes [21]

$$L(s;\lambda,\mu) = \|\dot{s}\|^2\delta^2 + \lambda(\|s\|^2 - 1) + \int_\delta^T \langle s, s_t + s_{-t}\rangle\, d\mu(t)$$

where λ is a nonzero scalar and $\mu(t)$ is a nonzero regular Borel measure on $[-T,T]$.

[1]See Appendix B. Had we required $\|s\|^2 \leqslant 1$, then λ would have had to be a nonnegative scalar.

Taking Frechet differentials in a direction $h \in \mathbb{S}[0, T]$, we obtain a necessary condition for (s, λ, μ) to minimize L as

$$L'_h = 2\langle \dot{s}, \dot{h} \rangle \delta^2 + 2\langle h, \lambda s \rangle + \int_\delta^T \langle h, s_t + s_{-t} \rangle \, d\mu(t)$$
$$= 0 \qquad \forall h \in \mathbb{S}[0, T].$$

After some simple manipulations this becomes

$$2\langle \dot{s}, \dot{h} \rangle \delta^2 + 2\langle h, \lambda s \rangle + \left\langle h, \int_\delta^T (s_t + s_{-t}) \, d\mu(t) \right\rangle = 0 \qquad \forall h.$$

Finally, applying a well-known lemma [37, p. 50], we see that the optimal signal must satisfy the second-order integrodifferential equation

$$-\delta^2 \ddot{s} + \lambda s + \frac{1}{2} \int_\delta^T (s_t + s_{-t}) \, d\mu(t) = 0$$

as well as the conditions stated in the problem, i.e.

$$\langle s, s_t \rangle = 0, \qquad |t| \geqslant \delta,$$
$$\|s\|^2 = 1,$$
$$s(0) = s(T) = 0.$$

We are unable to solve these necessary conditions directly. However, to prove the theorem we shall take $T = \delta$, so that the signal is zero outside $[0, \delta]$. It follows that $\psi(t) = 0$ outside the interval $(-\delta, \delta)$. For this case the necessary conditions reduce to

$$-\ddot{s} = \lambda s,$$
$$\|s\|^2 = 1,$$
$$s(0) = s(\delta) = 0.$$

One solution[2] is now given by a half sine pulse

$$s = \sqrt{\frac{2}{\delta}} \sin \frac{\pi t}{\delta},$$

[2]As suggested in the previous footnote, it is more natural to consider $\lambda > 0$, implying that trigonometric solutions are of interest.

and it is easy to see that

$$\beta^2\delta^2=\left(\frac{1}{2\pi}\right)^2\frac{\|\dot{s}\|^2}{\|s\|^2}\delta^2=\left(\frac{1}{2}\right)^2,$$

i.e.,

$$\beta\delta\geq\tfrac{1}{2}.$$

So this signal meets the requirements of the Theorem. ∎

It is easy to check that other solutions of the necessary conditions above yield signals which are pulses of sinusoids with radian frequency an integer multiple of π/δ. Such signals do not minimize $\|\dot{s}\|^2\delta^2$, but do meet the existence condition of the theorem.

All the signals found in this way are zero for $|t|>\delta$. It turns out that at least if $\dot{s}$ is continuous the requirement $\psi(t)=0$ for $|t|>\delta$ forces $s(t)$ to be zero outside an interval of length δ. As far as we are aware, this is a novel observation, and we include it for completeness.

THEOREM 2.3. *With $s\in\mathbb{S}[0,T]$ and with $\psi(t)=\langle s,s_t\rangle$, the equalities $\psi(t)=0$ $\forall|t|\geq\delta$, $s(t_1)s(t_2)\neq0$ imply $|t_1-t_2|<\delta$, and conversely.*

Proof. Let $[\mu,\nu]$ be the smallest interval containing the points in which $s(t)\neq0$. Then there exists $\varepsilon>0$ with $\varepsilon<(\nu-\mu)/2$ such that $\dot{s}$ is nonzero and of constant sign in $(\mu,\mu+\varepsilon]$ and $[\nu-\varepsilon,\nu)$. Consider

$$\psi(\nu-\mu-\varepsilon)=\int_0^T s(\tau)s(\tau+\nu-\mu-\varepsilon)\,d\tau$$

$$=\int_{-\mu-\varepsilon/2}^{T-\mu-\varepsilon/2} s\left(\sigma+\mu+\frac{\varepsilon}{2}\right)s\left(\sigma+\nu-\frac{\varepsilon}{2}\right)d\sigma$$

on setting $\sigma=\tau-\mu-\varepsilon/2$.

Taking account of the definitions of $[\mu,\nu]$, we have

$$\psi(\nu-\mu-\varepsilon)=\int_{-\varepsilon/2}^{\varepsilon/2} s(\sigma+\mu+\varepsilon/2)s(\sigma+\nu-\varepsilon/2)\,d\sigma$$

Because of the sign constraints on $\dot{s}$ and the fact that $s(\mu)=s(\nu)=0$, the integrand is of constant sign except at the endpoints, where it is zero. Therefore $\psi(\nu-\mu-\varepsilon)\neq0$. Consequently, $\nu-\mu-\varepsilon<\delta$ or $\nu-\mu<\delta+\varepsilon$, or since ε is arbitrary, $\nu-\mu\leq\delta$.

The converse statement of the theorem is easily proved. ∎

Several inequalities relating a function and its derivative are presented in [22]. One interesting example is Wirtinger's inequality, which when applied to our problem states that if $s(0)=s(\delta)=0$ and $\int_0^\delta s(t)\,dt=0$, then $\beta\delta \geqslant 1$. This has implications for signal design problems where the average value of the signal must be zero. Further results for periodic functions may also be found in [22].

Below we show how Wirtinger's inequality and Theorem 2.2 can be combined to show a converse for Theorem 2.2.

THEOREM 2.4. *With quantities as defined earlier, suppose for some signal* $s \in \mathbb{S}[0,T]$ *there obtains* $0<\beta\delta<\frac{1}{2}$. *Then* $\psi \in C[-T,T]$ *must have nonzero sidelobes on* $\mathfrak{T}=\{t\in[-T,-\delta]\cup[\delta,T]\}$.

Proof. To deduce a contradiction suppose there exists a signal with $0<\beta\delta<\frac{1}{2}$ such that ψ has zero sidelobes. By Theorem 2.3, the interval over which $s(\cdot)$ has nonzero support is at most δ. So without loss of generality, suppose $s\in\mathbb{S}[0,\delta]$. Define over $[0,2\delta]$ a signal

$$r(t)=s(t)=-s(2\delta-t).$$

It is trivial to check that

$$\int_0^{2\delta} r(t)\,dt=0,$$

so that by Wirtinger's inequality

$$\beta_r(2\delta)\geqslant 1.$$

However, it is also easy to check that $\beta_r=\beta_s$, and by the original assumption on s we have

$$\beta_s(2\delta)=\beta_r(2\delta)<1.$$

Hence we have a contradiction. ∎

Notice that, as shown earlier, the signal

$$s(t)=\sqrt{\frac{2}{\delta}}\ \sin\frac{\pi t}{\delta}$$

achieves $\beta_s\delta=\frac{1}{2}$, so we conclude that this signal has the smallest bandwidth for a given support, and if we choose $\beta\delta < \frac{1}{2}$, no signal exists for which ψ is zero on $\mathfrak{T}$.

REMARKS. There are several connections between the results presented above and classical uncertainty results. The best known uncertainty result for real signals concerns the product of second-moment bandwidth B and second-moment time spread T, $TB \geq \frac{1}{4}$ [26–28]. The optimum signal for this case is a Gaussian pulse. These results and the more abstract results of [34, 35] have many similarities with the Heisenberg uncertainty principle in quantum physics [3, 24]. Our results differ in that we are considering the time-spread properties of the autocorrelation of a finite-duration signal. Finite-duration signals are also considered in [33, 36].

With these results out of the way, we are now in a position to formulate realistic optimal-signal design problems. There can be several different formulations, depending upon which parameters are specified and which are to be optimized. The following examples illustrate this.

Problem 1. The bandwidth constraint B and the mainlobe width 2δ are specified with $0 < B\delta < \pi$ (i.e. $0 < \beta\delta < \frac{1}{2}$), the signal energy E is specified, and the sidelobe level is to be minimized. The problem becomes

$$\underset{s\in\mathbb{S}}{\text{minimize}} \qquad \max_{t\in\mathfrak{T}} |\langle s, s_t\rangle|,$$

where

$$\mathbb{S} \triangleq \{s\in\mathbb{S}[0,T] : \|\dot{s}\|^2 \leq B^2/E,\ \|s\|^2 = E\},$$

$$\mathfrak{T} \triangleq \{t\in[-T,T] : t\notin(-\delta,\delta)\}.$$

REMARKS.

(1) By Lemma 2.1 the peak power is automatically constrained by $P \leq \frac{1}{4}B^2ET$.

(2) The following generalization is often useful:

$$\underset{s\in\mathbb{S}}{\text{minimize}} \qquad \max_{t\in\mathfrak{T}} |\langle s, s_t\rangle - d(t)|$$

for some $d\in C[-T,T]$ and $\mathfrak{T} \triangleq \{t\in[-T,T]\}$.

(3) In many situations it is preferable to define

$$\mathbb{S} \triangleq \{s \in \mathbb{S}[0,T] : \|\dot{s}\|^2 \leqslant B^2 \|s\|^2,\ \|s\|_\infty^2 = P\}$$

(4) An equivalent formulation to Problem 1 is:

$$\begin{aligned} &\text{minimize} && g, \quad g \in R, \quad s \in \mathbb{S} \\ &\text{subject to} && |\langle s, s_t \rangle| \leqslant g, \quad t \in \mathfrak{T}. \end{aligned}$$

(5) It is possible to force s to be a bang-bang-type signal by defining

$$\mathbb{S} \triangleq \{s \in L^2[0,T] : \|s\|^2 = PT,\ \|s\|_\infty^2 = P\}.$$

Note that the signal-bandwidth constraint is no longer meaningful in this case, and we only consider $s \in L^2[0,T]$.

Problem 2. Constraining the sidelobe level and/or shape, minimize the signal bandwidth:

$$\begin{aligned} &\text{minimize} && \|\dot{s}\|^2, \quad s \in \mathbb{S}[0,T] \\ &\text{subject to} && |\langle s, s_t \rangle - d(t)| \leqslant \varepsilon(t), \end{aligned}$$

where $d, \varepsilon \in C[-T,T]$.

REMARKS.

(1) Clearly $d(0)$ and $\varepsilon(0)$ constrain $\psi(0)$ in that

$$d(0) - \varepsilon(0) \leqslant \|s\|^2 = \psi(0) \leqslant d(0) + \varepsilon(0).$$

(2) From Lemma 2.1 we see that minimizing $\|\dot{s}\|^2$ also keeps the peak power down, since $\|s\|^2$ and T are fixed. However, a peak-power constraint could be added if necessary, or it could be the cost functional.

(3) Various alternatives to the above are possible, for example

$$\begin{aligned} &\text{minimize} && -\|s\|^2 \\ &\text{subject to} && |\langle s, s_t \rangle| \leqslant 1, \qquad t \notin (-\delta, \delta), \\ &&& \|\dot{s}\|^2 \leqslant B^2 \|s\|^2. \end{aligned}$$

Problem 3. The same as Problem 2, except that now we want the matched-filter response to be satisfactory despite perturbation in the input signal shape, i.e.

$$\text{minimize} \qquad \|\dot{s}\|^2, \quad s\in\mathbb{S}[0,T]$$

$$\text{subject to} \qquad |\langle c_t, s\rangle - d(t)| \leqslant \varepsilon(t) \qquad \forall t\in[-T,T],$$

$$\forall c: |c-s| \leqslant \theta \geqslant 0.$$

This problem has application to real systems, where a matched filter/signal pair has been designed, but the actual signal is distorted in some way. It is important that small signal distortions do not cause large sidelobes to appear.

Before proceeding with a detailed mathematical analysis of the three problems presented above, we briefly remark on the question of the existence of optimal solutions.

The usual approach to establishing existence relies upon some version of the Weierstrass theorem, which effectively states that a q-continuous functional attains its extrema on a q-compact set, where q can refer to strong, weak, or weak-star. The constraint sets in each of the problems above are only closed and bounded, and without convexity properties it is extremely difficult to establish any q-compactness property. Similarly, the cost functionals are all continuous, but without convexity it is not easy to establish the more severe q-continuity results needed. For Problems 1, 2, and 3 above, however, we are able to establish the existence of a finite infimum with the use of Theorems 2.2 and 2.4. It is important to note that existence is no longer a problem when the finite-dimensional approximations are used for numerical results in Sec. 4.

3. OPTIMALITY CONDITIONS

As stated earlier, our aim in this paper is to investigate both theoretical and practical aspects of the problem of optimal autocorrelation synthesis. So far we have formulated several problems directly related to real problems in radar system design. In this section we shall move on to consider these problems further and obtain sets of necessary condition which an optimal signal must satisfy. Unfortunately the results are highly technical and not particularly revealing from a practical viewpoint. They are however relevant to the derivation of numerical algorithms, and this aspect is further discussed in Sec. 4.

The first problem we shall examine is Problem 2. As can be seen from the statement of the theorem following, the optimal signal must satisfy an integrodifferential equation with delay, which is extremely difficult to solve in closed form.

THEOREM 3.1. *If $s^0 \in \mathbb{S}[0,T]$ is optimal for Problem 2 and is a regular point*[3] *of the constraint set, then there exists a $\lambda^0 \in M[-T,T]$, the space of regular Borel measures on $[-T,T]$ such that*

$$\text{(i)} \qquad -\ddot{s}^0 + \frac{1}{2}\int_{-T}^{T}\left[s^0(t+\tau)+s^0(t-\tau)\right]d\lambda^0 = 0 \qquad \forall \tau \in [0,T],$$

$$\text{(ii)} \qquad \int_{-T}^{T}\left[\langle s^0, s_t\rangle - d(t)\right]d\lambda^0 = \|\lambda^0\|_\varepsilon,$$

where $\|\lambda^0\|_\varepsilon \triangleq \int_{-T}^{T}\varepsilon(t)|d\lambda^0|$.

Proof. The Lagrangian for Problem 2 is

$$L[s;\lambda_1,\lambda_2] = \|\dot{s}\|^2 + \int_{-T}^{T}[\langle s, s_t\rangle - d(t)]\,d\lambda_1 - \int_{-T}^{T}\varepsilon(t)\,d\lambda_1$$
$$+ \int_{-T}^{T}[-\langle s, s_t\rangle + d(t)]\,d\lambda_2 - \int_{-T}^{T}\varepsilon(t)\,d\lambda_2$$

where $\lambda_1, \lambda_2 \in M^+[-T,T]$ are positive measures.

Since both upper and lower constraints cannot be active simultaneously, it can be shown that λ_1 and λ_2 are mutually singular measures, and they can be combined into $\lambda = \lambda_1 - \lambda_2$; thus the Lagrangian becomes

$$L[s,\lambda] = \|\dot{s}\|^2 + \int_{-T}^{T}[\langle s, s_t\rangle - d(t)]\,d\lambda - \|\lambda\|_\varepsilon.$$

Taking the Frechet differential of L at s in a direction h, we obtain a necessary condition for L to be minimized in s as

$$\langle \dot{s}, \dot{h}\rangle + \frac{1}{2}\int_{-T}^{T}[\langle s, h_t\rangle + \langle h, s_t\rangle]\,d\lambda = 0, \qquad h \in \mathbb{S}[0,T].$$

After some straightforward manipulations and applying a well-known lemma from the calculus of variations [37, p. 47], it follows that $\ddot{s}$ exists and that

$$\left\langle h, -\ddot{s} + \frac{1}{2}\int_{-T}^{T}[s_t + s_{-t}]\,d\lambda \right\rangle = 0, \qquad h \in [0,T]$$

[3]See Appendix B for a brief description of relevant function of space optimization theory.

The theorem now follows by straightforward application of the function-space version of the Kuhn-Tucker theorem [21].

Below we establish several results required for the solution of Problem 1. The major difficulty here is the nondifferentiable minimax cost functional. We then apply these results to obtain the rather complicated necessary conditions as stated in Theorem 3.5. An apparently simpler set of conditions is presented in Theorem 3.4, but closer examination will reveal that these, like those in Theorem 3.1 above, are complicated by the measure λ^0. A considerable simplification of the results in Theorem 3.5 occurs if we assume that $\ddot{s}$ is continuous, and it has been used as the basis of a successful numerical algorithm as discussed in Sec. IV.

The following definitions are required below.

$$c(s) \triangleq \max_{t\in\mathfrak{T}} |\langle s, s_t\rangle| = \max_{t\in\mathfrak{T}} |\psi(s,t)|,$$

where $\psi(s,t) = \langle s, s_t\rangle = \psi(t)$. Note that $c(s)$ is continuous in s because $\mathfrak{T}$ is compact. Let

$$\mathfrak{T}^*(s) \triangleq \{t\in\mathfrak{T} : |\langle s, s_t\rangle| = c(s)\}.$$

The following theorem will be useful for solving Problem 1.

THEOREM 3.2.

$$\delta^+ c(\bar{s}; h) \triangleq \lim_{\lambda\to 0^+} \frac{1}{\lambda}\left[c(\bar{s}+\lambda h) - c(\bar{s})\right]$$

$$= \sup_{t\in\mathfrak{T}^*(\bar{s})} \delta^+ |\psi(\bar{s}; h, t)|.$$

Proof. See Appendix A. ■

LEMMA 3.3.

$$\delta^+ c(\bar{s}; h, t) = \delta^+ c(\bar{s}; h, -t) \qquad \forall t, h.$$

Proof. From the symmetry of $\psi(\bar{s}, t)$. ■

Lemma 3.3 tells us that there is no need to consider ψ over the whole interval $[-T, T]$. Thus we can redefine $\mathfrak{T}$ and $\mathfrak{T}^*$ as

$$\mathfrak{T} \triangleq \{t : \delta \leqslant t \leqslant T\}$$

$$\mathfrak{T}^*(s) \triangleq \{t\in\mathfrak{T} : |\langle s, s_t\rangle| = c(s)\}.$$

For example, if ψ achieves its maximum at $t=\xi\in\mathfrak{T}$, then $\mathfrak{T}^*=\{\xi\}$ and the directional Gateaux differential of $c(s)$ in a direction h is given, according to Theorem 3.2, by

$$\delta^+ c(s;h)=\delta^+|\psi(s;h,\xi)|.$$

Of course, ψ must also have an equivalent maximum at $t=-\xi$, but by Lemma 3.3 above the same directional differential will do for both points.

THEOREM 3.4. *If s^0 solves Problem 1 and is a regular point of the constraints, then there exists $\mu^0\neq 0$, $\gamma^0\geqslant 0$ and a measure $\lambda^0\in M[-T,T]$ with $d\lambda^0=0$ $\forall t\in\mathfrak{T}^*(s^0)$ and $\|\lambda^0\|\triangleq\int_{-T}^{T}|d\lambda^0|=1$ such that*

(i) $-2\gamma^0\ddot{s}+2\mu^0+\int_{-T}^{T}(s_t+s_{-t})\,d\lambda^0=0,$
(ii) $\mu^0(\|s^0\|^2-E)=0,$
(iii) $\gamma^0(\|\dot{s}^0\|^2-B^2/E)=0.$

Proof. Note the equivalence between Problem 1 and the formulation in Remark (4) afterward. Apply the Kuhn-Tucker conditions and follow the techniques used in Theorem 3.1. ∎

An alternative set of necessary conditions may be obtained by applying Theorem 3.3 to Problem 1. The following result is immediately clear.

THEOREM 3.5.

(i)

$$\max_{\xi\in\mathfrak{T}^*(s^0)}\left\{\begin{array}{l}\left\langle \dot{h}(\tau),2\gamma^0\dot{s}^0(\tau)+\int_\tau^T\left(s_\xi^0(t)+s_{-\xi}^0(t)+2\mu^0\right)dt\right\rangle\\ \quad \textit{if } \langle s^0,s_\xi^0\rangle>0\\ \quad \textit{or if } \langle s^0,s_\xi^0\rangle=0 \textit{ and } \langle s_\xi^0+s_{-\xi}^0,h\rangle>0,\\ \left\langle \dot{h}(\tau),2\gamma^0\dot{s}^0(\tau)+\int_\tau^T\left(-s_\xi^0(t)-s_{-\xi}^0(t)+2\mu^0\right)dt\right\rangle\\ \quad \textit{if } \langle s^0,s_\xi^0\rangle<0\\ \quad \textit{or if } \langle s^0,s_\xi^0\rangle=0 \textit{ and } \langle s_\xi^0+s_{-\xi}^0,h\rangle<0,\\ 0 \quad \textit{otherwise}\end{array}\right\}\geqslant 0$$

$$\forall h\in\mathbb{S}[0,T].$$

(ii) $\gamma^0(\|\dot{s}^0\|^2-B^2/E)=0.$
(iii) $\mu^0(\|s^0\|^2-E)=0.$

Proof. See Appendix C. ∎

REMARK. If we further assume that the second derivative of s is continuous, then the differential condition in Theorem 3.5 can be rewritten as

$$\max_{\xi\in\mathfrak{T}^*(s^0)}\left\{\begin{array}{l}\langle h,2\gamma^0\ddot{s}^0+s_\xi^0+s_{-\xi}^0+2\mu^0\rangle\\ \quad\text{if } \langle s^0,s_\xi^0\rangle>0\\ \quad\text{or if } \langle s^0,s_\xi^0\rangle=0 \text{ and } \langle s_\xi^0+s_{-\xi}^0,h\rangle>0,\\ \langle h,2\gamma^0\ddot{s}^0-s_\xi^0-s_{-\xi}^0+2\mu^0\rangle\\ \quad\text{if } \langle s^0,s_\xi^0\rangle<0\\ \quad\text{or if } \langle s^0,s_\xi^0\rangle=0 \text{ and } \langle s_\xi^0+s_{-\xi}^0,h\rangle<0,\\ 0 \quad\text{otherwise}\end{array}\right\}\geqslant 0 \qquad \forall h\in\mathbb{S}[0,T].$$

Finally we briefly examine Problem 3. This problem is intractable as it stands, and there does not exist any mechanism for determining necessary conditions for optimality. However, we are able to establish an equivalence between two signal sets which allows us to rewrite the problem in a form where the Gateaux differential ideas of Theorem 3.1 can be applied and necessary conditions obtained as in Theorem 3.7.

THEOREM 3.6. *The following signal sets are equivalent*:

$$S_1 \triangleq \{s\in\mathbb{S}[0,T]:|\langle s,s_t\rangle-d(t)|+\langle\theta_t,|s|\rangle\leqslant\varepsilon(t)\}\ \forall t\},$$

where $\theta(\cdot)$ is a function nonnegative everywhere;

$$S_2 \triangleq \{s\in\mathbb{S}[0,T]:|\langle c_t,s\rangle-d(t)|\leqslant\varepsilon(t)\ \forall t,$$

$$c(\cdot):|c(t)-s(t)|\leqslant\theta(t)\ \forall t\}.$$

Proof.

(1) $s\in S_2\Rightarrow s\in S_1$. Let $c=s+\delta$, $|\delta|\leqslant\theta\geqslant 0$; then for any fixed t,

$$|\langle c_t,s\rangle-d(t)|\leqslant\varepsilon(t)$$

$$\Rightarrow\quad |\langle s_t,s\rangle+\langle\delta_t,s\rangle-d(t)|\leqslant\varepsilon(t)$$

Now take $\delta(0)$ such that $\delta(\tau+t)=\operatorname{sgn}[\langle s_t,s\rangle-d(t)]\operatorname{sgn}s(\tau)\,\theta(\tau+t)$. Then

$$|\langle s_t,s\rangle-d(t)|+\langle\theta_t,|s|\rangle\leqslant\varepsilon(t).$$

(2) $s \in S_1 \Rightarrow s \in S_2$. If $c = s + \delta$ and $|\delta| \leqslant \theta \geqslant 0$,

$$|\langle c_t, s\rangle - d(t)| \leqslant |\langle s_t, s\rangle - d(t)| + |\langle \delta_t, s\rangle| \leqslant |\langle s_t, s\rangle - d(t)| + \langle \theta_t, |s|\rangle.$$

The result is then an immediate consequence of the definitions. ■

THEOREM 3.7. *If $s^0 \in [0, T]$ is optimal for Problem 3 and a regular point of the constraints, then there exist positive measures λ_1^0, λ_2^0 such that*[4]

$$\left\{\begin{array}{l} -2\ddot{s}^0 + \int_{-T}^{T} (s_t^0 + s_{-t}^0)\, d\lambda_-^0 + \int_{-T}^{T} \theta_t\, d\lambda_+^0 \\ \quad \text{if } \; s^0 > 0 \text{ or } s^0 = 0 \text{ and } [\ ] > 0, \\ 2\ddot{s}^0 - \int_{-T}^{T} (s_t^0 + s_{-t}^0)\, d\lambda_-^0 - \int_{-T}^{T} \theta_t \lambda_+^0 \\ \quad \text{if } \; s^0 < 0 \text{ or } s^0 = 0 \text{ and } [\ ] < 0 \end{array}\right\} = 0,$$

where $\lambda_+^0 = \lambda_1^0 + \lambda_2^0$ and $\lambda_-^0 = \lambda_1^0 - \lambda_2^0$, and such that

$$\int_{-T}^{T} (\langle s^0, s_t\rangle - d_t + \langle \theta_t, |s^0|\rangle - \varepsilon)\, d\lambda_1^0 = 0,$$

$$\int_{-T}^{T} (-\langle s^0, s_t^0\rangle + d_t + \langle \theta_t, |s^0|\rangle - \varepsilon)\, d\lambda_2^0 = 0.$$

Proof. The directional Gateaux differential of the Lagrangian can be written as

$$\delta^+ L_h = 2\langle \dot{s}, \dot{h}\rangle + \int_{-T}^{T} (\langle s, h_t\rangle + \langle h, s_t\rangle)\, d\lambda_1$$

$$+ \int_{-T}^{T} (-\langle s, h_t\rangle - \langle h, s_t\rangle)\, d\lambda_2$$

$$+ \int_{-T}^{T} d\lambda_1 (\langle \theta_t, \delta^+ |s; h|\rangle) + \int_{-T}^{T} d\lambda_2 (\langle \theta_t; \delta^+ |s; h|\rangle),$$

[4] The square brackets [ ] mean *everything in the above line.*

where

$$\delta^+|s;h| = \begin{cases} h & \text{if } s>0, \\ -h & \text{if } s<0, \\ |h| & \text{if } s=0. \end{cases}$$ ■

4. NUMERICAL ALGORITHM

In this section we present and discuss an algorithm for solving Problem 1. Preliminary tests with the algorithm on the PDP11/45 at the University of Newcastle have shown that it is capable of finding signals with "good" autocorrelation properties.

Rewriting Problem 1, we have to establish

$$\min_{s\in\mathbb{S}[0,T]} \max_{|t|\in[\delta,T]} |\langle s, s_t\rangle|$$

given

$$\|\dot{s}\|^2 \leqslant B^2,$$

$$\|s\|^2 = 1.$$

A gradient algorithm for solving this problem can be obtained directly from the necessary conditions stated in Sec. 3 and is given by

$$u^{k+1} = u^k + \alpha^k l^k,$$

$$s^{k+1} = \frac{u^{k+1}}{\|u^{k+1}\|},$$

$$\lambda^{k+1} = \max\{0, \lambda^k + \beta^k(\|\dot{s}\|^2 - B^2)\},$$

where $\alpha^k > 0$, $\beta^k > 0$ are scalar step sequences, and λ^k is a scalar sequence. The descent direction $l^k \in \mathbb{S}[0,T]$ is given by

$$l^k(\theta) = -\left[\int_0^\theta \left\{2\lambda^{k-1}\dot{s}^k(\tau) + \int_\tau^T [s_\xi(t) + s_{-\xi}(t)]\, dt\right\} d\tau\right]$$

when

$$\tau^*(s) \triangleq \{t \in [\delta, T] : |\langle s, s_t \rangle| = \max |\langle s, s_t \rangle|\}$$

$$\equiv \{\xi\}.$$

In case $\tau^*(s)$ is not simply a singleton [i.e. when the sidelobes of $\psi(t)$ achieve their maximum value at more than one point] we denote the descent direction for any point $\xi \in \tau^*(s)$ by l_ξ^k. Now the descent direction used in the algorithms is any l^k satisfying

$$\langle l^k, l_\xi^k \rangle \geqslant 0 \qquad \forall \xi \in \tau^*(s).$$

In order to implement this algorithm on a digital computer it is necessary to approximate the signal s by an n-dimensional vector $\mathbf{s} \in R^n$. Instead of simply discretizing the algorithm above, however, it is instructive to derive the discrete algorithm directly from a discrete version of Problem 1. Consider the problem

$$\underset{\mathbf{s} \in R^n}{\text{minimize}} \qquad \max_i \{|\mathbf{s}'\mathbf{F}_i\mathbf{s}|, i = -n, \ldots, 0, \ldots, n\}$$

$$\text{subject to} \qquad \mathbf{s}'\mathbf{G}\mathbf{s} \leqslant B^2,$$

$$\mathbf{s}'\mathbf{s} = 1,$$

where

$$\mathbf{G} \triangleq \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

approximates the signal time derivative at $t = i$ by $s_i - s_{i+1}$, and $\mathbf{F}_i$ is a shifting matrix such that

$$\mathbf{F}_i\mathbf{s} = [0, 0, \ldots, s_1, s_2, \ldots, s_{n-i}]'.$$

Now the Lagrangian for this problem is

$$L = \max_i \{|\mathbf{s}'\mathbf{F}_i\mathbf{s}|, i = -n, \ldots, 0, \ldots, n\} + \lambda[\mathbf{s}'\mathbf{G}\mathbf{s} - \mathbf{B}^2].$$

Proceeding as in the continuous case in Sec. 3, we finally obtain the following gradient algorithm:

$$\mathbf{u}^{k+1}=\mathbf{u}^k-\alpha^k\mathbf{l}_\xi^k,$$

$$\mathbf{s}^{k+1}=\frac{\mathbf{u}^k}{\|\mathbf{u}^k\|},$$

$$\lambda^{k+1}=\max\left\{0,\lambda^k+\beta^k(\mathbf{s}'\mathbf{G}\mathbf{s}-B^2)\right\},$$

$$\mathbf{l}_\xi^k=(\mathbf{F}_\xi+\mathbf{F}_{-\xi})\mathbf{s}^k+\lambda^k\mathbf{G}\mathbf{s}^k.$$

The above expression for $\mathbf{l}_\xi^k$ assumes that $|(\mathbf{s}^k)'\mathbf{F}_i\mathbf{s}^k|$ achieves a maximum at $i=\pm\xi$. If the maximum is achieved at several pairs of points, then any descent direction which makes a positive inner product with every $\mathbf{l}_\xi^k$ will do. More sophisticated schemes for multiple maxima are available [15, 16, 18–20].

Remarks. The algorithm was tried and found to work quite well, but difficulty was experienced with the step selection (α^k, β^k) and with the occurrence of multiple extrema. The most successful applications of the algorithm were when the initial signal was one with good sidelobe properties (13-bit Barker code, for example). In these cases the algorithm rapidly reduced the sidelobe level, but usually with fairly detailed amplitude modulations on the original signal. We believe that with further work and when carefully applied our algorithm will prove to be quite useful.

In order to overcome the problems of step selection and multiple extrema, two techniques were employed.

Firstly the algorithm was written in such a fashion that considerable user interaction took place. Thus the operator could let the program run n steps and then view the waveform, gradient signal, etc. on a graphical display. This feature enabled the user to determine the effects of varying the step sizes and to gauge if a local minimum had been reached. These interactive features appear to be very powerful in difficult optimization problems.

Secondly an adaptive random search algorithm was sometimes used to unjam the algorithm whenever a local minimum was reached.

5. DISCRETE-TIME RESULTS

It is instructive to consider the discrete-time version of the uncertainty result presented in Sec. 2. Define the bandwidth β of a discrete-time signal vector $\mathbf{s}'=(s_1\ldots s_N)$ as

$$\beta^2 \triangleq \left(\frac{1}{2\pi}\right)^2 \frac{\sum_{i=0}^{N}(s_i-s_{i+1})^2}{\sum_{i=1}^{N} s_i^2,}, \qquad s_0=s_{N+1}=0,$$

or equivalently

$$\beta^2=\left(\frac{1}{2\pi}\right)^2\frac{\mathbf{s}'\mathbf{G}\mathbf{s}}{\mathbf{s}'\mathbf{s}},$$

where

$$\mathbf{G}=\mathbf{D}\mathbf{D}'=\begin{bmatrix} 2 & -1 & 0 & 0 & \ddots \\ -1 & 2 & -1 & 0 & \ddots \\ 0 & -1 & 2 & -1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

We can state the following theorem.

THEOREM 2.5. *For all discrete-time signals consisting of N samples,*

$$\beta \geqslant \frac{1}{\pi}\sin\frac{\pi}{2(N+1)}.$$

Proof. Consider the optimization problem:

minimize $\quad \mathbf{s}'\mathbf{G}\mathbf{s}, \qquad s\in R^N$

subject to $\quad \mathbf{s}'\mathbf{s}=1.$

Clearly the optimum signal satisfies the eigenvector equation

$$\mathbf{G}\mathbf{s}^0=\lambda\mathbf{s}^0,$$

and it is well known that the eigenvalues of $\mathbf{G}$ are given by

$$\lambda_k=2-2\cos\left(\frac{k\pi}{N+1}\right), \qquad k=1,\ldots,N.$$

The smallest of these eigenvalues is

$$\lambda_{\min}=2\left[1-\cos\frac{\pi}{N+1}\right]=4\sin^2\frac{\pi}{2(N+1)}$$

With the associated eigenvector, it gives rise to $\beta^2=(1/2\pi)^2\lambda_{\min}$, or

$$\beta=\frac{1}{\pi}\sin\frac{\pi}{2(N+1)}.$$

For all signals, then, we must have

$$\beta\geqslant\frac{1}{\pi}\sin\frac{\pi}{2(N+1)}.$$

∎

Notice that for large N, the lower bound is approximately $1/2(N+1)$. Similarly, it is possible to establish a discrete-time version of Wirtinger's inequality and provide discrete-time versions of Theorems 2.3 and 2.4.

6. CONCLUSION

The problem of signal design given desired autocorrelation properties is important in a variety of real situations. The problem is not well understood, and considerable work remains to be done. In this paper we have examined certain aspects of optimal signal design via optimization theory. The results are by no means complete, but they do indicate that further work along these lines may lead to useful algorithms. Basically five aspects of the problem have been considered here. Firstly we established several uncertainty results relating signal properties to signal autocorrelation properties. Secondly, three fundamental signal design problems were formulated. Thirdly, basic necessary conditions were presented which characterize the solutions to these signal design problems. Fourthly, these conditions were employed to develop numerical algorithms for finding the desired signals, and finally certain aspects of discrete-time signal design were presented.

APPENDIX A. PROOF OF THEOREM 3.2

For $t\in\mathfrak{T}^*(s)$ and $\bar{t}\in\mathfrak{T}^*(\bar{s})$ it is clear that

$$|\psi(s,t)|=c(s)\geqslant|\psi(s,\bar{t})|$$

and

$$|\psi(\bar{s},t)| \leqslant c(\bar{s}) = |\psi(\bar{s},\bar{t})|.$$

Hence, subtracting these inequalities, it follows that

$$|\psi(s,t)| - |\psi(\bar{s},t)| \geqslant c(s) - c(\bar{s}) \geqslant |\psi(s,\bar{t})| - |\psi(\bar{s},\bar{t})|$$

Now let $s = \bar{s} + \lambda h$ for $\lambda > 0$; then

$$\frac{1}{\lambda}\left[|\psi(\bar{s}+\lambda h,t)| - |\psi(\bar{s},t)|\right] \geqslant \frac{1}{\lambda}\left[c(\bar{s}+\lambda h) - c(s)\right]$$

$$\geqslant \frac{1}{\lambda}\left[|\psi(\bar{s}+\lambda h,\bar{t})| - |\psi(\bar{s},\bar{t})|\right]$$

for all $t \in \mathfrak{T}^*(\bar{s}+\lambda h)$.

Now examining the second inequality and taking limits, we have

$$\liminf_{\lambda\to 0^+} \frac{1}{\lambda}\left[c(\bar{s}+\lambda h) - c(\bar{s})\right] \geqslant \liminf_{\lambda\to 0^+} \frac{1}{\lambda}\left[|\psi(\bar{s}+\lambda h,\bar{t})| - |\psi(\bar{s},\bar{t})|\right]$$

$$= \delta^+|\psi(\bar{s};\lambda h,\bar{t})|.$$

Since $\bar{t} \in \mathfrak{T}^*(\bar{s})$ is arbitrary, we have

$$\liminf_{\lambda\to 0^+} \frac{1}{\lambda}\left[c(\bar{s}+\lambda h) - c(\bar{s})\right] \geqslant \sup_{t\in\mathfrak{T}^*(\bar{s})} \delta^+|\psi(\bar{s};h,t)|.$$

We now examine the first inequality. Firstly note that for $\lambda > 0$ sufficiently small

$$|\psi(\bar{s}+\lambda h,t)| = |\psi(\bar{s},t)| + \delta^+|\psi(\bar{s};h,t)| + \rho(\lambda,t)$$

and

$$\lim_{\lambda\to 0^+} \rho(\lambda,t) = 0 \qquad \text{uniformly in } t.$$

Thus

$$\frac{1}{\lambda}\left[c(\bar{s}+\lambda h) - c(\bar{s})\right] \leqslant \sup_{t\in\mathfrak{N}(\mathfrak{T}^*(\bar{s}))} \left[\delta^+|\psi(\bar{s};h,t)| + \rho(\lambda,t)\right],$$

where $\mathfrak{N}(\mathfrak{T}^*(\bar{s}))$ is any neighborhood which contains $\mathfrak{T}^*(\bar{s})$, and because $\mathfrak{T}$ is compact, it follows that the point-to-set map $\mathfrak{T}^*(\bar{s})$ is continuous in the sense that it is always possible to find a $\delta>0$ such that $\mathfrak{T}^*(s)\subset\mathfrak{N}(\mathfrak{T}^*(\bar{s}))$ for $\|s-\bar{s}\|\leqslant\delta$. Taking limits above, we see that

$$\begin{aligned}\limsup_{\lambda\to0^+}\frac{1}{\lambda}\left[c(\bar{s}+\lambda h)-c(\bar{s})\right] &\leqslant \sup_{t\in\mathfrak{N}(\mathfrak{T}^*(\bar{s}))}\delta^+|\psi(\bar{s};h,t)| \\ &\leqslant \inf_{\mathfrak{N}(\mathfrak{T}^*(\bar{s}))}\left\{\sup_{t\in\mathfrak{N}(\mathfrak{T}^*(\bar{s}))}\delta^+|\psi(\bar{s};h,t)|\right\} \\ &= \sup_{t\in\mathfrak{T}^*(\bar{s})}\delta^+|\psi(\bar{s};h,t)|.\end{aligned}$$

The result follows immediately.

APPENDIX B

The following results are helpful in the derivations of Sec. 3.

Consider the following general nonconvex optimization problem:

$$\begin{array}{lll}\text{minimize} & f(s), & s\in S,\\ \text{subject to} & H_i(s)\leqslant 0, & i=1,\ldots,n,\\ & k_j(s)=0, & j=1,\ldots,m,\end{array}$$

where $f: S\to R$, $H_i: S\to Z_i$, $k_i: S\to R$, and S and Z_i are real normed linear spaces, and where $P_i\subseteq Z_i$ are convex cones which generate partial orderings on the Z_i denoted by $\leqslant$ and defined by

$$x\leqslant y \Leftrightarrow y-x\in P_i, \qquad x,y\in Z_i.$$

The Lagrangian for this problem, $L: S\times Z_1^*\times\cdots\times Z_n^*\to R$, is defined by

$$L(s;\lambda_1,\lambda_2\ldots\lambda_n,\mu_1,\ldots,\mu_n)=f(s)+\sum_{i=1}^{n}\lambda_i\cdot H_i(s)+\sum_{j=1}^{m}\mu_j k_j(s)$$

where $\lambda_i\in P_i^*\subseteq Z_i^*$, the dual positive cones on the dual spaces Z_i^*. Now if positive cones P_i have nonempty interiors and the mappings $H_i: S\to Z_i$ are Frechet differentiable, then an element $s^0\in S$ is defined as a *regular point* of the constraints $H_i(s)\leqslant 0$ if s^0 satisfies the constraints and there exists $r\in S$

such that

$$H_i(s^0)+H_i'(s^0;r)<0,$$

where the primed term refers to Frechet differential of H_i at s^0 with increment r.

THEOREM. *If $s^0 \in S$ solves the problem above and if s^0 is a regular point of the inequality constraints, then there exist Lagrange multipliers*

$$\lambda_i^0 \in P_i^* \subseteq Z_i^*, \qquad i=1,\dots,n$$

and $\mu_i^0 \in R$, $\mu_i^0 \neq 0$ such that

(1) $L(s,\lambda_n^0 \dots \lambda_n^0, \mu_1^0 \dots \mu_m^0)$ *is stationary at* s^0,
(2) $H_i(s^0) \leqslant 0$, $i=1,\dots,n$,
(3) $k_i(s^0)=0$, $i=1,\dots,m$,
(f) $\lambda_i^0 \cdot H_i(s^0)=0$, $i=1,\dots,n$.

Proof. See [21]. ■

APPENDIX C: PROOF OF THEOREM 3.5

The Lagrangian functional for Problem 1 can be written as

$$L(s;\gamma,\mu)=\max_{|t|\in[\delta,T]}|\langle s,s_t\rangle|+\gamma[\|\dot{s}\|^2-B^2/E]+\mu[\|s\|^2-E].$$

Now L is not differentiable with respect to s in the Frechet sense; however, it is possible via Theorem 3.4 to determine the directional Gateaux differential, and a necessary condition for L to be stationary at s^0 is that

$$\delta^+L(s;h,\gamma^0,\mu^0)\geqslant 0 \qquad \forall h \in \mathbb{S}[0,T].$$

Now it is easy to see that

(a) $$\delta^+(\|\dot{s}^0\|^2-B^2/E;h)=2\langle \dot{s}^0,\dot{h}\rangle,$$

(b) $$\delta^+(\|\dot{s}^0\|-E;h)=2\langle s^0,h\rangle,$$

(c) $$\delta^+(|\langle s^0,s_t^0\rangle|;h)=\begin{cases}\langle h,s_t^0+s_{-t}^0\rangle & \text{if } \langle s^0,s_t^0\rangle>0,\\ \langle h,-s_t^0-s_{-t}^0\rangle & \text{if } \langle s^0,s_t^0\rangle<0,\\ |\langle s_t^0+s_{-t}^0,h\rangle| & \text{if } \langle s^0,s_t^0\rangle=0,\end{cases}$$

where Lemma 3.3 has been applied to show (c).

Finally, combining the above and observing that

$$\langle h(t), x(t)\rangle = \left\langle \dot{h}(\tau), \int_{\tau}^{T} x(t)\, dt \right\rangle$$

we obtain the result.

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