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Necessary and Sufficient Conditions for Delay-Independent Stability of Linear Autonomous Systems

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***Abstract*—Strict quasi-diagonal dominance of the system matrix is known to be sufficient for a linear autonomous system with arbitrary time delays in off-diagonal interactions to be stable. A small perturbation of the matrix yields a perturbed system with the same dominance, and, hence, stability properties. In this paper, it is shown that quasi-diagonal dominance is also necessary for stability with respect to small perturbations and arbitrary off-diagonal time delays. Weaker necessary conditions are given for systems which are themselves stable for all time delays, but which have perturbations that are unstable for certain delays.**

I. Introduction

The linear autonomous time-delayed systems under consideration are described by

$$\dot{x}_i(t) = a_{ii}x_i(t) + \sum_{j\neq i} a_{ij}x_j(t - T_{ij}) \qquad (1.1)$$

where $t \in [0,\infty)$, $i = 1,\cdots,n$. Throughout, the abbreviation $\Sigma_{j\neq i}$ denotes $\Sigma^n_{j=1,j\neq i}$. The vector $x(t) = [x_1(t),\cdots,x_n(t)]^T$ will be called the instantaneous system state and the scalars $T_{ij} \geq 0$ are referred to as the time delays. By the system matrix A, we mean $(A)_{ij} = a_{ij}$, $i,j = 1,\cdots,n$. For any fixed set of time delays, define $T = \max_{i,j} T_{ij} < \infty$. Note that we do not allow delays in the "diagonal interactions," that is, $T_{ii} = 0$ for all i.

The solution of (1.1) depends upon the specification of an initial condition $x(s) = \phi(s)$, $s \in [-T, 0]$. It is sufficient for our purposes to assume that the given n vector function $\phi(\cdot)$ is bounded and measurable. For simplicity of statement, we have taken each component of $x(\cdot)$ to be specified on the same (maximal) initial interval, although this is not necessary.

For any initial condition ϕ, the solution $x(t)$ to (1.1) exists on the interval $[0,\infty)$, and there are constants k and α, depending only upon A and the T_{ij}'s, such that for all t

$$\|x(t)\| \leq ke^{\alpha t}\|\phi(\cdot)\| \qquad (1.2)$$

(see [1]). Here $\|\cdot\|$ denotes the Euclidean norm in R^n and $\|\phi(\cdot)\| = \sup_{s\in[-T,0]}\|\phi(s)\| < \infty$.

System (1.1) is said to be asymptotically stable if $\|x(t)\| \to 0$ as $t \to \infty$. For linear autonomous systems, this is equivalent to exponential stability, i.e, α in (1.2) can be taken to be negative [1]. Throughout this paper, when referring to stability, we shall therefore mean exponential stability.

For a fixed set of time delays, T_{ij}, $i,j = 1,\cdots,n$, $i \neq j$, a necessary and sufficient condition for (1.1) to be stable is that all the roots μ of the characteristic equation

$$\det\begin{bmatrix} \mu - a_{11} & -a_{12}e^{-\mu T_{12}} & \cdots & -a_{1n}e^{-\mu T_{1n}} \\ -a_{21}e^{\mu T_{21}} & \mu - a_{22} & & \\ \vdots & & \ddots & \vdots \\ -a_{n1}e^{-\mu T_{n1}} & \cdots & & \mu - a_{nn} \end{bmatrix} = 0 \qquad (1.3)$$

have negative real parts [1]. Even though there cannot be more than a finite number of roots with nonnegative real parts, the difficulty of determining the roots of nonalgebraic equations generally precludes the use of the above condition as a test for stability.

The search for an implementable test for stability has led the authors and others to the following (see [2], [3, Sect. 3], [10]).

Theorem 1: If there exist positive scalars d_i, $i = 1,\cdots,n$, such that for all i

$$d_i a_{ii} + \sum_{j\neq i} d_j|a_{ji}| < 0, \qquad (1.4)$$

then (1.1) is stable. □

A matrix A satisfying (1.4) for a suitable collection of d_i's is said to be (column) quasi-diagonal-dominant. This condition is equivalent to row quasi-diagonal-dominance, that is, the existence of positive scalars $\bar{d}_i$, $i = 1,\cdots,n$ such that

$$\bar{d}_i a_{ii} + \sum_{j\neq i} \bar{d}_j|_{ij}| < 0 \qquad (1.5)$$

(see [4]). We therefore refer to matrices satisfying (1.4) and (1.5) as quasi-diagonal-dominant, abbreviated to q.d.d. Clearly, the condition implies that $a_{ii} < 0$ for all i. Readers are referred to [1]–[3] and the references therein for wider ranging discussions of stability criteria for time-delayed systems.

Notes:

1) Throughout this paper, dominance is taken to be strict, i.e., equalities are not permitted in (1.4) or (1.5).

2) Conditions (1.4) and (1.5) are independent of the time delays T_{ij} and are therefore easily tested either by the Hicks conditions (see [5]) or by computing the inverse of $\bar{A}$, the matrix given by $\bar{a}_{ii} = a_{ii}$, $\bar{a}_{ij} = |a_{ij}|$ $i,j = 1,\cdots,n$, $i \neq j$; $\bar{A}^{-1}$ must have positive elements.

In [3] we raised the question as to whether any system having the form (1.1) and which is stable for arbitrary constant time delays must have a q.d.d. system matrix. The example given in Section III shows that this is not true. However, it is clear that a sufficiently small perturbation of a q.d.d. matrix A yields a q.d.d. perturbed matrix $\hat{A} = (\hat{a}_{ij})$. Consequently, the perturbed system (1.1) with $\hat{a}_{ij}$'s replacing a_{ij}'s is also stable for arbitrary time delays in off-diagonal interactions. For this stability of perturbed systems, it is, in fact, necessary as well as sufficient for the original (unperturbed) system matrix to be q.d.d. Our main result is as follows.

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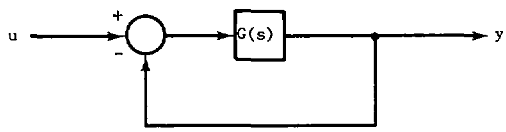

Fig. 1.

Theorem 2: If there exists $\epsilon>0$ such that all ϵ-perturbations of (1.1) are stable for all choices of time delays T_{ij}, $i,j=1,\cdots,n$, $i\neq j$, then the system matrix A is quasi-diagonal dominant.

A similar result for certain one-dimensional systems is given in [11].

Definition 1: Given $\epsilon>0$, an ϵ-perturbation of (1.1) is the system obtained from (1.1) by replacing each coefficient a_{ij} by $a_{ij}+\delta_{ij}$, $|\delta_{ij}|\leqslant\epsilon$, $i,j=1,\cdots,n$.

Note: As the systems concerned have constant coefficients, the above class of perturbations contains multiplicative perturbations $a_{ij}(1+\eta_{ij})$, $|\eta_{ij}|\leqslant\delta$ as a subclass, provided $\delta\leqslant\epsilon/\max_{ij}|a_{ij}|$. We shall always use multiplicative perturbations.

The solution to the original problem is more complex than that given by Theorem 2. In Section III, we show that if (1.1) is stable for arbitrary time delays, but admits ϵ perturbations which are not stable, then the system matrix A is rearrangeable in block lower triangular form, with each diagonal block being nonstrictly q.d.d. An example shows that this is the best possible result.

II. Proof of Theorem 2

The major thrust in our proof lies in establishing that if A satisfies the conditions of the theorem, then the nondelayed system

$$\dot{y}_i(t)=a_{ii}y_i(t)+\sum_{j\neq i}|a_{ij}|y_j(t) \tag{2.1}$$

$t\in[0,\infty)$ $i=1,\cdots,n$ is stable. The system matrix $\bar{A}$ of (2.1) has nonnegative off-diagonal elements; hence, to be stable, it must be quasi-diagonal-dominant [5]. The proof is completed by observing that $\bar{A}$ being q.d.d. is equivalent to A being q.d.d.

Our approach is based upon the Nyquist criterion for the stability of single-input single-output time-invariant feedback systems. This criterion is stated as follows.

Nyquist Stability Criterion

Let $G(s)$ be a transfer function for which the criterion is valid (see [6]); then the feedback system depicted in Fig. 1 is stable if and only if the Nyquist diagram (or plot) of $G(s)$ does not encircle or pass through the critical point $(-1,j0)$. [This version of the criterion holds for transfer functions which do not have poles in the right-hand half plane $\mathrm{re}(s)>0$. For transfer functions with poles in the right-hand half plane, an alternative statement is valid, namely, for stability, the number of encirclements of $(-1,j0)$ must equal the number of right-hand poles. We shall not need this version here (see Lemma 3 below).]

Note 1): The Nyquist diagram of $G(s)$ is the map of the imaginary axis (the Nyquist contour) into the complex plane under G. In the neighborhood of a pole, $j\omega_0$ say, of $G(s)$ on the imaginary axis, the contour is modified. Let an arbitrarily small $\epsilon>0$ be given, and let $\mathcal{C}$ be the semi-circle of radius ϵ, centered on $j\omega_0$, in the right-hand half plane. The contour is then $\{j\omega\colon \omega\geqslant\omega_0+\epsilon,\ \omega\leqslant\omega_0-\epsilon\}\cup\mathcal{C}$. The image of $\mathcal{C}$ under G is an arc of arbitrarily large radius.

As the transfer functions considered here cannot have more than a finite number of poles on the imaginary axis (see below and [1]), the above modification can be made at each pole. Then since the transfer functions are analytic apart from at their poles, we are assured that their Nyquist diagrams are connected.

Note 2): Strictly speaking, the "only if" statement of the Nyquist criterion should read: if the Nyquist diagram of $G(s)$ encircles or passes through the critical point $(-1,j0)$ a finite number of times, then the feedback system is unstable. An infinite number of encirclements may cause problems.

For our main results, we require the following in addition to the assumption that the Nyquist criterion is valid.

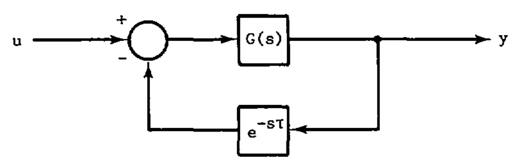

Fig. 2.

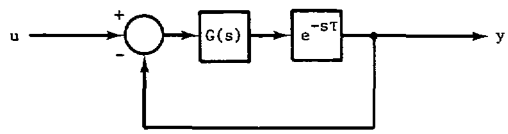

Fig. 3.

(2.2) $G(s)$ has at most a finite number of poles on the imaginary axis and is analytic apart from at its poles.

(2.3) $|G(j\omega)|\to 0$ as $|\omega|\to\infty$.

Note 3): If (2.2) and (2.3) hold, the Nyquist diagram of $G(s)$ can encircle the critical point at most a finite number of times.

Consider now the introduction of a time delay τ in the feedback path, as depicted in Fig. 2.

From the point of view of stability, the time delay can be in either the forward or feedback paths, so Fig. 2 can be redrawn as in Fig. 3.

The transfer function $H(s)=e^{-s\tau}G(s)$ satisfies (2.2) and (2.3) whenever $G(s)$ does.

Lemma 1: Let $G(s)$ satisfy conditions (2.2) and (2.3). Then if there is a $\omega_0\neq 0$ such that $|G(j\omega_0|\geqslant 1$, there exists a τ_0 such that the time-delayed feedback system is not stable.

Proof: The effect of a time delay τ is to rotate the Nyquist diagram, that is, for every ω, the point $H(j\omega)$ is $G(j\omega)$ rotated clockwise about the origin through an angle $\omega\tau$. Therefore, if $\omega_0\neq 0$ is such that $|G(j\omega_0)|\geqslant 1$, we can choose τ_0 such that the Nyquist diagram of $H(s)$ encircles or passes through $(-1,j0)$.

To see this more clearly, let ω_0 be the point in question. If $|G(j\omega_0)|>1$, then since the Nyquist diagram of $G(s)$ is connected (this follows from (2.2), see note 1) above), there are ω_1 and $\omega_2, \omega_1<\omega_0<\omega_2$ such that $|G(j\omega_1)|=|G(j\omega_2)|=1$ because $|G(j\omega)|\to 0$ as $|\omega|\to\infty$. Now ω_1 and ω_2 cannot both be zero, so we can choose $\bar{\omega}_0\neq 0$ with $|G(j\bar{\omega}_0)|=1$, i.e., $G(j\bar{\omega}_0)=e^{j\alpha}$ for some α. Choose $\tau_0>0$ such that $\alpha-\omega_0\tau_0=-\Pi$ if $\bar{\omega}_0>0$, $\alpha-\bar{\omega}_0\tau_0=\Pi$ if $\bar{\omega}_0<0$. Then $H(j\bar{\omega}_0)=-1$ so the Nyquist diagram of H passes through $(-1,j0)$. Obviously, if $|G(j\omega_0)|=1$, we put $\bar{\omega}_0=\omega_0$; then the above holds.

As the diagram of $H(s)$ can pass through or encircle $(-1,j0)$ only a finite number of times (note 3) above), the proof is complete. □

Note 4): The arcs of the Nyquist diagram of $G(s)$ corresponding to the semicircles around the poles of $G(s)$ on the imaginary axis are irrelevant to the above argument since the arcs have large radii $\gg 1$. On the other hand, if $G(s)$ does have a pole on the imaginary axis, then the conditions of the lemma are fulfilled and $H(s)$ gives an unstable feedback system for some time delays.

Let us now consider the time-delayed system (1.1). According to the outline at the beginning of this section, we want to prove that the conditions of Theorem 2 enable us to replace negative off-diagonal elements a_{ij} by their absolute values and maintain the stability properties of the system. If no off-diagonal elements are negative, we can proceed directly to Theorem 4 below.

Suppose that a_{kl}, $1\leqslant k$, $l\leqslant n$, $k\neq l$ is negative. By rearranging the indices in (1.1), we can label this element a_{12}. Then the single-input single-output system (D) on $[0,\infty)$ is defined by

$$\left.\begin{aligned}&\dot{x}_1(t)=a_{11}x_1(t)+u_{12}(t)+a_{13}x_3(t-T_{13})+\cdots+a_{1n}x_n(t-T_{1n})\\&\dot{x}_i(t)=a_{ii}x_i(t)+\sum_{j\neq i}a_{ij}x_j(t-T_{ij})\qquad i=2,\cdots,n\\&y_{12}(t)=-a_{12}x_2(t)\end{aligned}\right\}. \tag{2.4}$$

(D) has input $u_{12}(t)$, output $y_{12}(t)$, and transfer function $G(s)=-(J(s)^{-1})_{21}a_{12}$ where

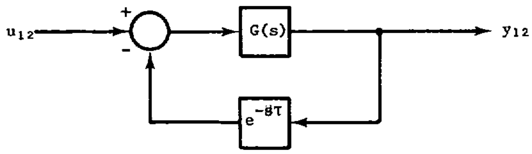

Fig. 4.

$$J(s)=\begin{bmatrix} s-a_{11} & 0 & -a_{13}e^{-sT_{13}} & -a_{1n}e^{-sT_{1n}} \\ -a_{21}e^{-sT_{21}} & s-a_{22} & -a_{23}e^{sT_{23}} & -a_{2n}e^{-sT_{2n}} \\ \vdots & & \ddots & \\ -a_{n1}e^{-sT_{n1}} & \cdots & & s-a_{nn} \end{bmatrix}. \quad (2.5)$$

Lemma 2: The transfer function $G(s)$ of (D) satisfies (2.2) and (2.3).

Proof:

1) The poles of $G(s)$ are given by the zeros of $\det J(s)$. From [1], there cannot be more than a finite number of these on the imaginary axis. It is clear from the form of (2.5) that $G(s)$ is analytic apart from at its poles.

2) Let $a=\max_{i,j}|a_{ij}|$ and $\epsilon>0$. Then for $|\omega|>a/\epsilon$, $J(j\omega)=j\omega(I+0(\epsilon))$, $J^{-1}(j\omega)=(j\omega)^{-1}(I+0(\epsilon))$, and $|G(j\omega)|=0(\omega^{-1})$. □

Note 5): That the Nyquist criterion is valid for the transfer function $G(s)$ of (D) is proved in [6].

If we now introduce feedback into (D) as depicted in Fig. 4 ($u_{12}(t)=-y_{12}(t-\tau)=a_{12}x_2(t-\tau)$), then the resulting feedback system is equivalent to (1.1) with $T_{12}=\tau$. The hypotheses of Theorem 2 imply that (1.1) is stable for arbitrary T_{12}, that is, the feedback system in Fig. 4 is stable for all delays τ in the feedback path.

Lemma 3: The transfer function $G(s)$ of (D) is analytic in the right-hand half plane, $\mathrm{Re}(s)>0$.

Proof: $G(s)=-(J(s)^{-1})_{21}a_{12}$. As noted in Lemma 2, $G(s)$ is analytic apart from at its poles where are given by $\det J(s)=0$. Let $\chi(s)$ denote the characteristic function of (1.1), i.e., the left-hand side of (1.3) (with s replacing μ); then

$$\chi(s)=\det J(s)-a_{12}e^{-sT_{12}}\det J_{12}(s)$$

where J_{12} is J with the first row and second column deleted. Suppose $\det J(s)$ has a zero at s_1 where $\mathrm{Re}(s_1)>0$; then since the zeros of $\det J(s)$ in the right half plane are isolated [1], choosing $0<\epsilon<\mathrm{Re}(s_1)$ sufficiently small, $|\det J(s)|\geq\delta>0$ on the circle $\{s_1+\epsilon e^{j\theta}: |\theta|\leq\pi\}$. Since $\det J_{12}(s)$ is analytic and the circle is contained in the positive right half plane, for T_{12} sufficiently large, $|-a_{12}e^{-sT_{12}}\det J_{12}(s)|<\delta$ on the circle. By Rouche's theorem [8], then $\chi(s)$ has a zero inside the circle, contradicting the stability of (1.1). Hence, $\det J(s)$ has no zeros in the right half plane and $G(s)$ is analytic there. □

This justifies our using the Nyquist condition in the form given above.

Proposition 1: The transfer function $G(s)$ of (D) satisfies $|G(j\omega)|\leq\eta<1$ for all $\omega\in(-\infty,\infty)$ for some η.

Proof: Let us first show that $|G(j\omega)|<1$ for all $\omega\in(-\infty,\infty)$ by supposing the contrary, i.e., the existence of ω_0 such that $|G(j\omega_0)|\geq1$. If this ω_0 is not zero, Lemmas 1 and 2 tell us that there is a time delay τ_0 such that the feedback system in Fig. 4 is unstable. However, this contradicts the hypotheses of Theorem 2; hence, such an ω_0 cannot exist.

The possibility $|G(0)|>1$ is ruled out because, by the analyticity of $G(s)$, it implies the existence of $\omega_0\neq0$ such that $|G(j\omega_0)|\geq1$. As $G(0)$ is real, we have two remaining possibilities to eliminate, $G(0)=-1$ and $G(0)=1$. As $G(0)=-1$ implies that the Nyquist diagram of $G(s)$ goes through the critical point, it cannot hold. To eliminate the possibility that $G(0)=1$, we invoke our hypothesis that perturbations of (1.1) are also stable for all time delays. Let $\tilde{A}$ denote the matrix obtained from A by replacing a_{12} by 0. $G(0)$ is given by

$$G(0)=-(-\tilde{A}^{-1})_{21}a_{12}.$$

(If $\tilde{A}$ is not invertible, then $\lim_{s\to0}|G(s)|=\infty$ and we have nothing to prove.)

Now let ϵ be the perturbation factor hypothesized in Theorem 2 and consider the perturbed system with matrix $\hat{A}$ where $\hat{a}_{12}=a_{12}(1+\epsilon)$, $\hat{a}_{ij}=a_{ij}$, $i,j\neq1,2$. The transfer function $\tilde{G}(s)$ of the single-input single output system (defined as previously) corresponding to the perturbed system satisfies

$$\hat{G}(0)=-(-\tilde{A}^{-1})_{21}\hat{a}_{12}=(1+\epsilon)G(0)=1+\epsilon>1$$

if $G(0)=1$. Then, arguing as above, we know that the perturbed system is unstable for certain time delays. This contradiction establishes that $G(0)\neq1$.

Therefore, $|G(j\omega)|<1$ for all $\omega\in(-\infty,\infty)$. As $G(j\omega)\to0$ when $|\omega|\to\infty$ and the Nyquist diagram of $G(s)$ is connected, there exists $\eta<1$ such that $|G(j\omega)|\leq\eta<1$ for all $\omega\in(-\infty,\infty)$. □

Lemma 4: If $|G(j\omega)|\leq\eta<1$ for all $\omega\in(-\infty,\infty)$, then the feedback system with transfer function $-G(s)$ and arbitrary feedback delay is stable.

Proof: The transfer function for the system is $H(s)=-G(s)e^{-s\tau}$ when τ is the delay. $H(s)$ has no poles on the imaginary axis, which is, therefore, the Nyquist contour for H. The Nyquist diagram for H is contained in $\{z: |z|\leq\eta\}$ as $|H(j\omega)|\leq\eta$ for $\omega\in(-\infty,\infty)$. □

Relating Lemma 4 to our system (D), the system with transfer function $-G(s)$ is obtained by replacing a_{12} by $-a_{12}$, i.e., $|a_{12}|$ (recall that $a_{12}<0$). Lemma 4 and Proposition 1 therefore imply that the system obtained from (1.1) by replacing a_{12} by $|a_{12}|$ is stable for arbitrary time delays in the off-diagonal interactions.

Note 6): The arbitrary nature of the time delay T_{12} follows from Lemma 4. The other T_{ij}'s, $i,j\neq1,2$, appearing in (2.4) and (2.5), are also arbitrary.

For sufficiently small $\epsilon>0$, ϵ-perturbations of the new system will also be stable for arbitrary time delays since the perturbations will yield feedback systems with transfer functions $\tilde{H}(s)$ satisfying $|\tilde{H}(j\omega)|\leq\eta_\epsilon<1$; $\omega\in(-\infty,\infty)$. This can readily be verified from (2.5).

Now the new system with $|a_{12}|$ replacing a_{12} satisfies the hypothesis of Theorem 2, although possibly for different ϵ from the original. The replacement of negative off-diagonal coefficients a_{ij} by their moduli can therefore be performed sequentially until none remains. We have proved the following.

Theorem 3: Subject to the hypothesis of Theorem 2, there exists a $\delta>0$ such that all δ perturbations of system (2.6) are stable for arbitrary time delays T_{ij}, $i,j=1,\cdots,n$, $i\neq j$.

$$\dot{y}_i(t)=a_{ii}y_i(t)+\sum_{j\neq i}|a_{ij}|y_j(t-T_{ij}) \quad (2.6)$$

□

Theorem 3 subsumes the result we need about system (2.1), viz., put all delays to zero and ignore the possibility of perturbations. Let $\bar{A}$ denote the system matrix of (2.1) and (2.6). As noted previously, $\bar{A}$ has nonnegative off-diagonal elements.

Theorem 4: The system matrix $\bar{A}$ of (2.1) and (2.6) is quasi-diagonally-dominant.

Proof: As $\bar{A}$ has nonnegative off-diagonal entries, it is a Metzler matrix [5, Definition A2]. Hence, because (2.1) and (2.6) are stable, $\bar{A}$ is quasi-diagonally-dominant [5, Corollary A1 and Theorem A3]. □

This completes the proof of Theorem 2.

III. Reduced Hypotheses

The authors originaly believed that stability of (1.1) for arbitrary time delays implied that the system matrix was quasi-diagonal-dominant, that is, the hypothesis regarding stability of perturbed systems is redundant. It transpires that in the absence of this hypothesis, a weaker dominance condition is necessary, namely, the system matrix must be rearrangeable in block lower triangular form with nonstrictly q.d.d. diagonal blocks. That the possibility of equality in some of the dominance conditions cannot be ruled out is shown by an example.

Theorem 5: If system (1.1) is stable for arbitrary time delays T_{ij}, $i,j=1,\cdots,n$, $i\neq j$, then for any $\epsilon>0$, there is an ϵ perturbation of (1.1) which has a quasi-diagonal-dominant system matrix.

Proof: The proof is very similar to that of Proposition 1 *et seq.* Let $G(s)$ be the transfer function defined by (2.4) and (2.5) where $a_{12}<0$ (if all of the off-diagonal entries in A are nonnegative, there is nothing to prove). Following the proof of Proposition 1, we note that if (1.1) is stable for arbitrary time delays, then $|G(j\omega)|<1$ for all $\omega\in(-\infty,\infty)$, $\omega\neq 0$ with the possibility that $G(0)=1$, i.e., $|G(0)|=1$.

Suppose $G(0)=1$. Let $H(s)$ be the transfer function of the perturbation of (1.1) defined by replacing a_{12} by ηa_{12} where $\eta=1-\delta$, $\delta>0$ sufficiently small, the other system coefficients being unchanged. Then $H(s)=\eta G(s)$ and $|H(j\omega)|\leq\eta<1$ for all $\omega\in(-\infty,\infty)$. By Lemma 4, the feedback delay is stable and we conclude that the system obtained from (1.1) by replacing a_{12} by $|\eta a_{12}|$ $(=\eta|a_{12}|)$ is stable for arbitrary off-diagonal time delays. (The delays T_{ij}, $i,j\neq 1,2$ in (2.4) and (2.5) are arbitrary and T_{12} is the feedback delay.)

If $G(0)\neq 1$, then $|G(0)|<1$ (see proof of Proposition 1) and the above goes through with $\eta=1$, i.e., $\delta=0$.

It is clear that iteration of the above enables us to replace all negative off-diagonal coefficients in (1.1) by a multiple $\eta<1$ of their moduli, while preserving the stability of the system. By Theorem 4, this system with nonnegative off-diagonal entries has a quasi-diagonally-dominant system matrix. This is equivalent to quasi-diagonal-dominance of the matrix $\hat{A}$ given by

$$\hat{a}_{ii}=a_{ii},\hat{a}_{ij}=\begin{cases} a_{ij} & a_{ij}\geq 0\\ \eta a_{ij} & a_{ij}<0,\end{cases}$$

$i,j=1,\cdots,n$, $i\neq j$. Note that the same perturbation $\eta<1$ can be applied to all negative off-diagonal elements. □

Theorem 6: If the system (1.1) is stable for arbitrary time delays, then there exists a rearrangement of the entries of $x(t)$ such that the system matrix A has a block lower triangular form with nonstrictly quasi-diagonal-dominant diagonal blocks. That is,

$$A=\begin{bmatrix} A_{11} & 0 & \cdots & & 0\\ A_{21} & A_{22} & 0 & \cdots & 0\\ & & \ddots & & \\ A_{r1} & \cdots & & & A_{rr}\end{bmatrix}$$

where each A_{ii}, $i=1,\cdots,r$ is nonstrictly q.d.d., i.e., there exists $d_j^i>0$, $j=1,\cdots,n_i$ such that

$$d_j^i a_{jj}^i+\sum_{\substack{k=1\\k\neq j}}^{n_i} d_k^i|a_{kj}^i|\leq 0 \qquad j=1,\cdots,n_i \tag{3.1}$$

where $A_{ii}=(a_{kj}^i)$, $k,j=1,\cdots,n_i$.

Proof: As the matrix A given in Theorem 5 is q.d.d., so too is the matrix $\hat{A}$ obtained from A by replacing all off-diagonal elements a_{ij} by ηa_{ij} for all $0<\eta<1$. Therefore, there exists a sequence of positive numbers $\{\eta_m\}$ with $\eta_m<1$ and $\eta_m\to 1$ as $m\to\infty$ and a sequence of n vectors $\{d^m\}$ where $d_j^m>0, j=1,\cdots,n$ for all m such that for all m

$$d_j^m a_{jj}+\eta_m\sum_{k\neq j}|a_{jk}|d_k^m<0 \qquad j=1,\cdots,n. \tag{3.2}$$

Note that this is a row quasi-diagonal-dominance condition.

Each vector d^m can be scaled so that $\max_j(d_j^m)=1$ and then each sequence $\{d_j^m\colon m=1,\cdots\}$, $j=1,\cdots,n$ is bounded, so by the usual diagonal procedure, we can select a subsequence $m_1,m_2,\cdots$ such that $d_j^{m_i}$ converges for each j. For convenience, we shall use the same index m for the subsequence. Let J denote the subset of $1,\cdots,n$ for which $\lim_{m\to\infty}d_j^m=d_j\neq 0$. As $\Sigma_{j=1}^n d_j^m>1$ and $d_j^m>0$, $\Sigma_{j=1}^n d_j\geq 1$ and $d_j\geq 0$ for all j so J cannot be empty.

For $j\notin J$, i.e., $d_j=0$, it is clear from (3.2) that $a_{jk}=0$ for all $k\in J$. Hence, if we rearrange the rows and columns of A so that $J=\{p+1,\cdots,n\}$[corresponding to a reordering of the entries of $x(t)$], then A can be written as $\begin{bmatrix}\bar{A}_{11} & 0\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}$ where $\bar{A}_{11}$ is $p\times p$, $\bar{A}_{21}$ is $(n-p)\times p$, and $\bar{A}_{22}$ is $(n-p)\times(n-p)$. Since $\lim_{m\to\infty}d_j^m=d_j>0$ for $j\in J$, $\bar{A}_{22}$ is nonstrictly quasi-diagonal-dominant, as claimed. Further, the first p equations of the reordering of (1.1) are independent of the remaining $n-p$, i.e., they form an independent system with system matrix $\bar{A}_{11}$. This system must be stable for arbitrary time delays in off-diagonal elements, whence, by applying the above argument, $\bar{A}_{11}$ with a suitable rearrangement of rows and columns has the form $\begin{bmatrix}\hat{A}_{11} & 0\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}$ where $\hat{A}_{22}$ is $q\times q$ for $q\geq 1$ and is nonstrictly q.d.d. The proof is completed iteratively, since at each step the dimension of the independent system is reduced by at least one. □

Unlike the necessary condition of Theorem 2, the above condition is not sufficient for stability because of the possibility of nonstrict dominance, that is, of equality in (3.1). Moreover, it is not sufficient for boundedness unless A itself is nonstrictly q.d.d. In this case, taking

$$V(t)=\sum_{i=1}^n\left\{d_ix_i(t)+\sum_{j\neq i}d_i\int_{t-T_{ij}}^t|a_{ij}x_j(s)|\,ds\right\}$$

we find that $\dot{V}(t)\leq 0$ and then $d_i>0$ implies that $x_i(t)$ is bounded for all i (cf. [3]). If A only has the block lower triangular form, then the system with matrix A_{11} has bounded solutions which are inputs to the systems with matrices $A_{22},\cdots,A_{rr}$. The integrals of these bounded solutions may be unbounded.

There do exist systems which are stable for arbitrary time delays, but which are neither column nor row quasi-diagonally-dominant, even in the nonstrict sense. Consider the system with matrix

$$B=\begin{bmatrix}-1 & 0 & 0\\ 1 & -1 & -1\\ 1 & 1 & -1\end{bmatrix}.$$

The characteristic equation (1.3) reduces to

$$(\mu+1)([\mu+1]^2+e^{-\mu T})=0 \tag{3.3}$$

where $T=T_{23}+T_{32}$. The roots of (3.3) are $\mu=-1$ and the roots of

$$[\mu+1]^2+e^{-\mu T}=0. \tag{3.4}$$

Suppose $\mu=\alpha+i\beta,\alpha,\beta$ real, solves (3.4); then

$$(\alpha+1)^2-\beta^2+e^{-\alpha T}\cos\beta T=0$$
$$2\beta(\alpha+1)-e^{-\alpha T}\sin\beta T=0,$$

i.e.,

$$(\alpha+1)^4+2\beta^2(\alpha+1)^2+\beta^4=e^{-2\alpha T}.$$

So $\alpha\leq 0$ and if $\alpha=0$, $\beta=0$, contradicting (3.4). Thus, $\alpha<0$ and the system (1.1) with matrix B is stable for arbitrary time delays. Now B is not row quasi-diagonal-dominant, since for arbitrary positive d_1,d_1,d_3, either $d_1+d_2-d_3>0$ or $d_1+d_3-d_2>0$. However, the block $[-1]$ is trivially diagonal dominant, while the block $\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}$ is nonstrictly diagonal dominant (and is not strictly q.d.d.) so B is stated in Theorem 6. B is, however, nonstrictly column q.d.d. A system which is neither column nor row q.d.d. is the six-dimensional one with matrix $\begin{bmatrix}B & 0\\ 0 & B^T\end{bmatrix}$ which, of course, has the same (but repeated) characteristic roots as the B system, and so is stable for arbitrary time delays.

Since B is not strictly q.d.d., there must be arbitrarily small ϵ perturbations of B which yield systems unstable for some time delays. Indeed, if in B we replace the block $\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}$ by $\begin{bmatrix}-1 & -C\\ C & -1\end{bmatrix}$ where $C>1$, (3.4) becomes

$$[\mu+1]^2+C^2e^{-\mu T}=0 \tag{3.5}$$

which has nonzero roots $i(c^2-1)^{1/2}$ provided $\tan[(c^2-1)^{1/2}T]=2(c^2-1)^{1/2}/(c^2-2)$.

IV. Conclusions

It has been established that for perturbations of a linear autonomous system to be stable independently of time delays in off-diagonal interactions, it is both necessary and sufficient for the system matrix to be quasi-diagonal-dominant. This result may be of considerable interest to modelers of large-scale systems for which the systems coefficients are known subject to error and in which largely unknown time delays occur.

A weaker diagonal-dominance condition is necessary when the system is stable, but some of its perturbations are not.

Quasi-diagonal-dominance is also necessary and sufficient for the "connective stability" of a nondelayed linear autonomous system [5]. Here connective stability means that any system with matrix $B=(b_{ij})$ obtained from the original which has matrix $A=(a_{ij})$ is stable provided $b_{ii}=a_{ii}$ and $|b_{ij}|\leq|a_{ij}|$, $i,j=1,\cdots,n$, $i\neq j$. We have shown that stability with respect to these large perturbations is equivalent to stability with respect to small perturbations and time delays.

Another apparently related equivalence is given in [9] where nonlinear feedback systems are considered. For the single-input single-output case, it is shown that stability in the presence of arbitrary nonlinear feedback (possibly with memory) lying in the sector $[-r,r]$ is equivalent to stability in the presence of arbitrary linear memoryless feedback in the sector $[-r,r]$ cascaded with an arbitrary time delay.

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State Feedback Decoupling with Stability of Linear Constant (A,B,C,D) Quadruples

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***Abstract*—The aim of this paper is to deal with the general decoupling problem of linear constant (A,B,C,D) quadruples from two points of view. The first one is devoted to a characterization of the control laws that decouple quadruples by state feedback, which has been lacking up until now in spite of the results previously obtained by Morse in [4]. Here, we give a complete characterization of solutions by state feedback by means of a necessary and sufficient condition of existence. The second objective of the paper is to check for stability of the closed-loop decoupled system when state feedback is available. This is done in the most general case, i.e., with no particular assumption on the open-loop system as well as on the decoupling partition.**

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I. Introduction

Over the past few years, the theory of decoupling has been well developed for linear multivariable systems described by the classical (A,B,C) triple representation [2], [8]. This is no longer true when the system includes a direct feedthrough, in other words, when it is described by an (A,B,C,D) quadruple representation. For the latter, previous results have already been obtained by using either a direct algebraic approach [3] or a geometric one [4]. For the latter, an incorrect result relative to the existence of state feedback decoupling has been stated by Morse [4, Lemma 2]. The first objective of this paper is to explain not only why it is incorrect, but also to provide a necessary and sufficient condition concerning the existence of state feedback solutions for the decoupling problem. The second one is to check stability of the closed-loop decoupled system under the assumption of state feedback. This is done in the most general case, that is to say, by breaking away from any particular assumption such as, for instance, the number of open-loop inputs equals the number of output blocks to be independently controlled.

The mathematical framework is purely geometric.

The paper is organized as follows.

In Section II, we specify the main notations and symbols used throughout the text, and we recall basic definitions from linear algebra which are necessary for understanding the paper.

In Section III, we recall fundamental geometric concepts including controllability subspaces and output controllability subspaces.

In Section IV, the concept of output controllability subspace is used to formulate the restricted (static compensation) decoupling problem (RDP). The main result of this section is provided by Theorem 4.1 which gives a necessary and sufficient condition in order that a solution by state feedback may exist.

In Section V, we deal with the stability properties of the closed-loop decoupled system under the assumption of decoupling by state feedback. The main result is provided by Theorem 5.1.

II. Preliminaries

A. Notations and Background Algebra

Below, capital letters in italics $A,B,C,\cdots$ denote matrices; script letters $\mathcal{A},\mathcal{B},\mathcal{C},\cdots$ denote finite dimensional vector spaces. The same symbol is used to denote both a matrix and its map. All matrices, maps, and vector spaces are defined over the field of real numbers. Elements of a vector space are denoted by $x,y,z,\cdots$. The zero space is written as 0. The dimension of a space $\mathcal{X}$ is denoted by $d(\mathcal{X})$.

We recall basic definitions from linear algebra [7]. Two spaces are called independent if their intersection is zero. The sum of two spaces, denoted by $+$, is the space spanned by the union of their respective bases.

The sum is called a direct sum, denoted by $\oplus$, if the two spaces are independent. If two spaces $\mathcal{V}$ and $\mathcal{W}$ are isomorphic, we write $\mathcal{V}\approx\mathcal{W}$. If $\mathcal{V}\subset\mathcal{W}\subset\mathcal{X}$, any subspace $\hat{\mathcal{V}}\subset\mathcal{X}$ such that $\mathcal{W}=\mathcal{V}\oplus\hat{\mathcal{V}}$ is called a completion of $\mathcal{V}$. If $\mathcal{V}\subset\mathcal{X}$, $\mathcal{X}/\mathcal{V}$ denotes the factor space of $\mathcal{X}$ relative to $\mathcal{V}$.

The notation M: $\mathcal{X}\to\mathcal{U}$ means that M is a map from $\mathcal{X}$ into $\mathcal{U}$. If M: $\mathcal{X}\to\mathcal{U}$ and $\mathcal{V}\subset\mathcal{X}$, the restriction of M to $\mathcal{V}$ is written as $M|\mathcal{V}$. If M is a map, $\{M\}$ or $\mathfrak{M}$ denotes the image of M and Ker M denotes its null space. $M^{-1}\mathcal{V}$ denotes the inverse image of $\mathcal{V}$ under M.

If A: $\mathcal{X}\to\mathcal{X}$, $\mathcal{B}\subset\mathcal{X}$, and $d(\mathcal{X})=n$, $\{A|\mathcal{B}\}=\mathcal{B}+\cdots+A^{n-1}\mathcal{B}$. If $\mathcal{V}\subset\mathcal{X}$ and $A\mathcal{V}\subset\mathcal{V}$, then $\mathcal{V}$ is called an A-invariant subspace. If $A\mathcal{V}\subset\mathcal{V}+\mathcal{B}$, $\mathcal{V}$ is called an (A,B)-invariant subspace. $\mathcal{V}$ is called cyclic relative to A if $\mathcal{V}=\{A|\{v\}\}$ for some $v\in\mathcal{V}$ and v is called a generator of $\mathcal{V}$. We define the spectrum of A, written $\sigma(A)$, to be the set of its complex eigenvalues.

If k is a positive integer, $\boldsymbol{k}=\{1,2,\cdots,k\}$. A family of subspaces $\mathcal{R}_i\subset\mathcal{X}$, $i\in\boldsymbol{k}$ is denoted by $\{\mathcal{R}_i\}_{\boldsymbol{k}}$. For this family,

$$\check{\mathcal{R}}_i=\sum_{j\neq i}\mathcal{R}_j \qquad i,j\in\boldsymbol{k}$$