and Applied Science in the Electrical Sciences and Engineering Department, and Assistant Dean for Graduate Studies in the School of Engineering and Applied Science. He has been engaged in research concerning the stability of distributed circuits, properties of nonlinear networks, theory of active circuits, digital signal processing, and analog circuit fault diagnosis. He is Editor of the book *Nonlinear Networks: Theory and Analysis*, IEEE Press, 1974.

Dr. Willson is a member of Eta Kappa Nu, Sigma Xi, Tau Beta Pi, the Society for Industrial and Applied Mathematics, and the American Society for Engineering Education. From June 1977 to June 1979 he served as Editor of the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS. He is the recipient of the 1978 Guillemin–Cauer Award of the IEEE Circuits and Systems Society, for coauthoring the best paper published in the TRANSACTIONS during the previous year.

Nonlinear Networks and Onsager–Casimir Reversibility

BRIAN D. O. ANDERSON, FELLOW, IEEE

***Abstract*—Time-invariant networks composed of transformers, linear resistors, and nonlinear reactive elements are studied, and it is shown that the usual noise model for the resistors implies that in an inductorless network, the capacitor charges have, as random processes, a microscopic reversibility property, and more generally, the capacitor charges and inductor fluxes have a generalized reversibility property, provided that the capacitor or inductor characteristics have odd symmetry.**

I. INTRODUCTION

THERE are at least three different ideas of thermodynamics and statistical mechanics which have been exemplified using network theory ideas. The first is the second law of thermodynamics, see e.g., [1], [2]. The second is the fluctuation-dissipation theorem, which in its earliest form was due to Nyquist, relating as it did the resistive part of the impedance of a passive network to the spectrum of its thermal noise voltage, see e.g., [3], [4]. The third idea is that of the Onsager reciprocal relations (an essentially thermodynamic concept) and microscopic reversibility (an essentially statistical mechanics concept more or less equivalent to the Onsager relations); these ideas are related to the concept of network reciprocity, see e.g., [1], [5]–[9] where some of these relations are exposited, and see e.g., [10], [11] for basic material on the Onsager relations and microscopic reversibility. One of the key ideas that comes out of this work is that in a linear resistor–capacitor network with all resistors at the one temperature and with no capacitor loops, the vector of capacitor charges constitutes a *reversible* random Markov process, i.e., not only is the process Markov and stationary, but the joint probability that at times t and t' the charge vector takes values a and b is the same as the joint probability that at times t and t' it takes values b and a. For a resistor–capacitor–inductor network, the idea is more complicated. As apparently first recognized by Casimir, [5], one must in comparing the probability densities, now of charges q and fluxes ϕ, allow for a sign reversal, so that

$$\begin{aligned} p(q(t)=a_1, \phi(t)=a_2, q(t')=b_1, \phi(t')=b_2) \\ =p(q(t)=b_1, -\phi(t)=b_2, q(t')=a_1, -\phi(t')=a_2). \end{aligned} \tag{1.1}$$

The reversibility property with the Casimir modification is sometimes termed dynamic reversibility.

We are interested in this paper in seeking nonlinear generalizations of these ideas, and in particular of the reversibility and dynamic reversibility properties. There are certainly contributions in the literature which have the goal of obtaining nonlinear network interpretations of one or more of the three thermodynamic ideas mentioned above. For example, [12]–[16] all study an interconnection of a resistor and a capacitor, one or both of which may be nonlinear (the resistor may even be a diode). In [16], *inter alia* reciprocity of single-element-kind nonlinear networks (which is normally equivalent to symmetry of the small signal resistance, inductance or capacitance matrix) is noted as being like an Onsager relation, but no results are obtained from this idea which apply to networks with more than one kind of element.

The main results of this paper are a verification that the reversibility idea is valid for networks with linear resistors and nonlinear capacitors, and a verification that the dynamic reversibility idea in its original form is valid for networks with linear resistors, nonlinear capacitors and nonlinear inductors with current flux characteristics

Manuscript received December 28, 1979; revised May 15, 1980. This work was supported by the Australian Research Grants Committee.
The author is with the Department of Electrical Engineering, University of Newcastle, New South Wales, Australia 2308.

0098-4094/80/1100-1051$00.75 ©1980 IEEE

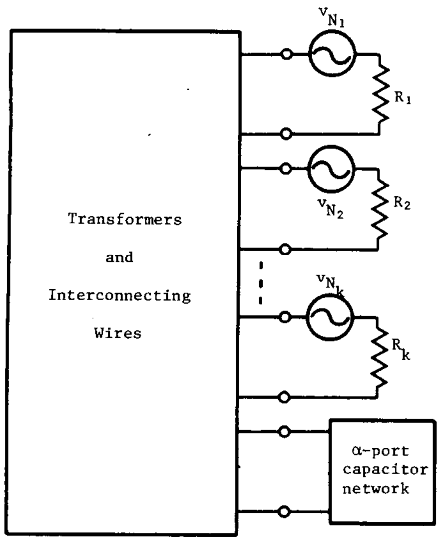

Fig. 2.1. Resistor–transformer nonlinear capacitor network, with resistor thermal noise sources.

possessing odd symmetry. (Of course, passivity and smoothness restrictions of some sort on the network elements are also needed.) What are the main steps? In Section II, we obtain noise models, representing the charge and flux processes as the solution of stochastic differential equations [18] excited by white noise. The functions appearing in these equations possess structure induced by the network structure. In Section III, we note a standard equation (the Fokker–Planck, or forward Kolmogorov equation, [18]) for the forward transition densities of the solution of a class of stochastic differential equations, including equations satisfied by the charge/flux processes; we also derive a new equation for the reverse transition densities. This equation (which is quite distinct from the backward Kolmogorov equation) has also been studied independently of network ideas in [19]). Because the reversibility and dynamic reversibility properties can be restated using transition densities as opposed to joint densities, this then allows us in Section IV to obtain the reversibility result for resistor–capacitor (actually transformer–resistor–capacitor) networks, and in Section V the dynamic reversibility result for resistor–capacitor–inductor networks. Section VI contains a brief remark on equipartition; though not connected with reversibility, the calculations of Section V allow the rapid stating of a nonlinear variant of the idea that the average stored energy in a linear inductor or capacitor is, under appropriate conditions, $1/2kT$, k being Boltzmann's constant and T the absolute temperature. It turns out that in the nonlinear case we can obtain a form of equipartition involving the sum of energy and coenergy.

II. Noise Models of Time-Invariant Networks with Nonlinear Reactances

2.1 *Resistor–Transformer–Capacitor Networks*

We consider time-invariant networks of the general form shown in Fig. 2.1, and we shall construct a stochastic

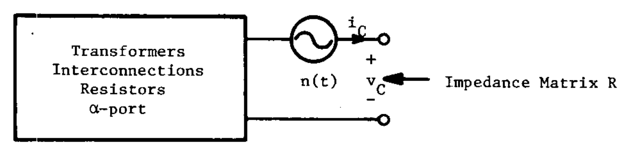

Fig. 2.2. Equivalent circuit for the nonreactive part of the circuit of Fig. 2.1.

differential equation for this model on the basis of a number of assumptions which we now set out.

Assumption 2.1: There are no capacitor loops, or linear constraints imposed on the capacitor voltages by virtue of the transformers and other interconnections.

Assumption 2.2: All resistors are linear and positive.

These assumptions mean that the capacitors see a network as depicted in Fig. 2.2, with an input impedance matrix which is constant, symmetric and positive definite. Call this matrix R.

Assumption 2.3: All resistors in the network are at T^0A and have associated with them a white noise, zero mean voltage with covariance $2kTR_i\delta(t-s)$, k being Boltzmann's constant, R_i the value of the resistance and $\delta(\cdot)$ the Dirac delta function. (These voltages are designated in Fig. 2.1 by v_{N_i}.)

We note that this is a standard noise model, though a more refined one is available, which gives a noise spectrum that is flat up to optical frequencies, and then decays to zero.

It is a consequence of Nyquist's theorem [3], but extended to multiport networks, or more generally of the fluctuation-dissipation theorem of statistical mechanics [4], that the Thévenin equivalent voltage vector at the terminals of the circuit of Fig. 2.2 is zero mean with covariance $2kTR\delta(t-s)$. We denote this vector by $n(t)$.

Having modelled the noise source and nondynamic part of the network, we turn now to the capacitive part. We require the following definition [20]:

Definition: $y=f(x)$ is a C^1-diffeomorphism of R^n onto itself if $f(\cdot)$ has a unique global C^1 inverse.

A necessary and sufficient set of conditions is:

i) $f(x)$ is continuously differentiable;

ii) The Jacobian $|\partial f/\partial x|$ is nonzero for all $x\in R^n$;

iii) $\|f(x)\|\to\infty$ as $\|x\|\to\infty$.

In now defining the capacitive network, we permit the possibility of (electrostatically) coupled capacitors.

Assumption 2.4: The capacitive network is defined by $v_C=f_C(q)$ where q and v_C are the charge and voltage vectors, respectively, f_C is a C^1-diffeomorphism and there exists a scalar energy storage function $E_C(q)$ such that $f_C(q)=\nabla E_C(q)$ and $\nabla^2E_C(q)-\varepsilon I$ is positive definite for all q and some $\varepsilon>0$.

These assumptions are reasonably standard, and in almost this form have been used frequently elsewhere [21]–[24]. As the name suggests, $E_C(q)$ is the energy stored by the capacitors when the charge vector is q. In the scalar case, they demand that the q–v characteristic be in the first and third quadrants, be monotonic, and that as $q\to\infty$, v_C and $E_C\to\infty$.

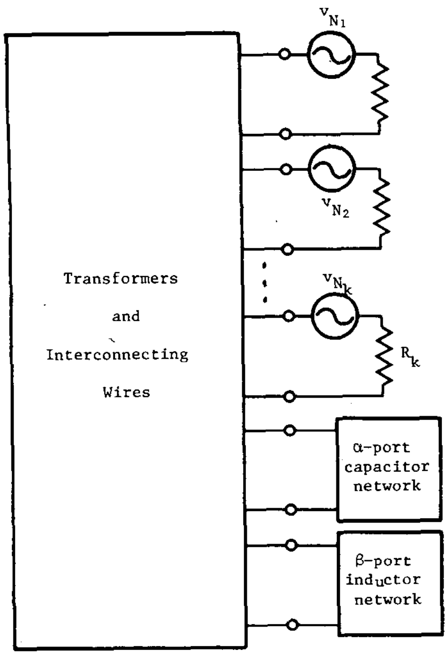

Fig. 2.3. Resistor–transformer-nonlinear capacitor-nonlinear inductor time-invariant network, with resistor thermal noise sources.

The equation describing the network now follows:

$$\begin{aligned}\dot{q}&=i_C\\&=-R^{-1}[v_C(t)-n(t)]\\&=-R^{-1}f_C(q)+R^{-1}n(t).\end{aligned}$$

Set $u(t)=[R^{1/2}\sqrt{2kT}\,]^{-1}n(t)$, where $R^{1/2}$ is a positive definite square root of R. This results in $E[u(t)u(s)]=I\delta(t-s)$ and

$$\dot{q}=-R^{-1}f_C(q)+R^{-1/2}\sqrt{2kT}\,u(t). \tag{2.1}$$

2.2 Resistor–Transformer–Capacitor–Inductor Networks

The treatment of resistor–transformer–inductor networks is so similar to resistor–transformer–capacitor networks that we shall omit it, considering just the more general case.

The arrangement considered is depicted in Fig. 2.3. Assumptions 2.1 through 2.4 remain in force. In addition, we require

Assumption 2.5: There are no inductor cut-sets, or linear constraints imposed on the inductor currents by virtue of the transformers and other interconnections.

This and Assumption 2.1 mean that the capacitors and inductors see a network as depicted in Fig. 2.4 with a hybrid matrix that, neglecting noise sources for the moment, forces relations among the port variables of the following form:

$$\begin{bmatrix}\hat{v}_C\\-\hat{i}_L\end{bmatrix}=\begin{bmatrix}H_{CC}&H_{CL}\\-H'_{CL}&H_{LL}\end{bmatrix}\begin{bmatrix}-\hat{i}_C\\\hat{v}_L\end{bmatrix}$$

which it will suit us to rewrite as

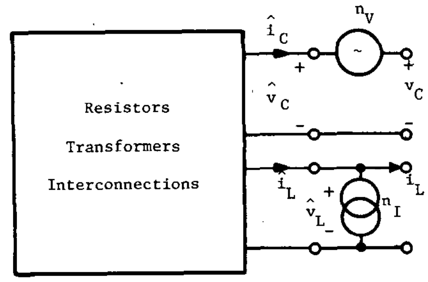

Fig. 2.4. Equivalent network for the nonreactive part of the circuit of Fig. 2.3.

$$\begin{bmatrix}\hat{v}_C\\\hat{i}_L\end{bmatrix}=\begin{bmatrix}-H_{CC}&H_{CL}\\-H'_{CL}&-H_{LL}\end{bmatrix}\begin{bmatrix}\hat{i}_C\\\hat{v}_L\end{bmatrix}. \tag{2.2}$$

Here, H_{CC} and H_{LL} are nonnegative definite impedance and admittance matrices respectively, while the overall matrix in (2.2) is guaranteed nonsingular.

Because of the noise voltages associated with each resistor, the equivalent network of Fig. 2.4 has Thévenin/Norton equivalent sources, and is as shown in the figure. Here $E[n_V(t)n_V Ls)]=2kTH_{CC}\delta(t-s)$, $E[n_I(t)n_I(s)]=2kTH_{LL}\delta(t-s)$, and $E[n_V(t)n_I(s)]=0$, for all t, s.

Last, we require the inductors to satisfy the same sort of conditions as the capacitors:

Assumption 2.6: The inductor network is defined by $i_L=f_L(\phi)$ where ϕ and i_L are the flux and current vectors, respectively, f_L is a C^1-diffeomorphism, and there exits a scalar energy storage function $E_L(\phi)$ such that $f_L\phi_L)=\nabla E_L(\phi)$ and $\nabla^2E_L(\phi)-\varepsilon I$ is positive definite for all ϕ and some $\varepsilon>0$.

The equations describing the network are now obtainable using (2.2) and the fact that $v_C=\hat{v}_C+n_V$, $i_L=\hat{i}_L+n_I$, as

$$\begin{aligned}\begin{bmatrix}\dot{q}\\\dot{\phi}\end{bmatrix}=\begin{bmatrix}i_C\\v_L\end{bmatrix}&=\begin{bmatrix}-H_{CC}&H_{CL}\\-H'_{CL}&-H_{LL}\end{bmatrix}^{-1}\begin{bmatrix}v_C-n_V\\i_L-n_I\end{bmatrix}\\&=\begin{bmatrix}-H_{CC}&H_{CL}\\-H'_{CL}&-H_{LL}\end{bmatrix}^{-1}\begin{bmatrix}f_C(q)\\f_L(\phi)\end{bmatrix}\\&\quad+\begin{bmatrix}H_{CC}&-H_{CL}\\H'_{CL}&H_{LL}\end{bmatrix}^{-1}\begin{bmatrix}H_{CC}^{1/2}&0\\0&H_{LL}^{1/2}\end{bmatrix}\sqrt{2kT}\,u(t)\end{aligned} \tag{2.3}$$

where again $E[u(t)u(s)]=I\delta(t-s)$.

III. Reversibility and the Reverse Time Kolmogorov Equation

Broadly speaking, reversible processes are those which have the same transition probabilities forward and backwards in time. For a Markov process, the definition is particularly straightforward. Let Y_t be a Markov process. (We shall use an upper case letter to denote a process, a corresponding lower case letter to denote a value taken by the process, and the subscript denotes a time argument.)

Then reversibility demands that the process be stationary and that

$$p_{Y_t|Y_s}(a|b)=p_{Y_s|Y_t}(a|b) \quad (3.1)$$

for all t and s. (Since $p_{Y_s}(b)=p_{Y_t}(b)$ by stationarity, (3.1) is equivalent to $p_{Y_t,Y_s}(a,b)=p_{Y_s,Y_t}(a,b)$ the condition given in Section 1.)

Now let us focus on a Markov process generated by a stochastic differential equation of diffusion type [18]

$$dy_t=f(y_t)\,dt+G\,dw_t. \quad (3.2)$$

This is in fact a specialized type of diffusion equation, the matrix G being a constant and the function $f(\cdot)$ not explicitly depending on t. (The fact that G is constant allows us to dispense with the question of whether this is an Ito or Stratonovich equation.) In the equation, w_t is a vector Wiener process, or, formally, dw_t/dt is a white noise process with covariance $I\delta(t-s)$. The function f: $R^n \to R^n$ has certain smoothness properties which are sufficient to guarantee existence and uniqueness properties for the equation solutions, see [18].

For such an equation, it is possible to write down a partial differential equation for $p_{Y_t|Y_s}(y_t,t|y_s,s)$ with $t>s$, known as the Fokker–Planck or first Kolmogorov equation. As we shall show, we can also write down an equation for $p_{Y_s|Y_t}$. Using these two equations, we may be able to prove a reversibility result of the type (3.1) for a particular $f(\cdot)$, without actually solving the equations. This will be our strategy for the stochastic differential equation associated with the resistor–transformer–capacitor network.

It is a standard result [18] that with smoothness assumptions on $f(\cdot)$, and when the densities exist and are smooth, the quantities $p_{Y_t|Y_s}(y_t,t|y_s,s)$ and $p_{Y_s}(y_s,s)$ satisfy the backward, and unconditioned forward, Kolmogorov equations

$$-\frac{\partial p(y_t,t|y_s,s)}{\partial s}=\sum_i f^i(y_s)\frac{\partial p(y_t,t|y_s,s)}{\partial y_s^i}+\frac{1}{2}\sum_{i,j}(GG')^{ij}\frac{\partial^2 p(y_t,t|y_s s)}{\partial y_s^i\partial y_s^j} \quad (3.3)$$

for $t\geqslant s$, and

$$-\frac{\partial p(y_s,s)}{\partial s}=\sum_i\frac{\partial}{\partial y_s^i}\left[f^i(y_s)p(y_s,s)\right]-\frac{1}{2}\sum_{i,j}(GG')^{ij}\frac{\partial^2 p(y_s,s)}{\partial y_s^i\partial y_s^j}. \quad (3.4)$$

(We have omitted the defining random variables in the interests of clarity.) The boundary condition for (3.3) is

$$\lim_{t\downarrow s}p(y_t,t|y_s,s)=\lim_{t\downarrow s}\delta(y_t-y_s).$$

The boundary conditions for (3.4) may simply be the density of $p(y_{s_0},s_0)$ at some initial time s_0. Now because

$$p_{Y_t,Y_s}(y_t,t,y_s,s)=p_{Y_t|Y_s}(y_t,t|y_s,s)p_{Y_s}(y_s,s) \quad (3.5)$$

we can attempt to obtain a partial differential equation for $p_{Y_t,Y_s}(y_t,t,y_s,s)$, regarding y_s,s as independent variables and y_t,t as parameters. Combining (3.3) through (3.5), we obtain

$$\frac{\partial}{\partial s}p(y_t,t,y_s,s)=\text{terms involving } f,\,g,\,p(y_t,t|y_s,s) \text{ and } p(y_s,s) \text{ and their } y_s \text{ derivatives.}$$

We eliminate every occurence of $p(y_t,t|y_s,s)$ on the right side, replacing it by $p(y_t,t,y_s,s)/p(y_s,s)$. The end result is, for $t\geqslant s$,

$$-\frac{\partial}{\partial s}p(y_t,t,y_s,s)=\sum_i\frac{\partial}{\partial y_s^i}\left[\bar f^i(y_s)p(y_t,t,y_s,s)\right.+\frac{1}{2}\sum_{i,j}(GG')^{ij}\frac{\partial^2 p(y_t,t,y_s,s)}{\partial y_s^i\partial y_s^j} \quad (3.6)$$

where

$$\bar f^i(y_s,s)=f^i(y_s)-\frac{1}{p(y_s)}\sum_j(GG')^{ij}\frac{\partial p(y_s,s)}{\partial y_s^j}. \quad (3.7)$$

If $p(y_s,s)$ depends explicitly on s, so does $\bar f^i(y_s,s)$. Now because $p(y_s,s|y_t,t)=p(y_t,t,y_s,s)/p(y_t,t)$, (3.6) also yields, for $t\geqslant s$

$$-\frac{\partial}{\partial s}p(y_s,s|y_t,t)=\sum_i\frac{\partial}{\partial y_s^i}\left[\bar f^i(y_s,s)p(y_s,s|y_t,t)\right]+\frac{1}{2}\sum_{i,j}(GG')^{ij}\frac{\partial^2 p(y_s,s|y_t,t)}{\partial y_s^i\partial y_s^j}. \quad (3.8)$$

The boundary condition is $\lim_{s\uparrow t}p(y_s,s|y_t,t)=\delta(y_s-y_t)$.

This equation, which is not standard, can be regarded as a reverse time Kolmogorov equation. Regarding t, y_t as fixed parameters and s, y_s as variables with s running backwards from t, (3.8) describes how the density $p(y_s,s|y_t,t)$ evolves in reverse time. The corresponding conditioned forward time equation is that for $p(y_t,t|y_s,s)$ where s, y_s are fixed parameters, and t, y_t are variables with t running forward from s. This equation describes how the density evolves in forward time and is different to (3.3), since the latter is concerned with regarding the conditioning quantities as variables rather than as parameters. The forward time equation is standard, [18] and is

$$-\frac{\partial p(y_t,t|y_s,s)}{\partial t}=\sum_i\frac{\partial}{\partial y_t^i}\left[f^i(y_s)p(y_t,t|y_s,s)\right]-\frac{1}{2}\sum_{i,j}(GG')^{ij}\frac{\partial^2 p(y_t,t|y_s,s)}{\partial y_t^i\partial y_t^j}. \quad (3.9)$$

In the next section, we shall find the reverse and forward time equations (3.8) and (3.9) associated with a resistor–capacitor network, and verify that the reversibility condition (3.1) holds.

IV. Reversibility for Resistor–Transformer–Capacitor Networks

In Section II, we derived a stochastic differential equation for the (vector) charge process on the capacitors of a network with linear resistors, transformers, and nonlinear capacitors, subject to certain constraints. In this section, we shall show using the theory of Section III that this charge process is (in the steady state) reversible. First, however, we must examine the steady state density of the charge vector.

The stochastic differential equation (2.1) is rewritten as

$$dq = -R^{-1}\nabla E_C(q)\,dt + R^{-1/2}\sqrt{2kT}\,dw. \tag{4.1}$$

The associated unconditioned forward Kolmogorov equation, see (3.4) is

$$-\frac{\partial p(q_s,s)}{\partial s} = -\sum_i \frac{\partial}{\partial q_s^i}\left[\sum_i (R^{-1})^{ij}\frac{\partial E_C(q_s)}{\partial q_s^j}q(q_s,s)\right] - kT\sum_{i,j}(R^{-1})^{ij}\frac{\partial^2 p(q_s,s)}{\partial q_s^i \partial q_s^j}. \tag{4.2}$$

Theorem 4.1: There exists a steady-state solution of the forward Kolmogorov equation given by

$$P_Q(q) = c\exp\left[-\frac{E_C(q)}{kT}\right] \tag{4.3}$$

where c is a normalizing constant.

Remark: The conclusion of the theorem is really predicted by statistical mechanics [17], [25]. Also note that if the capacitators are uncoupled, $E_C(q) = \Sigma_i E_{C_i}(q^i)$, where q^i is the charge on the ith capacitor C_i, and (4.3) implies that at any one instant of time, the entries of the charge vector are independent. (This is not the same as claiming that the charge vector process is a vector of independent processes.)

Proof: The right side of (4.2) is

$$-\sum_{i,j}\frac{\partial}{\partial q_s^i}(R^{-1})^{ij}\left\{\frac{\partial E_C(q_s)}{\partial q_s^j}p(q_s,s) + kT\frac{\partial p(q_s,s)}{\partial q_s^j}\right\}.$$

Replacing $p(q_s)$ by the expression in (4.3) yields zero for the quantity in { }, while the same expression in (4.3), being independent of s, gives zero for the left side of (4.2). This establishes the theorem. ∇∇∇

Theorem 4.1 now allows us to form the function $\bar{f}$ of the last sections and from it the equation for the reverse transition probabilities. We are interested in the charge process in the steady state. Thus the relevant unconditioned density to use is $p_Q(q)$. Equation (3.7) yields $\bar{f}$ independent of s:

$$\bar{f}^i(q) = -\sum_j (R^{-1})^{ij}\frac{\partial E_C(q)}{\partial q^j} - \frac{1}{p_Q(q)}\sum_j 2kT(R^{-1})^{ij}\frac{\partial p_Q(q)}{\partial q^j} \tag{4.4}$$

and using $p_Q(q)$ as given by (4.3), we obtain

$$\begin{aligned}\bar{f}^i(q) &= -\sum_j (R^{-1})^{ij}\frac{\partial E_C(q)}{\partial q^j} + 2\sum_j (R^{-1})^{ij}\frac{\partial E_C(q)}{\partial q^j}\\ &= \sum_j (R^{-1})^{ij}\frac{\partial E_C(q)}{\partial q^j}\\ &= -f^i(q).\end{aligned} \tag{4.5}$$

Using (3.8) and (3.9) and the specializations of this section, including (4.5), we have the reverse-time and forward equations, both valid for $t \geqslant s$

$$-\frac{\partial}{\partial s}p(q_s,s|q_t,t) = -\sum_i \frac{\partial}{\partial q_s^i}\left[f^i(q_s,s|q_t,t)\right] + kT\sum_{i,j}(R^{-1})^{ij}\frac{\partial^2}{\partial q_s^i \partial q_s^j}p(q_s,s|q_t,t) \tag{4.6}$$

and

$$-\frac{\partial}{\partial t}p(q_t,t|q_s,s) = \sum_i \frac{\partial}{\partial q_t^i}\left[f^i(q_t)p(q_t,t|q_s,s)\right] - kT\sum_{i,j}(R^{-1})^{ij}\frac{\partial^2}{\partial q_t^i \partial q_t^j}p(q_s,s|q_t,t) \tag{4.7}$$

and since

$$\lim_{s\uparrow t} p(q_s,s|q_t,t) = \delta(q_s - q_t) = \lim_{t\downarrow s} p(q_t,t|q_s,s)$$

it follows from (4.6) and (4.7) that the reverse-time evolution is the mirror image in time of the forward-time evolution, so that for $t \geqslant s$

$$p_{Q_s|Q_t}(a|b) = p_{Q_t|Q_s}(a|b).$$

This is the reversibility result (3.1).

V. Networks with Two Reactive Element Types

In Section II, we derived the basic form of stochastic differential equation describing networks with linear resistors, transformers, and nonlinear capacitor and inductor elements. Here, we shall evaluate the steady-state probability density, and compare the equations for the forward and backward transition probability densities. We shall discover that a modified form of reversibility holds.

Equation (2.3) is the relevant stochastic differential equation. To simplify the notation, let us set

$$y = \begin{bmatrix} q \\ \phi \end{bmatrix} \quad f(y) = \begin{bmatrix} f_C(q) \\ f_L(\phi) \end{bmatrix}$$

$$E(y) = E_C(q) + E_L(\phi)$$

$$H = \begin{bmatrix} H_{CC} & -H_{CL} \\ H'_{CL} & H_{LL} \end{bmatrix} \quad K = \begin{bmatrix} H_{CC} & 0 \\ 0 & H_{LL} \end{bmatrix} \tag{5.1}$$

so that

$$dy = -H^{-1}f(y)\,dt + H^{-1}K^{1/2}\sqrt{2kT}\,dw \tag{5.2}$$

with $w(t)$ being a vector Wiener process. The steady-state $p_Y(y)$ probability density satisfies

$$0 = \sum_i \frac{\partial}{\partial y^i}\left[-\sum_j (H^{-1})^{ij}f^j(y)p(y)\right] - kT\sum_{i,j}(H^{-1}KH^{-T})^{ij}\frac{\partial^2 p(y)}{\partial y^i \partial y^j}. \tag{5.3}$$

Notice that

$$H^{-1}KH^{-T} = \frac{1}{2}H^{-1}(H+H')H^{-T} = \frac{1}{2}\left[H^{-1} + H^{-T}\right]$$

and so

$$\sum_{i,j}(H^{-1}KH^{-T})^{ij}\frac{\partial^2 p(y)}{\partial y^i \partial y^j}=\frac{1}{2}\sum_{i,j}(H^{-1})^{ij}\frac{\partial^2 p(y)}{\partial y^i\partial y^j}+\frac{1}{2}\sum_{i,j}(H^{-T})^{ij}\frac{\partial^2 p(y)}{\partial y^i\partial y^j}$$
$$=\frac{1}{2}\sum_{i,j}(H^{-1})^{ij}\frac{\partial^2 p(y)}{\partial y^i\partial y^j}+\frac{1}{2}\sum_{i,j}(H^{-T})^{ij}\frac{\partial^2 p(y)}{\partial y^j\partial y^i}$$
$$=\frac{1}{2}\sum_{i,j}(H^{-1})^{ij}\frac{\partial^2 p(y)}{\partial y^i\partial y^j}+\frac{1}{2}\sum_{i,j}(H^{-T})^{ji}\frac{\partial^2 p(y)}{\partial y^i\partial y^j}$$
$$=\sum_{k,j}(H^{-1})^{ij}\frac{\partial^2 p(y)}{\partial y^i\partial y^j}. \quad (5.4)$$

Using (5.4) in (5.3) yields

$$0=\sum_{i,j}\frac{\partial}{\partial y^i}(H^{-1})^{ij}\left[f^j(y)p(y)+kT\frac{\partial p(y)}{\partial y^j}\right]$$

and then it is readily verified that

$$p_Y(y)=c\exp\left[-\frac{E(y)}{kT}\right]=c\exp\left[-\frac{E_C(q)}{kT}-\frac{E_L(\phi)}{kT}\right] \quad (5.5)$$

is a solution, where c is a normalizing constant. The argument is as in Theorem 4.1, (and the remark following that theorem applies with obvious change).

Now to get the equations for the reverse transition probabilities, we require the function $\bar{f}(y)$. Following (3.7), we have

$$\bar{f}^i(y)=-\sum_j(H^{-1})^{ij}f^j(y)-\frac{1}{p(y)}\sum_j 2kT(H^{-1}KH^{-T})^{ij}\frac{\partial p(y)}{\partial y^j}$$
$$=-\sum_j(H^{-1})^{ij}f^j(y)+\sum_j(H^{-1}+H^{-T})^{ij}f^j(y)$$
$$=\sum_j(H^{-T})^{ij}f^j(y). \quad (5.6)$$

We, therefore, need to compare the forward and reverse equations

$$-\frac{\partial p(y_t,t|y_s,s)}{\partial t}=\sum_i\frac{\partial}{\partial y_t^i}\left[-\sum_j(H^{-1})^{ij}f^j(y_t)p(y_t,t|y_s,s)\right]-kT\sum_{i,j}(H^{-1}KH^{-T})^{ij}\frac{\partial^2 p(y_t,t|Y_s,s)}{\partial y^i\partial y^j} \quad (5.7)$$

and

$$-\frac{\partial p(y_s,s|y_t,t)}{\partial s}=\sum_i\frac{\partial}{\partial y_s^i}\left[\sum_j(H^{-T})^{ij}f^j(y_s)p(y_s,s|y_t,t)\right]+kT\sum_{i,j}(H^{-1}KH^{-T})^{ij}\frac{\partial^2 p(y_s,s|y_t,t)}{\partial y_s^i\partial y_s^j}. \quad (5.8)$$

Because H is not symmetric, it is evident that forward transition probabilities are not the mirror images in time of the reverse transition probabilities. Nevertheless, there are two connections which we can explore. The first depends on a further assumption.

Assumption 5.1: The inductor characteristics are odd, i.e., $f_L(-\phi)=-f_L(\phi)$.

Define

$$\tilde{y}=\begin{bmatrix} q\\ -\phi\end{bmatrix}.$$

The stochastic differential equation for $\tilde{y}$ follows easily from that for y:

$$d\tilde{y}=-\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}H^{-1}\begin{bmatrix} f_C(q)\\ f_L(\phi)\end{bmatrix}+\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}H^{-1}K^{1/2}\sqrt{2kT}\,dw$$
$$=-\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}=H^{-1}\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}\begin{bmatrix} f_C(q)\\ f_L(-\phi)\end{bmatrix}+\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}H^{-1}K^{1/2}\sqrt{2kT}\,dw$$
$$=-H^{-T}f(\tilde{y})\,dt+H^{-T}\begin{bmatrix} H_{CC}^{1/2} & 0\\ 0 & -H_{LL}^{1/2}\end{bmatrix}\sqrt{2kT}\,dw$$

and so the forward transition density equation is

$$-\frac{\partial p(\tilde{y}_t,t|\tilde{y}_s,s)}{\partial t}=\sum_i\frac{\partial}{\partial\tilde{y}_t^i}\left[-\sum_j(H^{-T})^{ij}f^j(\tilde{y}_t)p(\tilde{y}_t,t|\tilde{y}_s,s)\right]-kT\sum_{i,j}(H^{-T}KH^{-1})^{ij}\frac{\partial^2 p(\tilde{y}_t,t|\tilde{y}_s,s)}{\partial\tilde{y}_t^i\partial\tilde{y}_t^j}.$$

Because $K=\frac{1}{2}(H+H')$, one has $H^{-1}KH^{-T}=H^{-T}KH^{-1}$, so that

$$-\frac{\partial p(\tilde{y}_t,t|\tilde{y}_s,s)}{\partial t}=\sum_i\frac{\partial}{\partial\tilde{y}_t^i}\left[-\sum_j(H^{-T})^{ij}f^j(\tilde{y}_t)p(\tilde{y}_t,t|\tilde{y}_s,s)\right]-kT\sum_{i,j}(H^{-1}KH^{-T})^{ij}\frac{\partial^2 p(\tilde{y}_t,t|\tilde{y}_s,s)}{\partial y_s^i\partial y_s^j}. \quad (5.9)$$

Comparing (5.8) [which has boundary condition

$\lim_{s\uparrow t} p(y_s, s|y_t, t) = \delta(y_s - y_t)]$ with (5.9) [which has boundary condition $\lim_{t\downarrow s} p(\tilde{y}_t, t|\tilde{y}_s, s) = \delta(\tilde{y}_t - \tilde{y}_s)]$, we see that $p(y_s, s|y_t, t)$ for fixed t and s varying backwards from t evolves as a mirror image of $p(\tilde{y}_t, t|\tilde{y}_s, s)$ for fixed s and t varying forwards from s, i.e.,

$$p(q_s = a_1, \phi_s = a_2 | q_t = b_1, \phi_t = b_2) = p(q_t = a_1, \phi_t = -a_2 | q_s = b_1, \phi_s = -b_2). \tag{5.10}$$

This is a *dynamic* reversibility result, of precisely the same type as is obtained for linear R, L, C networks, [5]–[9]. Of course in the linear case, the oddness Assumption 5.1 is automatically satisfied. A variant on this dynamic reversibility result, involving change of sign of the charges rather than fluxes, is available if the capacitor characteristics are assumed to be odd.

What if we do not have oddness of reactor characteristics? There is one relatively trivial observation we can make. Define a second network which is identical to the first, except that the inductor characteristics are given by

$$i_L = -f_L(-\phi). \tag{5.11}$$

Then (5.10) still holds with the left-hand probability referring to one network and the right-hand probability to the other network.

VI. A Note on Equipartition

The equipartition principle is not connected with reversibility. Nevertheless, the calculation of Section V leading to

$$p_Y(y) = c \exp\left[-\frac{E_C(q)}{kT} - \frac{E_L(\phi)}{kT}\right]$$

allows a rapid stating of a type of equipartition result. First, we explain what is not true. If the inductors and capacitors are uncoupled (so that $E_C(q) = \Sigma_i E_{C_i}(q^i)$, q^i denoting the change on the ith capacitor, and similarly for $E_L(\phi)$), simple calculations will show that in general we do not have

$$E[E_{C_i}(q^i)] = \frac{1}{2} kT \tag{6.1}$$

although in the linear case, (6.1) is true. (In fact (6.1) is true if the ith capacitor is linear and the others are not.) However we do have an interesting invariant:

$$E\left[q^i \frac{dE_{C_i}(q^i)}{dq^i}\right] = \int \cdots \int q^i \frac{dE_{C_i}(q^i)}{dq^i} p_Y(y)\, dy = c' \int_{-\infty}^{+\infty} q^i \frac{dE_c}{dq^i} \exp\left[-\frac{E_{C_i}(q^i)}{kT}\right] dq^i. \tag{6.2}$$

Here,

$$c' = \left[\int_{-\infty}^{+\infty} \exp\left[-\frac{E_{C_i}(q^i)}{kT}\right] dq^i\right]^{-1}. \tag{6.3}$$

The integral in (6.2) is easily evaluated by parts to yield

$$E\left[q^i \frac{dE_C(q^i)}{dq^i}\right] = kT. \tag{6.4}$$

Notice that $dE_C(q^i)/dq^i$ is simply the voltage across the capacitor when its stored charge is q^i. The left side of (6.4) is the mean of the stored energy plus coenergy. In the linear case, the coenergy equals the energy, and (6.1) can be recovered.

What if the capacitors are coupled? Then can evaluate

$$E[q' \nabla_{E_C}(q)] = \alpha kT. \tag{6.5}$$

Recall that α is the dimension of q.

VII. Conclusions

The two main contributions of the paper are easily stated. In networks with transformers, linear resistors, and nonlinear capacitors, the noise voltages on the resistors give rise to a random process defined by the capacitor charge vector which is reversible. If the network contains in addition nonlinear inductors, a dynamic reversibility result is available involving charges and fluxes if either the capacitor characteristics or the inductor characteristics have odd symmetry. When this symmetry is lacking, one can set up a related network with reverse transition probabilities that are the mirror image in time of the forward transition probabilities of the original network.

There is one very simple extension of the ideas, to cover the case of a network which can include gyrators as well as the other type of elements. Simple modifications of the arguments will show that if a second network is formed in which the gyrators are reversed, (and if two reactive types are present without odd characteristics, the replacement mentioned above and described in Section V is also made) then forward transition probabilities of one network are tied to the reverse transition probabilities of the other network.

There are a number of further problems which we are examining and which are linked to the ideas of this paper. What results are available if the resistors are nonlinear? The first difficulty here is to define an adequate noise model for a resistor. There is clearly controversy in the literature, [11]–[15]. Second, do the results enable the stating of a reciprocity type of property for a two port network containing elements of the type discussed in this paper? As noted in Section I, it is well known that in the linear case, statistical mechanics ideas can be used to interpret reciprocity [6], [7]; we are able to provide a property for a very wide range of networks at this point, but are seeking to make it wider still. Third, to what extent does the input–output mapping of a passive network with elements of the types used in this paper have a noise process at its ports with statistics which are determined just by that mapping? We know from the Nyquist result that in the linear one-port case, a network with impedance $z(s)$ at constant temperature has a stationary

Gaussian process at its ports of spectrum proportional to $\text{Re}[z(j\omega)]$, irrespective of the internal structure of the network, so long as it is built from linear passive elements. We are seeking a nonlinear version of this idea.

References

[1] J. Meixner, "Network theory in its relation to thermodynamics," in *Proc. Symp. on Generalized Networks*, (Polytechnic Institute of Brooklyn, NY) pp. 13–25, 1966.

[2] R. W. Brockett and J. C. Willems, "Stochastic control and the second law of thermodynamics," in *Proc. 1978 Decision and Control Conference*, (San Diego, CA), pp. 1007–1011.

[3] H. Nyquist, "Thermal agitation of electrical charge in conductors," *Phys. Rev.*, vol. 32, no. 1, pp. 110–173, 1928.

[4] H. B. Callen and R. F. Greene, "On a theorem of irreversible thermodynamics," *Phys. Rev.*, vol. 86, no. 5, pp. 702–710, 1952.

[5] H. B. G. Casimir, "On Onsager's principle of microscopic reversibility," *Rev. Mod. Phys.*, vol. 17, pp. 343–350, 1945.

[6] ——, "Reciprocity theorems and irreversible processes," *Proc. IEEE*, vol. 51, pp. 1570–1583, 1963.

[7] J. Meixner, "Thermodynamics of electrical networks and the Onsager-Casimir reciprocal relations," *J. Math. Phys.*, vol. 4, no. 2, pp. 154–159, 1963.

[8] B. D. O. Anderson and T. Kailath, "Forwards, backwards and dynamically reversible Markovian models of second-order processes," *IEEE Trans. Circuits Syst.*, to appear.

[9] J. C. Willems, "Time Reversibility in Deterministic and Stochastic Dynamical Systems," in *Proc. 1977 U.S.-Italy Seminar on Variable Structure Systems; Springer Lecture Notes in Economics and Mathematical Systems*, to appear.

[10] S. R. de Groot, "On the development of nonequilibrium thermodynamics," *J. Math. Phys.*, vol. 4, no. 2, pp. 147–153, 1963.

[11] S. R. de Groot and P. Mazur, *Nonequilibrium Thermodynamics*, Amsterdam, The Netherlands: North-Holland, 1962.

[12] N. G. Van Kampen, "Exact calculation of the fluctuation spectrum of a nonlinear model system," *J. Math. Phys.*, vol. 2, no. 4, pp. 592–601, 1961.

[13] D. K. C. MacDonald, "The Brownian movement of linear and nonlinear systems," *Phil. Mag.*, vol. 45, no. 360, pp. 63–72, 1954.

[14] C. T. J. Alkemade, "On the problem of Brownian motion of nonlinear systems," *Physica*, vol. 24, pp. 1029–1034, 1958.

[15] N. G. Van Kampen, "Nonlinear thermal fluctuations in a diode," *Physica*, vol. 26, pp. 585–604, 1960.

[16] W. Bernard and H. B. Callen, "Irreversible thermodynamics of a nonlinear *RC* system," *Phys. Rev.*, vol. 118, no. 6, pp. 1466–1470, 1960.

[17] G. Oster, A. Perelson, and A. Katchalsky, "Network thermodynamics," *Nature*, vol. 234, pp. 393–399, 1971.

[18] I. I. Gihman and A. V. Skorohod, *Stochastic Differential Equations*. Berlin, Springer, 1972.

[19] B. D. O. Anderson, "Reverse-time diffusion equation models," *Stochastic Processes and Their Applications*, to appear.

[20] R. S. Palais, "Natural operations on differential forms," *Trans. Amer. Math. Soc.*, vol. 92, pp. 125–141, 1959.

[21] L. O. Chua and Y. F. Lam, "A theory of algebraic *n*-ports," *IEEE Trans. Circuit Theory*, vol. 20, no. 4, pp. 370–381, 1973.

[22] C. Cherry, "Some general theorems for nonlinear systems possessing reactance," *Phil. Mag.*, vol. 42, no. 333, pp. 1161–1177, 1951.

[23] R. K. Brayton and J. K. Moser, "A theory of nonlinear networks, I and II," *Quart. Appl. Math.*, vol. 3, no. 2, pp. 1–33, pp. 81–104, 1964.

[24] N. T. Hung and B. D. O. Anderson, "Analysis and synthesis of nonlinear reciprocal networks containing two element types and transformers," *IEEE Trans. Circuits Syst.*, to appear.

[25] L. D. Landau and E. M. Lifschitz, *Statistical Physics*. Pergamon, Oxford, 1969.

Brian D. O. Anderson (S'62–M'66–SM'74–F'75), for photograph and biography see page 276 of the April 1980 issue of this TRANSACTIONS.