Insensitivity of a Class of Nonlinear Compartmental Systems to the Introduction of Arbitrary Time Delays

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Abstract—We examine a class of nonlinear compartmental systems and show that the introduction of arbitrary time delays, including nonconstant ones, in the action of one compartment on another, does not affect system properties such as stability, boundedness, and positivity. The results arise from a combination of techniques introduced separately for the studies of linear time-delay systems and nonlinear systems without delay and are directly testable. Some examples and estimates for the degree of stability are given.

I. INTRODUCTION

NONLINEAR compartmental systems are widely used as mathematical models of dynamical behavior found in chemical reactions, ecology, and human interactions (see [1]–[3]). It is well known that linear time-invariant systems whose interaction matrices have negative diagonal elements, positive off-diagonal elements and are (non strictly) column dominant, do not have any oscillatory solutions, that is, solutions are either unbounded or tend asymptotically to the equilibrium set [4]; further, nonnegative initial conditions and nonnegative inputs generate nonnegative solutions. These properties are carried over to the class of nonlinear systems, linearizations of which have the above structure; moreover, useful conditions guaranteeing the boundedness of nonnegative solutions are known [5], [15].

Comprehensive as these results may seem, they may well be of little practical use if the real system under consideration has a structure not well modeled by ordinary differential equations. An often observed feature in the real world is the existence of time delays in the interactions between compartments, due to finite processing times or geographical separation [6] and we therefore introduce these into the compartmental model. The analysis of the linear homogeneous (zero-input) case is carried out in [7] where it is shown that the stability property of column dominated systems persists under the introduction of arbitrary constant time delays in the off-diagonal terms. The significance of this for the modeler is that determining the stability of a real system depends only upon accurate identification of interaction coefficients and is completely insensitive to errors in the estimation of time delays, when these are constant. In fact, certain nonconstant delays can be tolerated [7], [16].

In this paper we show that these insensitivity results extend to the nonlinear compartmental systems discussed above. The material presented breaks down as follows. In Section II we give the system description and discuss positivity of and order relations among its trajectories. Section III gives a detailed account of stability for the linear, homogeneous, constant-delay case, being a slight extension of the results in [7]. In the subsequent sections these results are applied to nonlinear inhomogeneous systems. For the strictly column dominant case with a constant input the following chain of properties is established; a) all bounded trajectories converge exponentially to an equilibrium point; b) the existence of an equilibrium point implies that all trajectories are bounded; and c) for each constant input there exists a unique equilibrium point. This establishes the insensitivity of such systems to the introduction of time delays (the equilibrium points are delay independent).

When the inputs are time varying but bounded we use b) and c) above and the order relations among trajectories to obtain: a)′ any two trajectories corresponding to different initial conditions (with the same input of course) converge exponentially to one another, i.e., initial conditions are forgotten. Naturally, the limiting trajectories do depend upon the values of the time delays.

Non strictly column dominated systems have similar, weaker properties. Here we extend the results of [5] to the time-delay case. In particular, we note that equilibrium points need not exist nor need they be unique if they do exist.

The basic system structure introduced in Section II is quite simple and throughout the paper, section by section, we point out generalizations in structure which do not affect the results. Perhaps the most important of these, the time-varying delay case, is dealt with separately in Appendix III. As a result of this section by section treatment, we end with a very brief summary.

The stability of interconnected functional systems has a large literature (see [8]) but little is devoted to insensitivity to changes in delay. This may well be due to others' preoccupation with systems which are destabilized by delays and their attempts to characterize the onset of

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instability [9]. Moreover, the general usage of quadratic Lyapunov functions yields unwieldy stability criteria whereas here, by using differential inequalities and linear Lyapunov-like functions, very simple criteria are stated.

The occurrence of oscillations in time-delayed systems is discussed in [17] and [18].

II. System Description, Trajectory Nonnegativity, and Ordering

The systems of interest are described by nth-order inhomogeneous differential delay equations on the half line $[0,\infty)$.

$$\dot{x}_i(t) = -f_{io}(x_i(t)) + \sum_{\substack{j=1\\ j\neq i}}^{n} f_{ij}(x_j(t-T_{ij})) + u_i(t) \quad (2.1)$$

where $i=1,\cdots n$. The vectors $x(t)=[x_1(t),\cdots x_n(t)]'$ and $u(t)=[u_1(t),\cdots u_n(t)]'$ are called the instantaneous system state and input, respectively (the complete system state lies in an infinite dimensional space and is given by $x_t=\{x(s): t-T\leqslant s\leqslant t\}, T$ as below). The scalars $T_{ij}\geqslant 0$ are referred to as the time delays. We define $T=\max_{i,j}T_{i,j}<\infty$.

The solution of (2.1) depends upon the specification of an initial condition $x(s)=\phi(x)$, $s\in[-T,0]$. Here we shall assume that the given n vector function $\phi(\cdot)$ is continuous, even though it need only be measurable for (2.1) to be well defined. Note that for simplicity of statement, we take each component $x_i(\cdot)$ to be specified on the same (maximal) initial interval, though this too is not necessary. The input $u(\cdot)$ is a measurable n vector function on $[0,\infty]$, bounded on bounded intervals.

For our discussion to have meaning, we must assume the following: the functions f_{io}, f_{ij} are such that corresponding to any initial condition and input there is a unique solution of (2.1) on $[0,\infty)$. In this regard we shall assume throughout that these functions are locally Lipshitz continuous in $\mathbb{R}$. The solutions of (2.1) are called the trajectories of the system—systems with trajectories with finite escape time or with bounded trajectories failing to exist beyond a certain point are excluded (see [10], [15]).

Nonnegativity

A vector $y\in\mathbb{R}^n$ is said to be nonnegative [positive] if each of its components is nonnegative [positive], i.e., $y_i\geqslant 0$ [$y_i>0$] $i=1,\cdots,n$, and we write $y\geqslant 0$ [$y>0$]. (Note that $y\geqslant 0$ and $y\neq 0$ does not necessarily imply $y>0$.) The relation $\geqslant 0$ defines the usual partial ordering on $\mathbb{R}^n$. Nonnegative [positive] functions are those which are everywhere nonnegative [positive].

The results in this section, and some of the results in Section V, depend upon the following monotonicity hypothesis (M):

(M). The functions f_{io}, f_{ij} in (2.1) are all monotone nondecreasing locally Lipshitz continuous and satisfy $f_{io}(0)=f_{ij}(0)=0$, $i,j=1,\cdots n$.

For the remainder of this section we assume (M). The first implication is

Theorem 1: Trajectories $x(\cdot)$ corresponding to nonnegative initial conditions $\phi(\cdot)$ and nonnegative inputs $u(\cdot)$ are nonnegative.

Proof: Firstly suppose that both $\phi(0)>0$ and $u(\cdot)>0$ then $x(0)>0$ and there exists $\epsilon>0$ with $x(t)>0, t\in[0,\epsilon)$. If $x(\cdot)$ is not nonnegative, define

$$\bar{t} = \inf_{i,t}\{t: x_i(t)<0\}>0.$$

Then $x_i(\bar{t})=0$, where i is any index at which the infimum is achieved and

$$\dot{x}_i(\bar{t}) = \sum_{j\neq i}\left(x_j(\bar{t}-T_{ij})\right)+u_i(\bar{t})\geqslant u_i(\bar{t})>0 \quad (2.2)$$

so that $x_i(\bar{t}+\delta)>0$ for sufficiently small $\delta>0$, a contradiction.

If either $\phi(0)$ or $u(\cdot)$ are not positive we can select either one or two sequences $\{\phi^k\}$, $\{u^k\}$ with $\phi^k(0)>0$, $u^k(\cdot)>0$ and $\phi^k\to\phi$, $u^k\to u$ pointwise. (If one of $\phi(0)$ and $u(\cdot)$ is positive, one of the sequences is trivial.) If $u(\cdot)$ is bounded, we can select bounded $u^k(\cdot)$ and then, because of the local Lipshitz continuity of the function f_{io}, f_{ij}, the conditions of [10, th. 5.1] are met (the Lipshitz condition implies that the trajectories are unique, when they exist). We conclude that $x^k(\cdot)$, the trajectory corresponding to ϕ^k and u^k, converges uniformly to $x(\cdot)$ on every bounded interval. Hence as $x^k(\cdot)>0$, $x(\cdot)\geqslant 0$.

When $u(\cdot)$ is unbounded, put $u^m(t)=\min(u(t),m)$ then $u^m(t)=u(t)$ on $[0,t_m]$ for some increasing sequence t_m with $t_m\to\infty$ as $m\to\infty$. (Recall that u is bounded on bounded intervals). The above proof applies to u^m and x^m agrees with x on $[0,t_m]$. This completes the proof.

Note: Henceforth, as in (2.2), we shall write $\Sigma_{j=1,\,j\neq i}^{n}$ simply as $\Sigma_{j\neq i}$. Other sums are written out explicitly.

Trajectory Ordering

Theorem 2: If $x^1(\cdot)$ and $x^2(\cdot)$ are trajectories corresponding to ordered initial conditions and inputs, $\phi^1(\cdot)\geqslant\phi^2(\cdot)$ and $u^1(\cdot)\geqslant u^2(\cdot)$, then $x^1(\cdot)\geqslant x^2(\cdot)$.

The proof of Theorem 2 is similar to that of Theorem 1 and is omitted.

Remarks: (i) Theorem 1 depends only upon the properties $f_{io}(0)=0$ and $f_{ij}(\sigma)\geqslant 0$, for $\sigma\geqslant 0$. [to establish the inequality (2.2)]. In Theorem 2, the corresponding inequality

$$\begin{aligned}\dot{x}_i^1(\bar{t})-\dot{x}_i^2(\bar{t}) &= \sum_{j\neq i} f_{ij}\left(x_j^1(\bar{t}-T_{ij})\right)\\ &\quad -f_{ij}\left(x_j^2(\bar{t}-T_{ij})\right)+u_i^1(\bar{t})-u_i^2(\bar{t})\\ &\geqslant u_i^1(\bar{t})-u_i^2(\bar{t})>0\end{aligned}$$

depends upon the monotonicity of the functions f_{ij}.

(ii) In the absence of the local Lipshitz condition, Theorem 1 fails [11].

(iii) Theorem 2 will subsequently be used to demonstrate the boundedness of certain trajectories (Section V). Note that Theorem 2 does not depend upon the individual trajectories, etc., being nonnegative.

III. Stability of Linear Systems

Preparatory to deriving results for the nonlinear case, we consider a linear homogeneous version of (2.1)

$$\dot{x}_i(t) = -a_{io}(t)x_i(t) + \sum_{j\neq i} a_{ij}(t-T_{ij})x_j(t-T_{ij}) \quad (3.1)$$

$i=1,\cdots n$, with initial condition ϕ specified as before. Solutions to (3.1) exist on $[0,\infty)$ under mild assumptions on the coefficients $a_{io}(\cdot), a_{ij}(\cdot)$. It is necessary for the development in the following section that we admit time-varying coefficients, though the homogeneous nonlinear system obtained from (2.1) by putting $u(\cdot)\equiv 0$ is autonomous.

Let us assume that the coefficients $a_{io}(\cdot)$ are all nonnegative; then it can be shown that the solution of (3.1) corresponding to ϕ is dominated by that of

$$\dot{y}_i(t) = -a_{io}(t)y_i(t) + \sum_{j\neq i} |a_{ij}(t-T_{ij})| y_j(t-T_{ij})$$

$y_i(s) = |\phi_i(s)|$, $s\in[-T,0]$, $i=1,\cdots n$ (that is, $|x_i(t)| \leq y_i(t)$ $\forall t$) [7, lemma 2]. In studying stability we can, therefore, restrict attention to the nonnegative trajectories of systems (3.1) with nonnegative coefficients. The approach adopted here is a modification of that in [7] and is different from the usual Lyapunov method [8].

To demonstrate that solutions $x(t)$ of (3.1) tend asymptotically to zero, it is sufficient to prove that $\alpha(t)x(t)$ is bounded for some scalar function $\alpha(t)$ such that $\lim_{t\to\infty}\alpha(t)=\infty$. If $\alpha(\cdot)$ is differentiable and positive then $y(t)=\alpha(t)x(t)$ is positive and

$$\begin{aligned}\dot{y}_i(t) &= -(a_{io}(t) - \dot{\alpha}(t)/\alpha(t))y_i(t) \\ &\quad + \sum_{j\neq i}\left[\alpha(t)/\alpha(t-T_{ij})\right]a_{ij}(t-T_{ij})y_j(t-T_{ij}) \\ &= -b_{io}(t)y_i(t) + \sum_{j\neq i} b_{ij}(t-T_{ij})y_j(t-T_{ij}).\end{aligned}$$

Let

$$V(t) = \sum_{i=1}^{n}\left\{y_i(t) + \sum_{j\neq i}\int_{t-T_{ij}}^{t} b_{ij}(s)y_j(s)\,ds\right\}. \quad (3.2)$$

Each term in (3.2) is nonnegative, hence $V(t)\geq 0$ for all t. Now

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{n}\Big\{-b_{io}(t)y_i(t) + \sum_{j\neq i} b_{ij}(t-T_{ij})y_j(t-T_{ij}) \\ &\quad + \sum_{j\neq i}\left[b_{ij}(t)y_j(t) - b_{ij}(t-T_{ij})y_j(t-T_{ij})\right]\Big\} \\ &= \sum_{i=1}^{n}\Big\{-b_{io}(t)y_i(t) + \sum_{j\neq i} b_{ij}(t)y_j(t)\Big\} \\ &= \sum_{i=1}^{n}\Big\{-b_{io}(t) + \sum_{j\neq i} b_{ji}(t)\Big\}y_i(t).\end{aligned}$$

So if

$$b_{io}(t) - \sum_{j\neq i} b_{ji}(t) \geq 0, \qquad \text{for all } t, i=1,\cdots n \quad (3.3)$$

then $\dot{V}(t)\leq 0$, $V(t)$ is bounded (above by $V(0)$, below by 0) and $y(t)$ is bounded, as required. In terms of the original coefficients (3.3) becomes

$$a_{io}(t) - \dot{\alpha}(t)/\alpha(t) - \sum_{j\neq i}\left[\alpha(t+T_{ji})/\alpha(t)\right]a_{ji}(t) \geq 0, \qquad \text{for all } t, i=1,\cdots n. \quad (3.4)$$

This is similar to the column dominance property

$$a_{io}(t) - \sum_{j\neq i} a_{ji}(t) \geq \epsilon > 0, \qquad \text{for all } t, i=1,\cdots n \quad (3.5)$$

which is sufficient for asymptotic stability in the nondelayed case ($T_{ji}=0$) [12]. We show that (3.5) implies (3.4) for a wide range of systems, for suitable choices of $\alpha(\cdot)$, i.e., introduction of delays does not affect the stability of the system.

Bounded Coefficients

Let (3.5) hold and let all the coefficients be bounded on $[0,\infty)$. Define $c_1 = \sup_t \max_i a_{io}(t) < \infty$, $c_2 = \max_{i,j} T_{ij}$ and $c_3 = \inf_t \min_i [a_{io}(t) - \sum_{j\neq i} a_{ji}(t)] \geq \epsilon$.

If we seek $\alpha(t) = \exp(kt)$, for some $k>0$, (3.4) is implied by

$$g(k) = c_1 - k - (c_1 - c_3)\exp(kc_2) \geq 0. \quad (3.6)$$

Since $g(0)>0$ and g is continuous, there exists a positive k satisfying (3.6) and we conclude that the system (3.1) is exponentially stable, $\|x(t)\| \leq m\exp(-kt)$ for some constant m. A method for estimating k is given in Appendix 1.

Remarks: This case includes all constant coefficient systems and all the results on (3.1) in [7].

Unbounded Coefficients

As no general statement appears to cover this class, we indicate the nature of obtainable results by means of examples.

Example 1:

$$\begin{aligned}a_{io}(t) &= t+1, \\ a_{ij}(t) &= t, \\ T_{ij} &= 1, \qquad i,j=1,2,\ i\neq j.\end{aligned}$$

Equation (3.5) holds true and (3.4) is

$$t+1-\dot{\alpha}(t)/\alpha(t) - \left[\alpha(t+1)/\alpha(t)\right]t \geq 0.$$

This is not satisfied by $\alpha(t)=\exp(kt)$ for any $k>0$ since $t\cdot\exp(k) > t+1-k$ for large t. However $\alpha(t)=t+k$ yields

$$t+1-1/(t+k) - (t+k+1)(t)/(t+k) = (k-1)/(t+k)$$

which is nonnegative provided $k\geq 1$. The system trajectories, therefore, go asymptotically to zero at least as fast as $(t+k)^{-1}$. Also, it can be shown that they definitely are not exponentially stable (see Appendix 2). This corresponds to the accepted idea that the introduction of time delays reduces the degree of stability in a system.

Example 2:

$$\begin{aligned}a_{io}(t) &= t^2+1, \\ a_{ij}(t) &= t, \\ T_{ij} &= 1, \qquad i,j=1,2,\ i\neq j.\end{aligned}$$

Again (3.5) is true. Suppose we look for a monotone increasing function $\alpha(\cdot)$ satisfying (3.4) then

$$t^2+1-\left[\alpha(t+1)/\alpha(t)\right]t^2 \geqslant t^2+1-\dot{\alpha}(t)/\alpha(t) - \left[\alpha(t+1)/\alpha(t)\right]t^2 \geqslant 0$$

i.e., $\alpha(t+1) \leqslant \alpha(t)[1+1/t^2]$.

For integer $t=n$ we obtain by iteration

$$\alpha(n) \leqslant \alpha(1)\prod_{i=1}^{n-1}\left(1+\frac{1}{i^2}\right).$$

As the right-hand side converges, $\alpha(\cdot)$ is bounded and we do not have a positive test for stability. This illustrates a limitation in our approach since constructing a suitable non monotone α is clearly very difficult.

Remarks: (i) Though the trajectories of this system are bounded, we are not aware of any method for establishing whether they are stable or not.

(ii) The same applies to any system for which

$$\prod_{k=1}^{m}\left[a_{io}(k)/\sum_{j\neq i} a_{ji}(k)\right]$$

converges as $m\to\infty$ for at least one i, $1\leqslant i \leqslant n$.

Example 3: Let $a_{io}(t)$ be an arbitrary increasing unbounded function and $a_{ij}(t)\leqslant \epsilon a_{io}(t)$ for some $0\leqslant \epsilon<1$, $i,j=1,2$, $i\neq j$. Then if $0<k\leqslant(1/T)(-\log\epsilon)$, $\alpha(t)=e^{kt}$ satisfies (3.4) for large t.

It has been shown that for a wide class of linear time-delay systems (3.1), including all those with bounded coefficients, the column dominance condition (3.5),

$$a_{io}(t)-\sum_{j\neq i} a_{ji}(t)\geqslant\epsilon>0, \qquad \text{for all } t, i=1,\cdots n$$

is sufficient for stability. Such insensitivity to time delays is not true of all linear systems which are stable in nondelayed form, indeed in [9] Ladde introduced the notion of a maximal time delay for which stability is preserved. The converse question is answered in [19] where it is shown that a linear system which is stable for all off-diagonal delays, independently of small system parameter perturbations, is quasi-column dominant (see note (ii) below).

Notes: (i) The above results hold for systems with positive coefficients. For systems (3.1) with $a_{io}(\cdot)$ nonnegative but $a_{ij}(\cdot)$ of arbitrary sign, for $i\neq j$, the column dominance condition sufficient for their validity is $a_{io}(t)-\Sigma_{j\neq i}|a_{ji}(t)| \geqslant\epsilon>0$, for all t, $i=1,\cdots n$.

(ii) If for any system (3.1) there exists a positive vector $d\in\mathbb{R}^n$ such that

$$d_i a_{io}(t)-\sum_{j\neq i}^{n} d_j|a_{ji}(t)|\geqslant\epsilon>0$$

for all t, $i=1,\cdots n$, the system is said to be quasi-column dominant. Such systems have exactly the same stability properties as column dominant ones, as can be seen by replacing the function $V(\cdot)$ in (3.2) by

$$V_d(t)=\sum_{i=1}^{n} d_i\left\{y_{io}(t)+\sum_{j\neq i}\int_{t-T_{ij}}^{t} b_{ij}(s)y_j(s)\,ds\right\}.$$

When the a_{io} and a_{ij} are all positive and constant, the nondelayed system of the form (3.1) is stable if and only if it is quasi-column dominant [13].

(iii) The time delays need not be introduced exactly as in (3.1). The most general form for discrete delays is

$$\dot{x}_i(t)=-a_{io}(t)x_i(t)+\sum_{j=1}^{n}\sum_{k=1}^{m} a_{ijk}(t-T_{ijk})x_j(t-T_{ijk}) \tag{3.7}$$

for which dominance is expressed as

$$a_{io}(t)-\sum_{j=1}^{n}\sum_{k=1}^{m}|a_{jik}(t)|>0.$$

In particular, delayed versions of $x_i(t)$ can appear in the right-hand side of (3.7).

(iv) Distributed delays are dealt with in [7] and time varying delays are covered in Appendix 3.

IV. Asymptotic Behavior of Nonlinear Systems; Forgetting Initial Conditions

Recall that the system of interest is

$$\dot{x}_i(t)=-f_{io}(x_i(t))+\sum_{j\neq i} f_{ij}(x_j(t-T_{ij}))+u_i(t), \qquad i=1,\cdots n. \tag{2.1}$$

Throughout this section the functions $f_{io}(\cdot), f_{ij}(\cdot)$ will be continuously differentiable, hence, locally Lipshitz on the real line, and one of the following dominance conditions will hold, (note: (M) is not enforced here)

(D) $df_{io}(\sigma)/d\sigma-\sum_{j\neq i}|df_{ji}(\sigma)/d\sigma|\geqslant\epsilon>0,$ for all $\sigma\in\mathbb{R}$, $i=1,\cdots n$.

(D_r) The same as (D) but with $\epsilon=0$.

In applications, it may well be that

$$f_{io}=\sum_{i\neq j} f_{ji}+g_i.$$

(This exhibits f_{io} as a sum of flow terms to other components and the rest of the universe.) If the f_{ji} and g_i are monotone increasing, (D_r) holds. If $dg_i(\sigma)/d\sigma\geqslant\epsilon>0$, then (D) holds.

We firstly investigate the trajectories corresponding to constant inputs u.

Theorem 3: Subject to (D) any bounded trajectory of (2.1) corresponding to a constant input[1] converges asymptotically to an equilibrium point.

Proof: The derivative $v(t)=\dot{x}(t)$ of any solution to (2.1) satisfies the linear delay equation

$$\dot{v}_i(t)-a_{io}(x_i(t))v_i(t)+\sum_{j\neq i} a_{ij}(x_j(t-T_{ij}))v_j(t-T_{ij}) \tag{4.1}$$

where $a_{io}(\sigma)=df_{io}(\sigma)/d\sigma$ and $a_{ij}(\sigma)=df_{ij}(\sigma)/d\sigma$, $i=1,\cdots n$, $j\neq i$. As $x(\cdot)$ is bounded the coefficients in (4.1) are bounded and, because of (D), satisfy (3.5). Thus $v(t)=$

[1] We show in Theorems 6 and 9 that all trajectories corresponding to constant inputs are bounded.

$\dot{x}(t)$ goes to zero exponentially and the limit

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[x(0)+\int_0^t \dot{x}(s)\,ds\right]$$

exists and is finite. It is clear that this limit, $\bar{x}$ say, must belong to the equilibrium set $\{x: F(x)+u=0\}$, $F(x)$ being the n vector with ith entry

$$-f_{io}(x_i)+\sum_{j\neq i} f_{ij}(x_j). \qquad \square$$

The next theorem concerns bounded trajectories from different initial functions ϕ^1 and ϕ^2 but with the same time varying input $u(\cdot)$.

Theorem 4: Subject to (D), the difference between any two bounded trajectories of (2.1) corresponding to the same time varying input $u(\cdot)$ converges exponentially to zero.

Proof: Let the two trajectories concerned be $x(t)$ and $y(t)$ and set $z(t)=x(t)-y(t)$, so that

$$\dot{z}_i(t) = -\left[f_{io}(x_i(t))-f_{io}(y_i(t))\right] + \sum_{j\neq i}\left[f_{ij}(x_j(t-T_{ij}))-f_{ij}(y_j(t-T_{ij}))\right].$$

Define $a_{io}(\sigma)=df_{io}(\sigma)/d\sigma$ and $a_{ij}(\sigma)=df_{ij}(\sigma)/d\sigma$ as in the previous theorem and set

$$b_{io}(t)=\int_0^1 a_{io}(\beta x_i(t)+(1-\beta)y_i(t))\,d\beta$$

and

$$b_{ij}(t)=\int_0^1 a_{ij}(\beta x_j(t)+(1-\beta)y_j(t))\,d\beta$$

$i,j=1,\cdots n$, $i\neq j$. Then

$$\begin{aligned} b_{io}(t)(x_i(t)-y_i(t)) &= \int_0^1 d/d\beta\left[f_{io}(\beta x_i(t)+(1-\beta)y_i(t))\right]d\beta \\ &= f_{io}(x_i(t))-f_{io}(y_i(t)) \end{aligned}$$

and similarly

$$b_{ij}(t-T_{ij})(x_j(t-T_{ij})-y_j(t-T_{ij})) = f_{ij}(x_j(t-T_{ij}))-f_{ij}(y_j(t-T_{ij})).$$

The difference $z(t)$, therefore, satisfies the linear equation

$$\dot{z}_i(t) = -b_{io}(t)z_i(t)+\sum_{j\neq i} b_{ij}(t-T_{ij})z_j(t-T_{ij})$$

$i=1,\cdots n$. From (D) it is clear that

$$\begin{aligned} b_{io}(t)-\sum_{j\neq i}|b_{ji}(t)| &\geq \int_0^1\left(a_{io}-\sum_{j\neq i}|a_{ji}|\right)(\beta x_i(t)+(1-\beta)y_i(t))\,d\beta \\ &\geq \int_0^1 \epsilon\, d\beta = \epsilon > 0. \end{aligned}$$

As the trajectories are bounded, so are the above coefficients and we conclude from the results of Section III that $z(t)$ goes exponentially to zero. $\square$

Note: Theorem 4 strengthens Theorem 3 in that it shows the equilibrium point of the latter to be unique, i.e., independent of initial conditions.

These two theorems are direct extensions of those of Sandberg [12] to the time-delayed case. Condition (D) isolates a class of nonlinear systems whose stability properties are insensitive to the presence of time delays. In [15] Sandberg gives results for more general nondelayed systems including nondelayed versions of the systems discussed in note (ii) below. The quite complicated set of hypotheses used in [15] is implied by (D) for the problems considered here. Maeda *et al.* have studied the validity of the stability result for nondelayed systems when (D) is relaxed to (D_r), in [5] and [20]. For a constant input they prove that a bounded trajectory approaches a constant (steady state) as $t\to\infty$. We indicate here how this result carries over to the time-delayed case.

Theorem 5: Subject to (D_r) any bounded trajectory of (2.1) corresponding to a constant input approaches a steady state as $t\to\infty$.

Proof: Let the trajectory be $x(t)$ and define $v(t)=\dot{x}(t)$. The sets I_t and K_t are defined by $I_t=\{1\leq i\leq n: \dot{x}_i(t)>0, \text{ or } \dot{x}_i(t)=0,\ \ddot{x}_i(t)\geq 0\}$ $K_t=\{1,\cdots n\}/I_t$, and $V(t)$ is given by

$$V(t)=\sum_{i=1}^n\left[|v_i(t)|+\sum_{j\neq i}\int_{t-T_{ij}}^t |a_{ij}(x_j(s))v_j(s)|\,ds\right]$$

where the a_{ij} are as defined in the two previous theorems. Then $V(t)\geq 0$ and

$$\begin{aligned} \dot{V}(t) = &\sum_{i\in I_t}\Big[-a_{io}(x_i(t))v_i(t)+\sum_{j\neq i}a_{ij}(x_j(t-T_{ij}))v_j(t-T_{ij}) \\ &+\sum_{j\neq i}\{|a_{ij}(x_j(t))v_j(t)|-|a_{ij}(x_j(t-T_{ij}))v_j(t-T_{ij})|\}\Big] \\ &+\sum_{i\in K_t}\Big[a_{io}(x_i(t))v_i(t)-\sum_{j\neq i}a_{ij}(x_j(t-T_{ij}))v_j(t-T_{ij}) \\ &+\sum_{j\neq i}\{|a_{ij}(x_j(t))v_j(t)|-|a_{ij}(x_j(t-T_{ij}))v_j(t-T_{ij})|\}\Big] \\ \leq &\sum_{i\in I_t}\Big[-a_{io}(x_i(t))|v_i(t)|+\sum_{j\neq i}|a_{ij}(x_j(t))||v_j(t)|\Big] \\ &+\sum_{i\in K_t}\Big[-a_{io}(x_i(t))|v_i(t)|+\sum_{j\neq i}|a_{ij}(x_j(t))||v_j(t)|\Big] \\ = &\sum_{i=1}^n\Big[-a_{io}(x_i(t))+\sum_{j\neq i}|a_{ji}(x_i(t))|\Big]|v_i(t)| \\ \leq &\ 0. \end{aligned} \tag{4.2}$$

Thus V is bounded and has a limit as $t\to\infty$, V_∞ say, $V_\infty\geq 0$. Proof that $V_\infty=0$ proceeds along the lines given in [5, th. 2] with (4.2) above corresponding to [5, eq. (8)]. We omit the details as the procedure is quite lengthy. The proof is completed along the lines given in [20].

Notes: (i) In [5] attention is restricted to nonnegative trajectories of systems for which the functions f_{ij} are monotone increasing, hence all a_{ij} are nonnegative.

(ii) (D_r) implies that $a_{io}(\sigma)\geq 0$, for all $i=1,\cdots n$.

The methods applied in the above theorems can be used on a wider variety of systems than those specified by (2.1). A more general form for the right-hand side of (2.1) is

$$-f_{io}(x_i(t))+G_i(x_t), \qquad i=1,\cdots n \tag{4.3}$$

where $G_i(x_t)$ is a nonlinear functional on $x_t=\{x(s),\ t-T\leq s\leq t\}$. We consider the applicability of our methods to various forms of G_i.

a) $G_i(x_i)=\sum_{j=1}^{n}\sum_{k=1}^{m} f_{ijk}(x_j(t-T_{ijk})), \qquad i=1,\cdots n.$

Here the linear system for $v(t)=\dot{x}(t)$ in Theorem 3 has the same form as (3.7) with $a_{ijk}(t)=b_{ijk}(x_j(t))$ where $b_{ijk}(\sigma)=dt_{ijk}(\sigma)/d\sigma$. The dominance condition (D) is

$$b_{io}(\sigma)-\sum_{j=1}^{n}\sum_{k=1}^{m}|b_{jik}(\sigma)|>0, \qquad \text{for all } \sigma\in\mathbb{R},\ i=1,\cdots n.$$

b) $G_i(x_i)=\sum_{k=1}^{m} g_{ik}(x(t-T_{ik})), \qquad i=1,\cdots n$

where each g_{ij} is a continuously differentiable mapping of $\mathbb{R}^n$ into $\mathbb{R}$. The linear system for $\dot{x}(t)$ again has the form (3.7) but $a_{ijk}(t)=b_{ijk}(x(t))$ where $b_{ijk}(x)=\partial g_{ik}(x)/\partial x_j$ (and $T_{ijk}=T_{ik}$, for all $j=1,\cdots n$). Condition (D) now reads

$$b_{io}(x_i)-\sum_{j=1}^{n}\sum_{k=1}^{m}|b_{jik}(x)|>0$$

for $x=[x_1,\cdots,x_i,\cdots x_n]\in\mathbb{R}$, $i=1,\cdots n$. This is very restrictive, as it implies that each b_{jik} is bounded as a function of x_r, $r\neq i$, but it may be of some use for systems containing terms such as $x_1(t-T)x_2(t-T)$, especially if an *a priori* bound on $x(t)$ can be obtained (then the condition does not have to be checked over all of $\mathbb{R}^n$).

c) The most important class of systems to which our methods are inapplicable are those containing terms $x_1(t-T_1)x_2(t-T_2)$ say, where $T_1\neq T_2$, i.e., distinct delays appear in a multiplicative term. The difficulty with these lies in the terms

$$\int_{t-T}^{t} x_1(s)x_2(s)\,ds$$

entering our Lyapunov-like functions V. Whatever choice of T we take, we cannot cancel $x_1(t-T_1)x_2(t-T_2)$ with $-x_1(t-T)x_2(t-T)$ so $\dot{V}$ does not have the form $\dot{V}=$ function of present values of x only. Modifications will have to be made to V to deal with this problem.

d) The case of time-varying delays is considered in Appendix III.

V. Equilibrium Points and Boundedness of Trajectories

Theorems 3, 4, and 5 deal with the asymptotic behavior of bounded trajectories—in this section we shall establish their converses, dealing firstly with the constant input case. Denote the n vector with ith entry

$$-f_{io}(x_i)+\sum_{j\neq i} f_{ij}(x_j)$$

by $F(x)$; then for a constant input u, the equilibrium set of (2.1) is $\{x\colon F(x)+u=0\}$.

Theorem 6: Subject to (D_r), if for a fixed input u (2.1) has an equilibrium point x^* then all its trajectories corresponding to u are bounded.

Proof: Let $x(t)$ be any trajectory of (2.1) corresponding to u and set $z(t)=x(t)-x^*$, then for $i=1,\cdots n$,

$$\dot{z}_i(t)=-f_{io}(x_i(t))+f_{io}(x_i^*)+\sum_{j\neq i}\left[f_{ij}(x_j(t-T_{ij}))-f_{ij}(x_j^*)\right].$$

Define

$$V(t)=\sum_{i=1}^{n}\left[|z_i(t)|+\sum_{j\neq 1}\int_{t-T_{ij}}^{t}|f_{ij}(x_j(s))-f_{ij}(x^*)|\,ds\right].$$

Then $V(t)\geq 0$ and if we put $I_t=\{i\colon z_i(t)>0$ or $z_i(t)=0$ and $\dot{z}_i(t)\geq 0\}$ and $K_t=\{1,\cdots n\}/I_t$ (cf. Theorem 5), we obtain

$$\begin{aligned}
\dot{V}(t)=&\\
&\sum_{i\in I_t}\Big\{-f_{io}(x_i(t))+f_{io}(x_i^*)+\sum_{j\neq i}\left[f_{ij}(x_j(t-T_{ij}))-f_{ij}(x_j^*)\right]\\
&+\sum_{j\neq i}\left[|f_{ij}(x_j(t))-f_{ij}(x_j^*)|-|f_{ij}(x_j(t-T_{ij}))-f_{ij}(x_j^*)|\right]\Big\}\\
&+\sum_{i\in K_t}\Big\{f_{io}(x_i(t))-f_{io}(x_i^*)-\sum_{j\neq i}\left[f_{ij}(x_j(t-T_{ij}))-f_{ij}(x_j^*)\right]\\
&+\sum_{j\neq i}\left[|f_{ij}(x_j(t))|-f_{ij}(x_j^*)|-|f_{ij}(x_j(t-T_{ij}))-f_{ij}(x_j^*)|\right]\Big\}\\
&\leq\sum_{i\in I_t}\Big\{-|f_{io}(x_i(t))-f_{io}(x_i^*)|+\sum_{j\neq i}|f_{ij}(x_j(t))-f_{ij}(x_j^*)|\Big\}\\
&+\sum_{i\in K_t}\Big\{-|f_{io}(x_i(t))-f_{io}(x_i^*)|+\sum_{j\neq i}|f_{ij}(x_j(t))-f_{ij}(x_j^*)|\Big\}.
\end{aligned}$$

The first terms in the above sums follow from the previous line because of monotonicity of f_{io} (which is a consequence of (D_r)). So

$$\dot{V}(t)\leq\sum_{i=1}^{n}\Big\{-|f_{io}(x_i(t))-f_{io}(x_i^*)|+\sum_{j\neq i}|f_{ji}(x_i(t))-f_{ji}(x_i^*)|\Big\}.$$

Suppose $i\in I_t$; then

$$\begin{aligned}
|f_{ji}(x_i(t))-f_{ji}(x_i^*)|&=\left|\int_{x_i^*}^{x_i(t)} df_{ji}(\sigma)/d\sigma\,d\sigma\right|\\
&\leq\int_{x_i^*}^{x_i(t)}|df_{ji}(\sigma)/d\sigma|\,d\sigma
\end{aligned}$$

and because of monotonicity,

$$|f_{io}(x_i(t))-f_{io}(x_i)|=\int_{x_i^*}^{x_i(t)} df_{ji}(\sigma)/d\sigma\,d\sigma.$$

Therefore,

$$\begin{aligned}
&-|f_{io}(x_i(t))-f_{io}(x_i^*)|+\sum_{j\neq i}|f_{ji}(x_i(t))-f_{ji}(x_i^*)|\\
&\leq\int_{x_i^*}^{x_i(t)}-\left[df_{io}(\sigma)/d\sigma-\sum_{j\neq 1}|df_{ji}(\sigma)/d\sigma|\right]d\sigma\\
&\leq 0 \quad \text{by } (D_r).
\end{aligned}$$

The same is true when $i\in K_t$ (the integrations go from $x_i(t)$ to x_i^* in this case). Hence $V(t)\leq 0$, and $V(t)$, $z(t)$, and $x(t)$ are bounded. □

Notes: (i) This is more than just a time-delay version of [5, th. 3] since there is only nonnegative trajectories corresponding to nonnegative inputs to nonnegative systems ($f_{ij}(\sigma)\geq 0$, for $\sigma\geq 0$) are considered.

(ii) Theorems 3 and 6 together show the equivalence of asymptotic stability and equilibrium point existence for systems satisfying (D).

For time-varying inputs two distinct approaches are possible; either we attempt to generalize the method used above or we use the trajectory ordering properties given in Section II. The first yields

Theorem 7: Let $u(t)$ be an input such that there exists an absolutely continuous bounded function $x^*(t)$ satisfying $F(x^*(t))+u(t)=0$ for almost all t and $\int_0^\infty|\dot{x}^*(t)|dt<\infty$. Then subject to (D_r) any trajectory of (2.1) corresponding to $u(t)$ is bounded.

Proof: Using $x^*(t)$ in place of x^* in $z(t)$ and $V(t)$, follow the proof of Theorem 6. In the expression for $\dot{V}(t)$, in addition to the negative terms, there is

$$\sum_{i=1}^{n}|\dot{x}^*(t)|$$

i.e.,

$$\dot{V}(t)\leqslant\sum_{i=1}^{n}|\dot{x}_i^*(t)|=|\dot{x}^*(t)|.$$

The result follows directly from our hypothesis. □

The restrictions imposed on the inputs in Theorem 7 can be severe. For a linear system $u(t)=-Ax^*(t)$ everywhere implies that $u(\cdot)$ is absolutely continuous and of bounded variation on $[0,\infty)$, i.e., it converges to a limit as $t\to\infty$. A more satisfactory result obtains if we look at systems with monotone increasing rate functions f_{ij}.

Theorem 8: Suppose the system (2.1) satisfies (M) and (D_r) and that $u(t)$ is bounded and measurable, with upper and lower bounds v and w such that $\{x: Fx+w=0\}$ and $\{x: Fx+v=0\}$ are both nonempty. Then all trajectories corresponding to $u(t)$ are bounded.

Proof: Let ϕ be any initial condition for (2.1) and denote the trajectories corresponding to ϕ and inputs v, $u(t)$, and w by $x_v(t)$, $x(t)$, and $x_w(t)$, respectively. By Theorem 2, since the rate functions are monotone increasing, $x_v(t)\geqslant x(t)\geqslant x_w(t)$, while by Theorem 6, $x_v(t)$ and $x_w(t)$ are bounded. □

The above results can be extended to the variations of (2.1) discussed in Section IV.

VI. The Existence of Equilibrium Points

To complete our study of asymptotic properties it is evident from Theorems 6, 7, and 8 that we should give directly testable conditions for the existence of equilibrium points.

Theorem 9: If (D) holds then for every $u\in\mathbb{R}^n$ there is a unique solution x to $F(x)+u=0$.

Proof: As a consequence of (D), the matrix $F_x(x)$ is diagonally dominant and has an inverse $F_x^{-1}(x)$ for all $x\in\mathbb{R}^n$, i.e., F is a local C^1 diffeomorphism.

Further, for any x,

$$\begin{aligned}\|F(x)\|_1 &= \sum_{i=1}^{n}|F_i(x)|=\sum_{i=1}^{n}\Big|f_{io}(x_i)-\sum_{j\neq i}f_{ij}(x_j)\Big|\\ &\geqslant\sum_{i=1}^{n}\Big[|f_{io}(x_i)|-\sum_{j\neq i}|f_{ij}(x_j)|\Big]=\sum_{i=1}^{n}\Big[|f_{io}(x_i)|-\sum_{j\neq i}|f_{ji}(x_i)|\Big]\\ &=\sum_{i=1}^{n}\left\{\left|\int_0^{x_i}\frac{df_{io}}{d\sigma}d\sigma\right|-\sum_{j\neq i}\left|\int_0^{x_i}\frac{df_{ji}}{d\sigma}d\sigma\right|\right\}.\end{aligned}$$

If $x_i\geqslant 0$, the positivity of $df_{io}/d\sigma$ (from (D)) implies that the ith term is larger than or equal to

$$\int_0^{x_i}\frac{df_{io}}{d\sigma}d\sigma-\sum_{j\neq i}^{n}\int_0^{x_i}\left|\frac{df_{ji}}{d\sigma}\right|d\sigma\geqslant\int_0^{x_i}\epsilon\, d\sigma=\epsilon x_i=\epsilon|x_i|.$$

(ϵ as in (D)).

Similarly if $x_i<0$, the ith term is larger than or equal to

$$\int_{x_i}^{0}\frac{df_{io}}{d\sigma}d\sigma-\sum_{j\neq i}^{n}\int_{x_i}^{0}\left|\frac{df_{ji}}{d\sigma}\right|d\sigma\geqslant\int_{x_i}^{0}\epsilon\, d\sigma=-\epsilon x_i=\epsilon|x_i|.$$

Hence $\|F(x)\|_1\geqslant\epsilon\|x\|_1$ which implies that $\|F(x)\|_1\to\infty$ as $\|x\|_1\to\infty$ (F is a proper map, [14]). It follows that F is C^1 diffeomorphism onto $\mathbb{R}^n$, [14, corollary]. □

This result is an application of Palais global inverse theorem [14] widely used in nonlinear network theory. If instead of (D), the weaker condition (D_r) holds existence can still be demonstrated subject to F being a proper map, $\|F(x)\|_1\to\infty$ as $\|x\|_1\to\infty$ (see [5]).

VII. Summary

For completeness let us restate what has been proved. For constant inputs, all the trajectories of a nonlinear time-delay system (2.1) for which (D) holds converge exponentially to a fixed point, unique for each input. For the same system, provided the rate functions are monotone increasing, all the trajectories corresponding to a bounded measurable input are bounded and converge exponentially on one another. Weaker results hold when (D) is replaced by (D_r).

These developments have been given for autonomous systems (f_{io}, f_{ij} time-invariant) and constant but arbitrary time delays (for variable time delays see Appendix 3). The nonautonomous case presents the major remaining task.

Appendix 1
Estimates of Stability

Linear systems with bounded coefficients are at least exponentially stable, with degree k given by the solution of

$$g(k)=c_1-k-(c_1-c_3)\exp(kc_2)=0 \tag{A.1}$$

where $g(\cdot)$ is continuous and monotone decreasing. Now $g(0)=c_3>0$ and $g(c_3)<0$ so the solution to (A.1) lies in $(0,c_3)$. On this interval we have

$$\exp(kc_2)\leqslant mk+1, \qquad m=\left[\exp(c_3c_2)-1\right]/c_3$$
$$g(k)\geqslant c_1-k-(c_1-c_3)(mk+1)$$
$$=k(-1-m(c_1-c_3))+c_3.$$

So if

$$\bar{k}=c_3/(1+m(c_1-c_3)) \tag{A.2}$$

then the solution to (A.1) is greater than or equal to $\bar{k}$, which therefore provides a first estimate for the degree of stability. Improved estimates can be obtained by repeating the above linear approximation on $(\bar{k},c_3)$ and so on.

When no delays are present the solution to (A.1) is given by $k=c_3$, which for certain systems is the best possible estimate of exponential stability, i.e., it is the real part of an eigenvalue of the system. As a function of the maximum time delay, c_2, solutions to (A.1) are decreasing since $\partial g/\partial c_2<0$, for $k>0$. This confirms again the reduction in degree of stability accompanying the introduction of time delays.

Appendix 2

In example (1) of Section 2, the system is symmetric so if we choose initial conditions $x_1(s)=x_2(s)=\phi(s)$, $s\in[-1,0]$ then $x_1(t)=x_2(t)$, for all t and we can consider the equation

$$\dot{x}(t)=-(t+1)x(t)+(t-1)x(t-1).$$

Suppose a solution of this is exponentially stable, i.e., $|x(t)|\leq me^{-ct}$, for some $m,c>0$. Then $y(t)=e^{(c/2)t}x(t)$ is also exponentially stable and

$$\dot{y}(t)=-(t+1-c/2)y(t)+(t-1)\exp(c/2)y(t-1).$$

Put

$$V(t)=y(t)+e^{c/2}\int_{t-1}^{t}\tau y(\tau)\,d\tau\geq 0$$

(y remains nonnegative provided $y(1)\geq 0$—we can always arrange the trajectory to start at any $t\geq 0$.) Now

$$\dot{V}(t)=\left[t(\exp(c/2)-1)-1+c/2\right]y(t)>0$$

for $t\geq(1-c/2)(\exp(c/2)-1)=t_0$. Thus starting the system at t_0, with a nontrivial initial condition, for $t\geq t_0$ $V(t)\geq V(t_0)>0$. On the other hand, y is exponentially stable so directly from the definition, $V(t)\to 0$ as $t\to\infty$, a contradiction. The system is, therefore, not exponentially stable.

Appendix 3
Time-Varying Delays

For many real systems the time delays T_{ij} will not be constant but may vary as functions of time or even of state, x, or input, u. We shall here suppose that they are given functions of t, with the property $0\leq T_{ij}(t)\leq T$, for all i,j, $t\geq 0$. This ensures that the system equation

$$\dot{x}_i(t)=-f_{io}(x_i(t))+\sum_{j\neq i}f_{ij}(x_j(t-T_{ij}(t))),\qquad i=1,\cdots n \quad \text{(A.3)}$$

is well defined for initial conditions ϕ specified on a finite interval $[-T,0]$.

The linear system for $v(t)=\dot{x}(t)$ is

$$\dot{v}_i(t)=-a_{io}(t)v_i(t)+\sum_{j\neq i}\left(1-\dot{T}_{ij}(t)\right)\cdot a_{ij}(t-T_{ij}(t))v_j(t-T_{ij}(t)) \quad \text{(A.4)}$$

where $a_{io}(t)=df_{io}(\sigma)/d\sigma$ evaluated at $x_i(t)$ and similarly for $a_{ij}(t)$. Introducing $y(t)=\alpha(t)v(t)$ as in Section III, for some $\alpha(\cdot)>0$,

$$\dot{y}_i(t)=-b_{io}(t)y_i(t)+\sum_{j\neq i}\left(1-\dot{T}_{ij}(t)\right)\cdot b_{ij}(t-T_{ij}(t))y_j(t-T_{ij}(t))$$

$$b_{io}(t)=a_{io}(t)-\frac{\dot{\alpha}(t)}{\alpha(t)}$$

$$b_{ij}(t)=a_{ij}(t)\left[\alpha(t+T_{ij}(t))/\alpha(t)\right]. \quad \text{(A.5)}$$

Recall that solutions of (A.5) are dominated by nonnegative solutions of (A.5) with nonnegative coefficients; if $\dot{T}_{ij}(t)<1$ this involves replacing b_{ij} by $|b_{ij}|$, $i\neq j$ (the b_{io} are assumed nonnegative). For nonnegative coefficients and solutions, set

$$V(t)=\sum_{i=1}^{n}\left[y_i(t)+\sum_{j\neq i}\int_{t-T_{ij}(t)}^{t}b_{ij}(s)y_j(s)\,ds\right]\geq 0$$

then

$$\dot{V}(t)=\sum_{i=1}\left[-b_{io}(t)+\sum_{j\neq i}b_{ji}(t)\right]y_i(t)\leq 0$$

provided

$$b_{io}(t)-\sum_{j=1}^{n}b_{ji}(t)\geq 0,\qquad \text{for all } i,t. \quad \text{(A.6)}$$

If (A.4) is column dominant,

$$a_{io}(t)-\sum_{j\neq i}a_{ji}(t)\geq\epsilon>0,\qquad \text{for all } i,t$$

and all the functions $a_{io}(\cdot)$, $a_{ij}(\cdot)$, and $T_{ij}(\cdot)$ are bounded, there is a $k>0$ such that (A.6) holds with $\alpha(t)=\exp(kt)$ and (A.4) is exponentially stable. Hence, directly:

Theorem 10: If (A.3) satisfies (D) and the time-delays $T_{ij}(\cdot)$ are bounded and have derivatives bounded above by unity, then the bounded trajectories of (A.3) are asymptotically stable.

Conditions under which the other theorems carry over to the time-varying delay case are more restrictive: for example, to obtain Theorem 5 we appear to need $\dot{T}_{ij}\leq 0$, for all i,j. We shall not pursue this matter in the present paper, but refer to [16].

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