Analysis and Synthesis of Nonlinear Reciprocal Networks Containing Two Element Types and Transformers
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Abstract—This paper presents results for some analysis and synthesis problems of nonlinear reciprocal networks. In particular, for special systems such as lossless reciprocal networks and nonlinear RC reciprocal networks, synthesis techniques are introduced. The central feature of these techniques is the method of finding an appropriate stored energy function for the synthesizing network from prescribed network equations. Under certain assumptions, the stored energy function can be determined directly from the controllability and observability matrices of a system obtained by linearization around any state of the original system.

1. INTRODUCTION

This paper is concerned with the problem of synthesizing a nonlinear reciprocal network from given equations describing that network. The results obtained are restricted to lossless or RC networks which can contain constant turns-ratio transformers, and with the describing equations being of state-variable form, with entries of the state vector corresponding to inductor fluxes and/or capacitor voltages.

There are two ideas in the study which appear crucial, those of state function and the viewing of a network as an interconnection of a nondynamic or memoryless reciprocal system and a dynamic, but lossless reciprocal system.

In the study of nonlinear reciprocal networks, the concept of state functions has already proven very useful as an analysis tool [1]. The most familiar example (but not the only example) of a state function is the stored energy function which may be defined for any device exhibiting the attribute of conservation of energy, and a major part of the paper is devoted to proving that the synthesis of some special passive reciprocal networks may be reduced to the problem of determining an appropriate total stored energy function for a network synthesizing the prescribed network equations. As it turns out, for these networks—lossless reciprocal networks and nonlinear RC reciprocal networks—a unique stored energy function can be obtained.

As noted above, another important feature in the study of the synthesis of nonlinear reciprocal networks is to view the synthesizing network as a cascade interconnection of a memoryless reciprocal system and a dynamic lossless system [2]. A network synthesizing prescribed network equations may contain some or all of the following nonlinear reciprocal time-invariant network elements: inductive elements, capacitive elements, and nondynamic elements. Here nondynamic elements which are memoryless comprise ideal transformers and resistive elements in the usual sense.
The ideas of this paper also draw on two streams of prior work. There has of course been much work on passive, reciprocal linear network synthesis and in recent years, state-variable approaches have been developed, e.g., [3]-[5], presumably providing a better jump-off point for nonlinear theories. Such nonlinear theories have in part been developed for passive synthesis; for example, [6] introduces a nonlinear generalization of the state-variable characterization of operator passivity provided in the linear case by the Kalman-Yakubovic Lemma [7]-[8]. This idea is applied in [2] to nonlinear synthesis. Again, Willems has studied a number of questions relating to the stored energy of nonlinear passive systems [9].

The second stream is embodied more in such work as that of Chua and Lam [10], Brayton and Moser [11], and Varaiya and Verma [12]. This work in a sense is concerned with building up network properties from element properties, and in this may have more of an analysis than synthesis flavor. It has been highly successful in developing properties of single-element kind networks.

The structure of this paper is as follows. In Section II, we review the notions of reciprocity, passivity, and lossless for nonlinear network elements. Section III is concerned with developing synthesis procedures for lossless nonlinear reciprocal networks. Section IV deals with the synthesis of nonlinear resistor, capacitor, transformer (RCT) reciprocal complete networks, defined in more detail in that section. A systematic synthesis procedure is presented and finally illustrated with a simple example. Section V studies general RCT networks and Section VI, in addition to containing concluding remarks, mentions some open questions for future research.

II. NONLINEAR RECIPROCITY, PASSIVITY, AND LOSSLESSNESS

Most of this section is concerned with reviewing definitions and standard properties, and explaining the rationale for some standing assumptions.

A. Nonlinear Reciprocity

We shall regard the n-port networks considered in this paper as interconnections of resistors, capacitors, inductors, transformers, and independent voltage or current sources. In order to define the reciprocity concept for such networks [10]-[15] we shall first define the concepts for three classes of algebraic n-ports, viz. the resistive (or nondynamic) n-port, the inductive n-port, and the capacitive n-port. The general resistive n-port is usually characterized by a set of n algebraic relations between the currents i and the voltages U across every port. In the inductive n-port, the voltages U are replaced by the magnetic flux linkages A, with dA/df = o. In the capacitive n-port, the currents i are replaced by the electric charges q, with dq/df = i.

The various n-ports are not restricted to being interconnections of 2-terminal elements. Coupling either by ideal transformers or coupling internal to a multiport element is permitted. We require that such n-ports can be represented by a constitutive relation of the form

\[ R(y, x) = 0 \]  \hspace{1cm} (2.1)

between two n-vectors y, x with, in the case of a resistive n-port, \([x, y] = [i, v]\), in the case of an inductive n-port, \([x, y] = [A, i]\), and in the case of a capacitive n-port, \([x, y] = [q, v]\).

Now suppose that it is possible to parametrize the relations in terms of an m-vector s, which is not necessarily a vector of all currents, or all voltages, etc., so that the relation becomes

\[ y = y(s) \text{ and } x = x(s) \]  \hspace{1cm} (2.2)
where $x,y \in \mathbb{R}^n$ are $C^1$ functions, $s \in \mathbb{R}^m$ for $0 < m < n$, and the rank of the $2n \times m$ Jacobian matrix $\frac{\partial (x,y)}{\partial s}$ is equal to $m$. In this case the $n$-port is said to have dimension $m$ and will be denoted by $N(m)$.

The following reciprocity definition was first introduced in [10] as a generalization of the definition in [19].

**Algebraic n-Port Reciprocity Definition:** An algebraic $n$-port $N(m)$ represented by the parametric form (2.2) is reciprocal if and only if its associated reciprocity matrix

$$R(s) = \left( \frac{\partial x}{\partial s} \right)^T \left( \frac{\partial y}{\partial s} \right) \Delta J_x^T$$

is symmetric where the Jacobian matrix $J_x(s)$ for a vector $f(x)$ is defined as

$$J_x(s) = \left[ \frac{\partial f_1}{\partial s_1}, \ldots, \frac{\partial f_m}{\partial s_m} \right]$$

Closely related to the concept of reciprocity are the state functions and the potential functions which may be defined as follows.

A $C^1$-function $g: \mathbb{R}^n \to \mathbb{R}^m$ is said to be a state function if the Jacobian matrix $J_x(s)$ is symmetric for all $s \in \mathbb{R}^n$. As a consequence, the line integral of $g(s)$ between any two points is independent of the path of integration. In order to obtain a state function for a reciprocal algebraic n-port, the following result is available.

Let $N(n)$ be a reciprocal $n$-port represented by the parametric form (2.2) with $x$ a $C^2$-function on $\mathbb{R}^n$. Then the function

$$g(s) = \left[ \frac{\partial x(s)}{\partial s} \right] y(s)$$

is a state function on $\mathbb{R}^n$ [10].

The potential function $\psi(s)$ for this reciprocal $n$-port is then defined in terms of the state function $g(s)$ as

$$\psi(s) = \int_0^s g(s') ds'.$$

Hence the relationship between the potential function $\psi$ and the state function $g$ of a reciprocal $n$-port may be expressed as

$$g(s) \Rightarrow \left( \frac{\partial \psi}{\partial \phi} \right) \Delta \nabla \phi.$$

**Remarks**

1. A resistive reciprocal $n$-port may be expressed explicitly as

$$y = v(s) \quad \text{and} \quad x = i(s)$$

where $v$, $i$ are voltage and current vectors, and $s$ is a parameter vector. It is clear that a state function and a potential function can be obtained from (2.4) and (2.5), and are by no means unique, depending on the choice of $s$:

(a) If $s = [i_1,i_2]$, then $R(s) = J_x^T J_x$, and reciprocity is equivalent to symmetry of the incremental resistance matrix.

(b) If $s = [i_1,v_2]$, the associated reciprocity matrix is

$$R(s) = \begin{bmatrix} \frac{\partial (i_1,v_2)}{\partial (i_1,v_2)} & \frac{\partial (i_1,v_2)}{\partial (i_1,v_2)} \\ \frac{\partial (i_1,v_2)}{\partial (i_1,v_2)} & \frac{\partial (i_1,v_2)}{\partial (i_1,v_2)} \end{bmatrix} = \begin{bmatrix} \frac{\partial i_1}{\partial i_1} & \frac{\partial i_1}{\partial v_2} + \frac{\partial v_2}{\partial i_1} \\ 0 & \frac{\partial v_2}{\partial v_2} \end{bmatrix}.$$ 

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So reciprocity is equivalent to the following matrix (an incremental hybrid matrix with sign change) being symmetric (a result well known in the linear case):

\[
\begin{bmatrix}
\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\
\frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}
\end{bmatrix}
\]  

(2.9)

Resistive n-ports include ideal linear transformers. An ideal transformer can be defined by \( v_i = T_{pp} \), \( i_o = -T'_{pp} \) for some matrix \( T \), so that \( s = [i_v, o]^T \) is a suitable parameter. One finds \( R(s) \) in (2.8) to be zero. Therefore, a linear transformer n-port is reciprocal. Note that in general, a nonlinear transformer n-port is not reciprocal [15]. State and potential functions can also be calculated.

(2) An inductive reciprocal n-port is expressed in the parametric form

\[ y = i(s) \quad \text{and} \quad x = \lambda(s) \]

where \( i, \lambda \) are current and flux linkage vectors.

Usually, a convenient choice for \( s \) is

\[ s = x = \lambda(s) \]

and in this case, the associated reciprocity matrix, \( R(s) = J_{ij} J_{ij} = J_{ij} J^{'}_{ij} \) is symmetric. As a consequence, \( y = i(s) \) is a state function and the corresponding potential function \( W_m(s) \) is defined as

\[ i(s) = \nabla W_m(s), \quad W_m(0) = 0. \]  

(2.11)

(3) Similarly, a capacitive reciprocal n-port may be expressed as

\[ y = v(s) \quad \text{and} \quad x = q(s) \]

where \( v, q \) are voltage and charge vectors. A convenient choice for \( s \) is

\[ s = x = q(s) \]

and the associated reciprocity matrix \( R(s) = J_{ij} J_{ij} = J_{ij} J^{'}_{ij} \) is symmetric. Therefore, \( y = v(s) \) is a state function and the corresponding potential function \( W_m(s) \) is

\[ v(s) = \nabla W_m(s), \quad W_m(0) = 0. \]  

(2.13)

(4) A constant current source \( I \) and a constant voltage source \( V \), may be considered as a nonlinear inductive element and a nonlinear capacitive element respectively, with

\[ W_m = I'\lambda \quad \text{and} \quad W_m = V'q. \]

Consider an arbitrary n-port, comprising an interconnection of resistive, inductive and capacitive multiports. We shall term the n-port reciprocal when the constituent algebraic multiports are individually reciprocal.

If the constituent multiports are all of the one kind or of the one kind other than for ideal transformers, the resulting interconnection is also an algebraic multiport. Some analysis will show that, as an interconnection of reciprocal multiports [15], [10] it possesses the reciprocity property of (2.3). So the definition of reciprocity of an interconnected network in terms of its constituent subnetworks is consistent with the definition of reciprocity for algebraic multiports. In Section V, a particular case will be considered.

B. Nonlinear Passivity and Losslessness

Passivity and reciprocity are of course different concepts. For example, a constant voltage source is reciprocal but not passive, and a gyrator is passive but not reciprocal. Here we briefly define nonlinear passivity and losslessness and show that despite the difference of the properties, reciprocity plays a very important role in determining several fundamental aspects of passivity and losslessness.
Passivity and losslessness may be defined in terms of energy and power as follows.

An n-port \( N \) is said to be passive if, with \( N \) unexcited at \( t=0 \),
\[
\psi(t) - \int_0^t \langle v(r), i(r) \rangle \, dr, \quad \forall t > 0
\]
for all continuous voltage–current pairs \((v, i)\).

An n-port \( N \) is said to be lossless if it is passive and if the average power
\[
P_{av}[v(t), i(t)] = \lim_{T \to \infty} \frac{1}{T} \int_0^T \langle v(t), i(t) \rangle \, dt = 0
\]
for all bounded continuous voltage–current pairs of \( N \).

The lemma below relates reciprocity, passivity, and losslessness. For these lemmas, we require the notion of a diffeomorphism; we say that \( y = f(x) \) is a \( C^1 \)-diffeomorphism \( \mathbb{R}^n \) onto itself if [16]:
1) \( f(x) \) is continuously differentiable.
2) \( |f(x)| \neq 0 \) for all \( x \in \mathbb{R}^n \).
3) \( \|f(x)\| \to \infty \) as \( \|x\| \to \infty \).

Note that these properties imply that \( f^{-1} \) exists and is also continuously differentiable. In particular, when \( n = 1 \), the function \( f \) must have nonzero slope everywhere.

**Lemma 1:** Let \( L(C) \) be a reciprocal inductive (capacitive) n-port represented by \( i = f(h) \) \((v = j_q)\) where \( i, h \) are the current and flux linkage vectors, and \( f(\cdot) \) is a \( C^1 \)-diffeomorphism with \( f(0) = 0 \). Then \( L(C) \) is passive if and only if \( W_m > 0 \), \( \forall q \in \mathbb{R}^n \) and \( \forall q \in \mathbb{R}^n \) where \( W_m(q) \) is the potential function associated with the state function \( j_q \).

**Proof:** It suffices to consider the inductor case.

**Sufficiency:** As \( L \) is reciprocal, from (2.11) there exists a potential function \( W_m(\lambda) \) so that
\[
f(\lambda) = \nabla W_m(\lambda)
\]
with \( W_m(0) = 0 \). If
\[
\nabla^2 W_m(\lambda) = J(\lambda) > 0, \quad \forall \lambda \in \mathbb{R}^n
\]
then \( f(\lambda) \) is nondecreasing in the sense that \( \langle f(\lambda) - f(\lambda_0), \lambda - \lambda_0 \rangle > 0 \) for all \( \lambda, \lambda_0 \). Hence \( L \) is passive.

**Necessity:** If \( L \) is passive, then \( W_m(\lambda) = 0, \forall \lambda \in \mathbb{R}^n \). From (2.11) as \( W_m(0) = 0 \), the potential function \( W_m(\lambda) \) has its minimum point at \( \lambda = 0 \), so \( \nabla W_m(0) = 0 \). Furthermore, as \( i = f(h) = \nabla W_m(h) \) is a \( C^1 \)-diffeomorphism, \( J(h) = \nabla^2 W_m(h) \) is continuous, symmetric, and invertible.

Now for arbitrary but fixed \( \lambda \) and \( d \in \mathbb{R}^n \) and variable \( \alpha \in \mathbb{R} \), we have
\[
W_m(\lambda + \alpha d) = W_m(\lambda) + \alpha \nabla W_m(\lambda)^T d + \frac{1}{2} \alpha^2 \nabla^2 W_m(\lambda) d + o(\alpha^2).
\]
Taking \( \alpha = 0 \), we obtain
\[
W_m(0,0) = \frac{1}{2} \alpha^2 \nabla^2 W_m(0) d + o(\alpha^2).
\]
Now \( \nabla^2 W_m(0) \) is nonsingular and symmetric and so has all real nonzero eigenvalues. If \( \nabla^2 W_m(0) \) is not positive definite, there exists \( d \) such that \( d^T \nabla^2 W_m(0) d < 0 \), and then by taking \( \alpha \) small enough, we contradict the passivity. So \( \nabla^2 W_m(0) \) is positive definite. Since \( \nabla^2 W_m(\cdot) \) is continuous, symmetric, and nonsingular the eigenvalues of \( \nabla^2 W_m(\cdot) \) for fixed \( \lambda \in \mathbb{R}^n \) and \( \alpha \in [0, 1] \) vary continuously with \( \alpha \), are real for all \( \alpha \), zero for no \( \alpha \), and are positive for \( \alpha = 0 \). Hence they are positive for \( \alpha = 1 \), i.e., \( \nabla^2 W_m(\lambda) > 0 \).
**Lemma 2.** Let \( R \) be a reciprocal resistive \( n \)-port represented by \( i=g(v) \) where \( i, v \) are the current and voltage vectors, and \( g(·) \) is a \( C^1 \)-diffeomorphism with \( g(0)=0 \). Then \( R \) is passive if and only if \( J_f(v)=\nabla^2 W_f(v)>0, \forall v \in \mathbb{R}^n \) where \( W_f(v) \) is the potential function associated with the state vector \( g(v) \).

**Proof.** Sufficiency: As \( R \) is reciprocal, \( J_f(v) \) is symmetric. Therefore, \( i=\nabla W_f(v) \) with \( i(0)=0 \) where \( W_f(v) \) is the potential function. If \( \nabla^2 W_f(v)>0, \forall v \in \mathbb{R}^n \), then \( i=g(v) \) is nondecreasing on \( \mathbb{R}^n \). Hence \( R \) is passive [10].

Necessity: If \( R \) is passive, then \( \langle g(v), g(v) \rangle>0, \forall v \in \mathbb{R}^n \) [10]. Define \( \alpha = \frac{d}{d} \) for fixed \( \alpha \in \mathbb{R}^n \) and variable \( \alpha \in \mathbb{R}^n \). Then a Taylor series analysis yields

\[
\langle d\alpha, g(\alpha) \rangle = \alpha^T d^2 \left[ \nabla^2 W_f(0) \right] d + o(\alpha^2)
\]

and the proof then follows as for the preceding lemma, mutatis mutandis.

**Lemma 3.** Let \( L(C) \) be a reciprocal inductive (capacitive) \( n \)-port represented by \( i=f(A) \) \( (v=j(q)) \) where \( i, A(A, q) \) are the current and flux linkage (voltage and charge) vectors, and \( f(·) \) is a \( C^1 \)-diffeomorphism with \( f(0)=0 \). In addition \( L(C) \) is passive. Then \( L(C) \) is lossless.

**Proof.** The proof is obvious from (10).

**Remarks**

(1) We can summarize a number of the preceding ideas for algebraic \( n \)-ports as follows. If the \( n \)-port \( N \) can be described by a \( C^1 \)-diffeomorphism

\[
\nu=f(x)
\]

then \( N \) is reciprocal if and only if the Jacobian matrix \( J_f(x) \) is symmetric. In addition, the network \( N \) is also passive if and only if \( J_f(x) \) is positive definite, and is lossless if it is an inductive or capacitive \( n \)-port.

(2) A linear transformer \( n \)-port is also passive and lossless, as is well known, and easy to check using the definition.

(3) Note that by our definition, a lossless element must be passive. By Lemma 3, with \( C^1 \)-diffeomorphisms between appropriate port vectors, passive reciprocal inductive \( n \)-port and capacitive \( n \)-port are lossless. Henceforth in this paper, we shall regard a general lossless reciprocal network as an interconnection of passive inductive and capacitive \( n \)-ports, described by \( C^1 \)-diffeomorphisms and linear transformers.

(4) In a dynamic network which contains only reciprocal capacitors and inductors, the total stored energy function is defined as \( \phi(A, q)=W_f(A)+W_q(q) \). As an immediate consequence of Lemma 1, with \( C^1 \)-diffeomorphisms between the port vectors of the constituent algebraic multiports, the network is passive if and only if \( \nabla^2 \phi(x)>0, \forall x \in \mathbb{R}^{n \times n}, \) where

\[
\nabla^2 \phi(x)=\begin{bmatrix}
\frac{\partial^2 W_f}{\partial A^2} & 0 \\
0 & \frac{\partial^2 W_q}{\partial q^2}
\end{bmatrix}
\]

**III. LOSSLESS NONLINEAR RECIPROCAL NETWORKS**

In this section, we study lossless nonlinear reciprocal networks which are interconnections of ideal linear transformers, and inductor and capacitor multiports representable using \( C^1 \)-diffeomorphisms as in Lemma 1. We shall use the fact that the input–output mapping of ideal linear transformers has a very simple form in order to show how one may reduce the synthesis problem for lossless nonlinear reciprocal networks to the problem of determining an appropriate potential function.
A. Analysis Results

Consider a general lossless reciprocal network which is composed of a reciprocal inductive multiport \( L \), a passive reciprocal capacitive multiport \( C \), a linear transformer multipport \( T \), and voltage sources \( u \). The schemes depicted in Fig. 1, in which \( T_c \) denotes the linear transformer multipport \( T \) together with all the interconnections.

Suppose that \( L \) is represented by the \( C^1 \)-diffeomorphism \( i_L = f_i(\lambda) \), with \( i_L, \lambda \in \mathbb{R}^m \) the current and flux linkage vectors, and \( C \) by the \( C^1 \)-diffeomorphism \( q_C = f_c(\varphi) \) where \( q_C, \varphi \in \mathbb{R}^n \) are the voltage and charge vectors. Let the corresponding potential functions be \( W_m(\lambda) \) and \( W_c(\varphi) \).

Suppose also that it is possible to specify independently \( i_L, q_C, \) and \( u \), i.e., interconnections and \( T \) do not force a linear relation among the entries of \( i_L, q_C, \) and \( u \). Further, suppose that specification of \( i_L, q_C, \) and \( u \) at any instant of time uniquely determines all other variables in the network at that time. As a consequence, there exists a constant matrix \( T \) such that

\[
\begin{bmatrix}
\dot{i}_L \\
q_C
\end{bmatrix} = T
\begin{bmatrix}
\dot{\lambda} \\
u
\end{bmatrix}
= T \textbf{v}_T. 
\tag{3.1}
\]

The detailed argument leading to (3.1) is developed, for example, [4, see sec. 4.3], for the case when \( T_c \) contains resistors, but the lossless case is easy to recover.

Partition the matrix \( T \) as

\[
T = \begin{bmatrix}
T_1 & T_2 \\
T_2 & 0
\end{bmatrix}
\tag{3.2}
\]

Equations (3.1) then can be rearranged as

\[
\begin{bmatrix}
\dot{i}_L \\
q_C
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
-T_1 & 0
\end{bmatrix}
\begin{bmatrix}
i_L \\
q_C
\end{bmatrix} + \begin{bmatrix}
T_1 & T_2 \\
0 & 0
\end{bmatrix} u
\tag{3.3}
\]

Choosing the state vector \( x \) of the network state equations to contain inductor flux linkages \( i_L \) and capacitor charges \( q_C \), we obtain with total stored energy function \( \Phi(x) = W_m(\lambda) + W_c(\varphi) \) the relations

\[
\begin{bmatrix}
i_L \\
q_C
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial \lambda} W_m(\lambda) \\
\frac{\partial}{\partial \varphi} W_c(\varphi)
\end{bmatrix} = \nabla \Phi
\]

and

\[
\dot{x} = \begin{bmatrix}
d\lambda/dt \\
dq_C/dt
\end{bmatrix} = \begin{bmatrix}
q_C \\
i_L
\end{bmatrix}.
\tag{3.4}
\]

Therefore, the state-space network equations for the

\[
\dot{x} = \begin{bmatrix}
0 & T_1 \\
-T_1 & 0
\end{bmatrix} \nabla \Phi + \begin{bmatrix}
T_1 & T_2 \\
0 & 0
\end{bmatrix} u \quad y = \begin{bmatrix}
T_1 & 0
\end{bmatrix} \nabla \Phi.
\tag{3.5}
\]

B. Synthesis Results

When the state vector of a lossless reciprocal \( p \)-port network is known to comprise flux linkage and capacitor charges, it is clear from analysis results that its state-space network equations should have the following form:

\[
\begin{aligned}
\dot{x} &= f(x) + Gu, \\
y &= h(x)
\end{aligned}
\tag{3.6}
\]

where \( f(\cdot) : R^{m+p} \rightarrow R^{m+p}, \ G \in R^{m+p \times n}, \ h(\cdot) : R^{m+p} \rightarrow R^n, \ h(0) = 0, \) \( T_c \) denoting the linear transformer multipport \( T \) together with all the interconnections.

The synthesis problem is to determine whether equations (3.6) can be obtained by constructing an appropriate lossless reciprocal network. In other words, one is asked to search for a network consisting of a passive reciprocal capacitive \( n \)-port, a passive reciprocal inductive \( m \)-port, a linear transformer multipport and voltage sources so that its state-space equations coincide with (3.6). It is clear that if we can deduce from (3.6) an appropriate total stored energy function \( \phi(x) \) and a constant matrix \( T \) defining the network \( T_c \), then the synthesis problem is almost solved as one can construct the corresponding reciprocal network. Since any linear transformer multipport is lossless, it remains to prove that the capacitive and inductive \( n \)-port
Theorem 1: A necessary and sufficient condition for equations (3.6) to be realized by a lossless reciprocal network of the type described at the start of the section, is that there exist a constant matrix \( T = \begin{bmatrix} T_1 & 0 \\ T_2 & T_1 \\ 0 & 0 \end{bmatrix} \) and a \( C^2 \)-function \( \phi(.) \), with \( \nabla^2 \phi(x) > 0 \) for all \( x \), \( \phi(0) = 0 \) and \( \phi(0) = 0 \) such that

\[
G = \begin{bmatrix} T_2 \\ 0 \end{bmatrix}, 
\begin{bmatrix} T_2 & 0 \end{bmatrix} \right) \nabla \Phi \left( f(x) = \begin{bmatrix} 0 & T_1 \\ -T_1 & 0 \end{bmatrix} \phi \right)
\]

(3.7)

Proof: Sufficiency: Suppose that \( T = \begin{bmatrix} T_1 & 0 \\ T_2 & T_1 \\ 0 & 0 \end{bmatrix} \) and \( \phi \) exist such that equations (3.7) are satisfied. Then \( T \) gives adequate information for the linear transformer \( n \)-port and the interconnection structure of the network. In addition, the reciprocal capacitive and inductive \( n \)-ports may be obtained directly from \( f(x) \) where

\[
\begin{bmatrix} J_0(\lambda) \\ J_0(\phi) \end{bmatrix} = \begin{bmatrix} I_c \\ Q \end{bmatrix} = \nabla \phi.
\]

As \( \nabla^2 \phi(x) > 0 \) for all \( x \), \( f_1(\lambda) \) and \( f_2(\lambda) \) are \( C^1 \)-diffeomorphisms with \( J_0(0) = 0 \), \( f_2(0) = 0 \) since

\[
\begin{bmatrix} 0 & T_1 \\ -T_1 & 0 \end{bmatrix} \phi(0) = f(0) = 0.
\]

Hence the synthesizing network is lossless.

Necessity: Conversely, if (3.6) are realized by a lossless reciprocal network, its network equations have the form as described by (3.5). Consequently, there exist \( T \) and \( \phi(x) \) so that (3.7) are satisfied. In addition, as the system in lossless, it is also passive and consequently, \( \nabla^2 \phi(x) > 0 \) for all \( x \), while also \( \phi(0) = 0 \), \( \phi(0) = 0 \).

It is clear that from (3.7) that \( T_2 \) can be found directly from \( G \), and \( T_1 \) as shown later is readily obtained if \( \phi(x) \) is known. There are a number of ways that can be used to deduce the total stored energy function \( \phi(x) \) from the given functions \( G, f, \) and \( h \). However, the most interesting practical method which is very similar to one synthesis technique widely used for linear networks [4] relies on the assumption that the system derived from (3.6) by linearization around any nominal state is completely controllable [19]. The linearized system' has the form:

\[
\Delta x = J_2 \Delta x + G \Delta u
\]

(3.8)

and this system is completely controllable if the controllability matrix

\[
V_c = \begin{bmatrix} G & JG & \cdots & J^{m+n-1}G \end{bmatrix}
\]

has full rank. The original system is then said to possess incremental controllability if \( V_c \) has full rank for all linearizations.

Lemma 1: Assume that the system described by (3.6) is incrementally controllable. Then the total stored energy function \( \phi(x) \) may be uniquely defined from

\[
\nabla^2 \phi = \left( V_c V_c^T \right)^{-1}
\]

(3.9)

where

\[
\phi(0) = 0, \quad \phi(0) = 0
\]

\[
V_c = \begin{bmatrix} J_0 & -J_2 J_0 & \cdots & (-J_2)^{m+n-1} J_0 \end{bmatrix}
\]

(3.10)

Proof: Equations (3.7) imply

\[
J_2 \phi = G \nabla^2 \phi f = \begin{bmatrix} 0 & T_1 \\ -T_1 & 0 \end{bmatrix} \phi
\]

(3.10)

and the second equation implies

\[
J_2 \nabla^2 \phi = -\nabla^2 \phi f.
\]

(3.11)
Combining Equations (3.10) and (3.11), a solution for $V^2\phi$ can be obtained easily:

\[
V^2\phi G = J_0
\]
\[
V^2\phi J_0 G = -J_0 V^2\phi G
\]
\[
\dots
\]
\[
V^2\phi J^{**-1} G = (-J_0)^{**-1} V^2\phi G = (-J_0)^{**-1} J_0
\]

Consequently, we have

\[
V^2\phi V' = \bar{P}_0
\] (3.12)

Since the system is incrementally controllable, $V'$ has full rank, and there exists an inverse $V' V$. Thus $V^2\phi$ is uniquely defined by (3.9). Furthermore since $\phi(0) = 0$ and $V\phi(0) = 0$, the total stored energy function $\phi(x)$ is uniquely defined by integration.

In brief, the synthesis problem may be solved as follows. Consider an incremental controllable system with the following state-space equations

\[
x = f(x) + Gu
\]
\[
e = h(x)
\]

where $f(0) = 0$, $h(0) = 0$. The problem is to determine $T$ and $\phi(x)$ so that (3.7) are satisfied.

By Lemma 1, the total stored energy function for the system can be uniquely determined from

\[
V^2\phi(x) = (V_x V')(V_x V')^{-1}
\]

and $\phi(0) = 0$, $\bar{V}\phi(0) = 0$. If $V'$ is square, it is clear that

\[
\bar{V}^2(\phi) = \bar{V}_0 V^{-1}
\]
The constant matrix $T$ can be easily recovered using

\[
\begin{bmatrix}
0 \\
-T_1 \\
0
\end{bmatrix} = G
\]

and

\[
\begin{bmatrix}
0 \\
T_2 \\
0
\end{bmatrix} = (J)(\bar{V}^2\phi)^{-1}.
\] (3.13)

Hence it only remains to check that $\bar{V}^2\phi(x) > 0 \forall x \in \mathbb{R}^{m*}$ in order to ensure that the synthesizing network is passive.

There are in fact two other ways by which $\phi(x)$ can theoretically be found. Recognizing that $\phi(x)$ is the energy stored by the network when the vector of inductor currents and capacitor voltages is $x$, it is clear that

\[
\phi(x) = \int_0^T \nu'(t) u(t) dt
\] (3.14)

where $x(0) = 0$ (i.e., the network stores no energy at the initial time), and $u(\cdot)$ is a voltage excitation causing the state at time $T$ to be $x$. Of course, for a given $x$, it is not obvious what $u(\cdot)$ will achieve that $x$ at time $T$, starting with $x(0) = 0$. Put another way, the calculation of $\phi$ is reduced to a nonlinear controllability problem.

Second, one can recognize that if $x$ is the initial state vector in the network, and if the network is resistively loaded under zero input conditions, then the resistive load will absorb the network energy, and as $t \to \infty$, the total power dissipated by the resistor should equal the initially stored energy. A resistive load of $1 \Omega$ at each port corresponds to $u = -v$, and so

\[
\phi(x) = \int_0^\infty h^2[x(t)] dt
\] (3.15)

with

\[
x = f(x) - Gh(x), \quad x(0) = x_0
\] (3.16)

To avoid the possibility of some of the inductors and capacitors not being connected to the ports, and thus not being discharged by the resistor, some form of nonlinear observability assumption is needed.
The system equations to be synthesized is

\[
\dot{x} = \begin{bmatrix} 3 \left( x_1 + \frac{x_2^2}{3} \right) \\ -3 \left( 2x_1 + \frac{x_2^2}{3} \right) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = 2x_1 + \frac{x_2^2}{3}.
\]

One easily obtains

\[
J_x = \begin{bmatrix} 0 & 3(1 + x_2^2) \\ -3(2 + x_2^2) & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 2 + x_2^2 & 0 \\ 0 & 1 + x_2^2 \end{bmatrix}
\]

so that the incremental controllability matrix \( V_x \) is

\[
V_x = [G, J_x, J_y] = \begin{bmatrix} 1 & 0 \\ 0 & -3(2 + x_2^2) \end{bmatrix}
\]

Since \( V_x \) has full rank for all \( x \) the system is incrementally controllable. Using Lemma 1, we obtain

\[
\bar{V}_x = [J_x - J_y J_y^T] = \begin{bmatrix} 2 + x_2^2 & 0 \\ 0 & -3(2 + x_2^2)(1 + x_2^2) \end{bmatrix}
\]

\[
V^2 \theta = \bar{V}_x V_x^{-1} = \begin{bmatrix} 2 + x_2^2 & 0 \\ 0 & 1 + x_2^2 \end{bmatrix}
\]

As \( \phi(0) = 0 \) and \( V \phi(0) = 0 \), the total stored energy function is determined uniquely:

\[
\phi(x) = x_1^2 + \frac{x_2^4}{12} + \frac{x_2^4}{2} + \frac{x_3^4}{12}
\]

with

\[
\begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \bar{V} \theta(x) = \begin{bmatrix} 2x_1 + \frac{x_2^2}{3} \\ x_2 + \frac{x_2^2}{3} \end{bmatrix}
\]

The constant matrix is easily recovered as follows:

\[
\begin{bmatrix} T_3 \\ 0 \end{bmatrix} = G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

and

\[
\begin{bmatrix} 0 & T_1^T \\ -T_1 & 0 \end{bmatrix} = J_y (V \theta)^{-1} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}
\]

Hence

\[
T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\]

In addition, since \( V \phi(x) > 0 \) for all \( x \in \mathbb{R}^3 \), the given equations can be synthesized by a lossless reciprocal network which is described in Fig. 2.
A. Analysis Results

Consider a general passive reciprocal RC complete network which is composed of passive reciprocal capacitors, passive reciprocal resistors, linear transformers, and current sources, as shown in Fig. 2. The capacitors are represented by \( v_i(f_i(q)) \) where \( v_i, q \in \mathbb{R}^n \) are the voltages and charges of all capacitors, the resistors are represented by \( iR = f_i(p_i) \) where \( i, p_i \in \mathbb{R}^n \) are the currents and voltages of all resistors, and \( f_i(.), f_i(.) \) are \( C_1 \)-diffeomorphisms with \( f_i(0) = 0, f_i(0) = 0 \).

The completeness implies that for some matrices \( F_i \) and \( F_i \) determined by \( T_i \), only:
\[
\frac{d}{dt} = F_i \Phi(x) + F_i u \quad \text{and} \quad y = F_i \Psi(x),
\]

where \( \Phi(x) \) is the total stored energy function.

The state-space equations for a general RC complete network then become:
\[
x = f(x) + G u, \quad y = h(x)
\]

where \( f(.) : R^n \rightarrow R^n, h(.) : R^n \rightarrow R, u, y \in R^n, G \in R^{n \times n} \).

The synthesis problem is to determine a complete RC network with current sources so that its state-space equations are the same as (4.7); obviously, one needs to recover the corresponding total stored energy function \( \Phi(x) \) and the potential function \( W_i(x) \).

**Theorem 2**: A necessary and sufficient condition for (4.7) to be realized by a passive reciprocal RC complete network (of the type described above) and driven by current sources is that there exist a constant matrix \( F = [F_1, F_2] \) and \( C_1 \)-functions \( \Phi(x) \rightarrow R, W_i(x) \rightarrow R \) with \( \Phi(0) = 0, \quad W_i(0) = 0 \), such that:

\[
\frac{d}{dt} = F_1 \Phi(x) + F_2 u, \quad y = F_1 \Psi(x), \quad h(x) = F_1 \Phi(x), \quad f(x) = -F_1 \Phi(x)
\]

where \( \Phi(x) \) is the total stored energy function.

**Proof**: Sufficiently: Suppose that \( F = [F_1, F_2] \), \( \Phi(x) \), and \( W_i(x) \) exist with the stated properties, and are such that Equations (4.8) are satisfied. Then \( F \) gives adequate information for the interconnection structure of the network, and the capacitors and resistors are sufficiently represented by their potential functions \( \Phi(x) \) and \( W_i(x) \). In addition, with \( C_1 \)-diffeomorphisms between appropriate vectors, and \( \nabla \Phi(x) > 0, \quad \nabla W_i(x) / \nabla^2 > 0 \), the network is passive.

Necesarily: Conversely, if (4.8) are realized by a passive reciprocal RC complete network, its network equations have the form as described (4.7). Consequently, there exist \( F, \Phi(x) \), and \( W_i(x) \) such that (4.8) are satisfied. The remaining conditions on \( \Phi(x) \) and \( W_i(x) \) are fulfilled through the passivity, reciprocity and \( C_1 \)-diffeomorphism assumption.
Similarity to the results obtained for lossless reciprocal networks, the total stored energy function \( \phi(x) \) can be obtained directly from the incremental controllability and observability matrices of the given system. It can also be shown that \( F \) and \( W_F(v_0) \) may be recovered easily from \( \phi(x) \). These results rely on the assumption that the system is incrementally controllable.

**Lemma 2:** With the notation as above, assume that the system described by (4.7) is incrementally controllable. Then the total stored energy function \( \psi \) for the synthesizing network may be uniquely defined from
\[
\psi = (V^2_x V_y) (V^2_x V^2_y)^{-1}.
\]

where
\[
\phi(0) = 0, \quad \psi(0) = 0 \quad \phi(0) = 0.
\]

\[
V_x = \begin{bmatrix} J & I \end{bmatrix} \quad (J_j)^{n-1} J_k
\]
\[
V_y = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad J_j G
\]

**Proof:** Equations (4.8) imply
\[
\psi = \mu^T \psi \phi(x)
\]

or
\[
\psi = G^T \psi \phi(x).
\]

In addition, with \( \sigma_k = F^T \psi \phi(x) \),
\[
\frac{\partial \sigma_k}{\partial x} = -F_j \frac{\partial \sigma_k}{\partial v} \frac{\partial \sigma_k}{\partial x} = -F_j \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial v} F_j \psi.
\]

Hence
\[
\psi = \mu^T \psi \phi(x).
\]

Combining (4.10) and (4.12), the unique solution for \( \phi(x) \) may be obtained from
\[
(V^2 \phi) V_x = V_0
\]

or from
\[
V^2 \phi = (V^2_x V_y) (V^2_x V^2_y)^{-1}
\]

with \( \phi(0) = 0 \) and \( \psi(0) = 0 \).

Therefore, the procedure to solve the synthesis problem for a passive reciprocal RC complete network may be briefly summarized as follows:

Consider an incrementally controllable system with the following state-space equations
\[
\dot{x} = f(x) + Gu, \quad y = h(x)
\]

where \( f(0) = 0, h(0) = 0 \). The problem is to calculate \( F = [F_{1g} F_{2g}] \) and the potential functions \( (x) \) and \( W_F(v_k) \). By Lemma 2, the total stored energy function can be uniquely determined from
\[
\psi = (V^2_x V_y) (V^2_x V^2_y)^{-1}
\]

and
\[
\psi(0) = 0, \quad \psi(0) = 0.
\]

The matrix \( F \), the capacitor voltage \( v_c \), and the potential function \( W_F(v_k) \) can be easily recovered from (4.8) and (4.11). First we take
\[
F_1 = G, \quad F_2 = F - I.
\]

Now (4.11) requires
\[
\frac{\partial W_F(v_k)}{\partial v_k} = -J_j (V^2 \phi)^{-1}
\]

while as we also know, \( \sigma_k = F_j \psi \phi(x) = \psi(0) \) for the choice \( F_1 = I \). Because of the \( C^1 \)-diffeomorphism property, \( x = \phi(0) \) for some \( v_k \), allowing the right-hand side of (4.16), which is a symmetric matrix function of \( x \), to be rewritten as a function of \( v_k \):
\[
\frac{\partial W_F(v_k)}{\partial v_k} = \psi(0).
\]
Then \( W_x(x) \) can be obtained by integration.

Note that other choice of \( F_1 \) are possible and lead to different \( W_x(x) \). These choices correspond to the possibility of varying the transformer with consequent variation of the resistive network to leave the input/state/output behaviour invariant.

C. Example

We synthesise the following equations:

\[
\dot{x} = \left[ -2\left(x_1 + \frac{x_1^3}{3} + 2x_2 + \frac{x_2^3}{3}\right) \\
-2\left(x_1 + \frac{x_1^3}{3}\right) - 3\left(2x_2 + \frac{x_2^3}{3}\right) \right] + \left[ 1 \right]^T
\]

The incremental controllability matrix \( V_c \) and the incremental observability matrix \( V_o \) are:

\[
V_c = \begin{bmatrix} J_c & J_o \end{bmatrix} = \begin{bmatrix} 1 & -2(1+x_1^2) \\
0 & -2(1+x_1) \end{bmatrix}
\]

\[
V_o = \begin{bmatrix} J_o & J_o \end{bmatrix} = \begin{bmatrix} 1 + x_1 \ -2(1 + x_1)^2 \\
0 \ -2(1 + x_1)^2 \end{bmatrix}
\]

Since \( V_c \) has full rank for all \( x \), the system is incrementally controllable. Using Lemma 2, we obtain

\[
V_o = V_c^{-1} \begin{bmatrix} 1 + x_1^2 \\
0 \end{bmatrix}
\]

As \( \phi(0)=0 \) and \( \nabla \phi(0)=0 \), the total stored energy function is

\[
\phi(x) = \frac{x_1^3}{3} + \frac{x_1^4}{12} + x_2 + \frac{x_2^3}{3}
\]

with

\[
\nabla \phi(x) = \begin{bmatrix} x_1 + \frac{x_1^3}{3} \\
2x_2 + \frac{x_2^3}{3} \end{bmatrix}
\]

The matrix \( F \) and the potential function \( W_x(x) \) are received as follows:

\[
F = \begin{bmatrix} 1 \\
0 \end{bmatrix}
\]

\[
F_1^T \frac{\delta^2 W_x}{\delta u^2} \frac{\delta W_x}{\delta u} = -f(\nabla^2 \phi)^{-1}
\]

\[
= \begin{bmatrix} 2 & 2 \\
2 & 3 \left(1 + \left(2x_1 + \frac{x_1^3}{3}\right)^2\right) \end{bmatrix}
\]

Although we could take \( F_1 = f \), some simplification is achieved by making \( \delta^2 W_x/\delta u^2 \) diagonal. As a consequence, we obtain

\[
F_1 = \begin{bmatrix} 1 \\
0 \end{bmatrix}
\]

and then

\[
\nabla \phi(x) = \begin{bmatrix} x_1 + \frac{x_1^3}{3} + 2x_2 + \frac{x_2^3}{3} \\
2x_2 + \frac{x_2^3}{2} \end{bmatrix}
\]
and
\[ \frac{\partial^2 W_v}{\partial v_i^2} = F \begin{bmatrix} 2 & 2 \\ 2 & 3(1+v_i) \end{bmatrix} (P^D)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1+3v_i^2 \end{bmatrix}. \]

Hence
\[ \begin{bmatrix} i_{n_1} \\ i_{n_2} \end{bmatrix} = \frac{\partial W_v}{\partial v_i} = v_i + \frac{v_i^3}{3}. \]

In addition, since \( v^2 \) and \( \frac{\partial^2 W_v}{\partial v_i^2} \) are positive definite, the (4.19) can be synthesized by the passive reciprocal RC complete network with
\[ F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \]
depicted in Fig. 4.

We shall assume that \( R \) is a reciprocal, passive, resistive multipart defined by a \( C^1 \) diffeomorphism \( i_i = g(v_i) \), and \( T \) is a network of constant ideal transformers. Our goal is to show that the interconnection has a certain hybrid description, reflecting the reciprocity and passivity properties. Exhibiting existence of the hybrid description is a task in itself.

Being lossless, \( T \) has a hybrid matrix description; suppose the excitations are as depicted in Fig. 6. We shall show that the interconnections has a hybrid description with excitation variables \( e_1, e_2 \).

For some constant \( \theta_{0j} \), we have
\[ \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} T_{31} & T_{32} \\ T_{33} & T_{34} \end{bmatrix} \begin{bmatrix} 0 \\ e_4 \end{bmatrix}, \]
\[ \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{13} & T_{14} \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix}. \]

The properties assumed for \( R \) guarantee, see [20], that there exists a hybrid description for \( R \) of the form:
\[ \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} h_2(-i_2, e_2) \\ h_3(-i_2, e_2) \end{bmatrix}, \]
\[ \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_4(-i_2, e_2) \\ h_4(-i_2, e_2) \end{bmatrix}. \]
Let us define
\[
H = \begin{bmatrix}
\frac{\partial h_1(x,y)}{\partial x} & \frac{\partial h_1(x,y)}{\partial y} \\
\frac{\partial h_2(x,y)}{\partial x} & \frac{\partial h_2(x,y)}{\partial y}
\end{bmatrix}
\tag{5.3}
\]
Reciprocity implies symmetry of \(\partial h_1/\partial x\) and \(\partial h_2/\partial y\) and also that
\[
\frac{\partial h_2}{\partial x} = -\left(\frac{\partial h_1}{\partial y}\right) \tag{5.4}
\]
Passivity implies that
\[
\frac{\partial h_3}{\partial x} > 0, \quad \frac{\partial h_4}{\partial y} > 0 \tag{5.5}
\]
for all \(x\) and \(y\). Also, see [20],
\[
\lim_{|x|\to\infty} \left||h_3(x,y)|| = \infty \right. \quad \lim_{|y|\to\infty} \left||h_4(x,y)|| = \infty. \tag{5.6}
\]
Now define
\[
k(x,y) = \begin{bmatrix}
h_3(x,y) \\
h_4(x,y)
\end{bmatrix} = \begin{bmatrix}
h_3(x,y) \\
h_4(x,y)
\end{bmatrix} + \begin{bmatrix}
0 & -T_{34} \\
T_{34} & 0
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}. \tag{5.7}
\]
Then (5.4), (5.5), and (5.6) hold with \(h_3, h_4\) replaced by \(k_3, k_4\). Now for each \(x, y\), the Jacobian of \([k_3, k_4] \) has the form
\[
\begin{bmatrix}
K_{33} & K_{34} \\
-K_{34} & K_{44}
\end{bmatrix}
\]
with \(K_{33} = K_{33} > 0, K_{34} = K_{34} > 0\). Its determinant, viz.,
\[
|K_{33}| = K_{33} + K_{34}K_{44}
\]
is accordingly nonsingular for all \(x, y\) and thus the mapping \((x, y) \to (k_3, k_4)\) is globally invertible.

Now observe that (5.2) and the transformer equations force
\[
\begin{bmatrix}
\frac{\partial h_1}{\partial x} \\
\frac{\partial h_2}{\partial y}
\end{bmatrix}
+ \left[\begin{array}{cc}
0 & -T_{34} \\
T_{34} & 0
\end{array}\right]
\left[\begin{array}{c}
x \\
y
\end{array}\right]
= \left[\begin{array}{c}
0 \\
0
\end{array}\right].
\]
so we have
\[
\begin{bmatrix}
-i_3 \\
v_4
\end{bmatrix}
= \left(\begin{bmatrix}
T_{31} & 0 \\
0 & T_{34}
\end{bmatrix} \right) \begin{bmatrix}
v_1 \\
l_2
\end{bmatrix}
\]
and on using the transformer equations again, we have
\[
\begin{bmatrix}
i_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix}
C & -T_{21} \\
T_{21} & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
l_2
\end{bmatrix}
+ \begin{bmatrix}
T_{31} & 0 \\
0 & T_{34}
\end{bmatrix} \begin{bmatrix}
0 \\
v_1
\end{bmatrix}.
\]
Accordingly, we have exhibited a hybrid description for the interconnection of \(T\) and \(R\), the existence of which stems from the fact that any \(T\) must have a hybrid description, and from an assumption that \(R\) is passive and reciprocal and defined by a \(C^1\)-diffeomorphism.

It is not hard to check explicitly that this new description satisfies reciprocity and passivity constraints. With \(h_1, h_2\) defined by
\[
\begin{bmatrix}
i_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix}
h_1(v_1, l_2) \\
h_2(v_1, l_2)
\end{bmatrix} \tag{5.9}
\]
we have
\[
\begin{bmatrix}
\frac{\partial h_1}{\partial v_1} & \frac{\partial h_1}{\partial l_2} \\
\frac{\partial h_2}{\partial v_1} & \frac{\partial h_2}{\partial l_2}
\end{bmatrix}
= \begin{bmatrix}
0 & -T_{34} \\
T_{34} & 0
\end{bmatrix}
+ \begin{bmatrix}
T_{31} & 0 \\
0 & T_{34}
\end{bmatrix}
= \left(\begin{bmatrix}
0 & -T_{34} \\
T_{34} & 0
\end{bmatrix} \right)^{-1}
\begin{bmatrix}
T_{31} & 0 \\
0 & T_{34}
\end{bmatrix}. \tag{5.10}
\]
and this formula can be used to check the reciprocity and passivity.

Given the left side of (5.10) and the fact that the left side is expressible in the form shown on the right, the fact that

\[
H + \begin{bmatrix} 0 & -T_m \\ T_m & 0 \end{bmatrix}
\]

is nonsingular allows one to perform a decomposition of the type shown on the right of (5.10). Many such decompositions will exist, but will differ only trivially.

B. Resistor-Capacitor-Transformer Networks—Analysis

Consider a passive reciprocal network comprising a passive reciprocal capacitive n-port \( C \), a passive reciprocal resistive \( p \)-port \( R \), and a transformer/interconnection \( (n + n + p) \)-port \( T \), as shown in Fig. 7.

Throughout this section, we shall assume that the networks \( T \) and \( R \) have the properties described in the last section, while the assumptions on \( C \) guarantee, for example, the existence of a strictly convex stored energy function, zero together with its gradient at the origin.

Suppose that those ports of \( T \) which are connected to \( C \) can be taken as voltage excited, and the ports connected to the generators \( u \) can be taken as current excited in providing a hybrid matrix description of \( T \). Let \( \tilde{R} \) denote the interconnection of \( T \) and \( R \), see Fig. 8. Then as the last subsection showed, there is a hybrid description for \( \tilde{R} \) of the form:

\[
\begin{align*}
\dot{x} &= h_1(q, u) \\
y &= h_2(q, u).
\end{align*}
\]

(5.11)
The reciprocity and passivity of \( \tilde{R} \) impose certain restrictions of \( h_1, h_2 \). These restrictions, given in effect by (5.10) and (5.3) through (5.6), imply the existence of a potential function

\[
W_c(q, u) = \int_0^u \left[ h_1(q, u - h_2(q, u)) \right] \, du.
\]

(5.12)

With \( \phi \) the stored energy function of the capacitor and \( x = q \), we have

\[
q_t = \nabla \phi(x)
\]

(5.13)
and then (5.11) can be written

\[
\begin{align*}
\dot{x} &= -\left( \frac{\partial W_c(q, u)}{\partial q} \right)_{q = \nabla \phi(x)} \\
y &= -\left( \frac{\partial W_c(q, u)}{\partial u} \right)_{q = \nabla \phi(x)}.
\end{align*}
\]

(5.14)

C. Synthesis Results

The analysis results show that if the state vector is known to comprise capacitor charges, the state-space network equations for a general passive reciprocal RC network have the following form:

\[
\begin{align*}
x &= f(x, u) \\
y &= h(x, u)
\end{align*}
\]

(5.15)
where \( f: \mathbb{R}^{n+u} \to \mathbb{R}^x \) and \( h: \mathbb{R}^{n+u} \to \mathbb{R}^y \) satisfy certain side conditions.

The synthesis problem can be posed in two ways. First, one can search for a network consisting of a lossless reciprocal multipar and a passive reciprocal nondynamic multipar, driven by current sources. Thus one seeks separate descriptions of \( C \) and \( R \) of Fig. 8. Second, one can seek descriptions of \( C, T, \) and \( R \). The first problem is solved by recovering \( \phi(x) \) and \( W_c(q, u) \), the second by recovering \( \phi(x) \) and obtaining a description of \( \tilde{R} \) in the decomposed form of (5.8), or equivalently (5.10).
Theorem 3: A necessary and sufficient condition for (5.15) to be realized by a passive reciprocal resistor-capacitor-transformer network is that there exist $C^2$-functions $\phi(x)$ and $W_f(x,u)$ such that

$$
\begin{align*}
\phi(x,u) &= -\left(3 \frac{\partial W_f(x,u)}{\partial u}\right)|_{u=\phi(x)} \\
\phi(x,u) &= -\left(3 \frac{\partial W_f(x,u)}{\partial u}\right)|_{u=\phi(x)}
\end{align*}
(5.16)
$$

with $\phi(0) = 0$, $\nabla\phi(0) = 0$, $\nabla^2 \phi(x) > 0$, $W_f(0,0) = 0$, $\partial W_f/\partial u(0,0) = 0$, and

$$
\begin{bmatrix}
\frac{\partial^2 W_f}{\partial x^2} & \frac{\partial^2 W_f}{\partial x \partial u} \\
\frac{\partial^2 W_f}{\partial x \partial u} & \frac{\partial^2 W_f}{\partial u^2}
\end{bmatrix}
$$

expressible in the form of the right side of (5.10), where $H$, defined in (5.3), satisfies (5.4) through (5.6).

Proof: Sufficiency: Suppose that $\phi(x)$ and $W_f(x,u)$ exist with the listed properties such that (5.16) are satisfied. It is obvious that a passive reciprocal RC network can be constructed.

Necessity: Conversely, if equations (5.15) are realized by a passive reciprocal RC network, its network equations have the form as described by (5.14). It is clear that the potential functions $\phi(x)$ and $W_f(x,u)$ satisfy (5.16).

For the synthesis problem, one must recover the appropriate functions $\phi(x)$ and $W_f(x,u)$ from $\phi(x,u)$ and $h(x,u)$. We shall now show how, first, the stored energy function $\phi(x)$ can be obtained and then how $W_f(x,u)$ can be obtained.

Similarly to the study of lossless reciprocal networks, the system described by (5.6) is assumed to be incrementally controllable, i.e., the linearized system derived from the system described by (5.6), which is

$$
\begin{align*}
\Delta x &= \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial u} \Delta u \\
\Delta y &= \frac{\partial h}{\partial x} \Delta x + \frac{\partial h}{\partial u} \Delta u
\end{align*}
(5.17)
$$

is required to be completely controllable, i.e.,

$$
V_0 \begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial u}
\end{bmatrix} 
= \begin{bmatrix}
\frac{\partial h}{\partial x} \\
\frac{\partial h}{\partial u}
\end{bmatrix}
$$

has full rank.

Lemma 3: Assume that the system described by (5.15) is incrementally controllable. Then the total stored energy function $\phi(x)$ may be uniquely determined from

$$
\nabla^2 \phi = \begin{bmatrix}
\nabla^2 \phi \nabla^2 \phi \cdots \nabla^2 \phi
\end{bmatrix}^{-1}
\begin{bmatrix}
V_0 V_0^T (V_0 V_0^T)^{-1} \\
\nabla^2 \phi
\end{bmatrix}
(5.18)
$$

with $\phi(0) = 0$, $\nabla\phi(0) = 0$, where

$$
V = \begin{bmatrix}
\frac{\partial h}{\partial x} \\
\frac{\partial h}{\partial u}
\end{bmatrix}
$$

Proof: Equations (5.16) imply

$$
\frac{\partial h}{\partial x} = \frac{\partial^2 W_f}{\partial x^2} + \frac{\partial^2 W_f}{\partial x \partial u} \\
\frac{\partial h}{\partial u} = \frac{\partial^2 W_f}{\partial x \partial u} + \frac{\partial^2 W_f}{\partial u^2}
$$

Since

$$
\frac{\partial^2 W_f}{\partial x \partial u} = \frac{\partial^2 W_f}{\partial u \partial x}
$$

we obtain

$$
\nabla^2 \phi \left(\frac{\partial h}{\partial x}\right) = -\left(\frac{\partial h}{\partial u}\right)
(5.19)
$$

In addition, we have

$$
\frac{\partial f}{\partial x} = -\frac{\partial^2 W_f}{\partial x^2} - \frac{\partial^2 W_f}{\partial x \partial u} \\
\frac{\partial f}{\partial u} = -\frac{\partial^2 W_f}{\partial x \partial u} - \frac{\partial^2 W_f}{\partial u^2}
$$

$$
\nabla^2 \phi \left(\frac{\partial f}{\partial x}\right) = -\left(\frac{\partial f}{\partial u}\right)
(5.20)
$$
The conclusion of the lemma follows from (6.19) and (6.20) as in the case of lossless reciprocal networks.

Once the function \( \phi(x) \) has been obtained, the potential function \( W(x, u) \) has to be recovered. For this purpose, we need to assume incremental observability, which is equivalent to \( \psi \) having full rank and in turn equivalent to \( V_\psi \phi(x) \) having full rank. But, as we know, this is consistent with an assumption that \( \psi = \psi(x) \) is invertible, i.e., we have \( x = \psi(u) \) for some function \( \psi \). Now in the course of proving (6.19) and (6.20), we showed that

\[
\begin{bmatrix}
\frac{\partial^2 W}{\partial u^2} & \frac{\partial^2 W}{\partial u \partial \psi} \\
\frac{\partial^2 W}{\partial \psi \partial u} & \frac{\partial^2 W}{\partial \psi^2}
\end{bmatrix} = \begin{bmatrix}
-\left( \frac{\partial^2 \phi}{\partial u^2} \right) V_\psi^{-1} - \left( \frac{\partial^2 \phi}{\partial \psi \partial u} \right) \\
-\left( \frac{\partial^2 \phi}{\partial \psi^2} \right) \frac{\partial^2 \phi}{\partial u^2} - \left( \frac{\partial^2 \phi}{\partial \psi \partial u} \right)
\end{bmatrix}.
\]

(5.21)

The right-hand side of the equality can be expressed as a function of \( \psi \) and \( u \) since \( x = \psi(u) \). Then integration using the conditions \( \partial W/\partial u(0, 0) = 0 \), \( \partial W/\partial \psi(0, 0) \), \( W(0, 0) = 0 \) yields \( W \).

C. Example

We shall synthesize the following network equations:

\[
\begin{align*}
\dot{x} &= f(x, u) = \begin{bmatrix}
-\frac{4}{11} \left( x_1 + \frac{x_1^3}{3} \right) - \frac{1}{11} \left( 2x_1 + x_2 \right) + \frac{5}{11} u \\
-\frac{1}{11} \left( x_1 + \frac{x_1^3}{3} \right) - \frac{3}{11} \left( 2x_1 + x_2 \right) + \frac{4}{11} u
\end{bmatrix} \\
y &= \frac{5}{11} \left( x_1 + \frac{x_1^3}{3} \right) + \frac{4}{11} \left( 2x_1 + x_2 \right) + \frac{21}{11} u + u^2.
\end{align*}
\]

(5.15)

The incremental controllability and observability matrices \( V_x \) and \( V_\psi \) can be determined as follows:

\[
\begin{align*}
\frac{\partial f}{\partial u} &= \frac{5}{11} \\
\frac{\partial f}{\partial x} &= \begin{bmatrix}
-\frac{4}{11} (1 + x_1^2) - \frac{1}{11} (2 + x_1) \\
-\frac{1}{11} (1 + x_1^2) - \frac{3}{11} (2 + x_1)
\end{bmatrix} \\
\left( \frac{\partial \phi}{\partial x} \right) &= \begin{bmatrix}
\frac{5}{11} (1 + x_1^2) \\
\frac{4}{11} (2 + x_1^2)
\end{bmatrix}
\end{align*}
\]

\[
V_x = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial u} & \frac{\partial f}{\partial \psi}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{5}{11} (1 + x_1^2) - \frac{20}{121} (1 + x_1^2) - \frac{4}{121} (2 + x_1^2) \\
\frac{4}{11} (2 + x_1^2) - \frac{5}{121} (1 + x_1^2) (2 + x_1^2) - \frac{12}{121} (2 + x_1^2)^2
\end{bmatrix}
\]

It is easily checked that \( V_x \) has full rank for all \( x_1, x_2 \); using Lemma 3, we obtain

\[
V_\psi = V_x^2 V_x^\top = \begin{bmatrix}
1 + x_1^2 & 0 \\
0 & 2 + x_1^2
\end{bmatrix}.
\]
With \( \phi(0)=0 \) and \( V(0)=0 \), the corresponding energy function \( \phi(x) \) is

\[
\phi(x) = \frac{x_1^2}{2} + \frac{x_1^4}{12} + \frac{x_2^2}{2} + \frac{x_2^4}{12}
\]

and

\[
v_0 = \nabla \phi(x) = \left[ \frac{x_1 + x_1^3}{2x_1 + x_1^3/3} \right]. \quad (5.16)
\]

The potential function \( W_c(v_0, u) \) is also easily recovered. We have, using (5.12),

\[
\begin{bmatrix}
\frac{\partial^2 W_c}{\partial v_0^2} & \frac{\partial^2 W_c}{\partial v_0 \partial u} \\
\frac{\partial^2 W_c}{\partial u \partial v_0} & \frac{\partial^2 W_c}{\partial u^2}
\end{bmatrix} = \begin{bmatrix}
4/11 & 1/11 & -5/11 \\
1/11 & 3/11 & -4/11 \\
-5/11 & -4/11 & -57/11 - 3u^2
\end{bmatrix}
\]

from which

\[
\frac{\partial^2 W_c(v_0, u)}{\partial v_0^2} = \begin{bmatrix}
4/11 v_0 + 1/11 v_0^3 & -5/11 v_0 \\
1/11 v_0 + 3/11 v_0^3 & -4/11 v_0
\end{bmatrix}
\]

and

\[
\frac{\partial^2 W_c(v_0, u)}{\partial u^2} = -5/11 v_0 - 4/11 v_0^3 - 57/11 - 6u^2.
\]

Observing (5.1) and (5.3), we see that the equations of the hybrid coupling network \( R \) are

\[
\begin{align*}
\dot{x}_1 &= \frac{4}{11} x_1 + \frac{1}{11} x_1^3 - \frac{5}{11} u \\
\dot{x}_2 &= -\frac{1}{11} x_0 - \frac{3}{11} x_0^3 - \frac{4}{11} u \\
y &= \frac{5}{11} x_0 + \frac{4}{11} x_0^3 + \frac{57}{11} u + 6u^2. \quad (5.17)
\end{align*}
\]

It is not difficult to synthesize (5.17). One way is to observe that (5.17) are equivalent to

\[
\begin{align*}
v_0 &= -3x_0 + x_3 + u \\
v_0 &= -x_3 + 4x_3 + u \\
y &= -x_3 - 6u + u^3. \quad (5.18)
\end{align*}
\]

Together, (5.16) and (5.18) are synthesized by the network of Fig. 9.

If we proceed according to the second synthesis problem, viz., identifying \( R \) and \( T \), we have for the hybrid equations of \( R \), see (5.10),

\[
\begin{bmatrix}
\frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\
\frac{\partial h_1}{\partial x_0} & \frac{\partial h_1}{\partial x_3} \\
\frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \\
\frac{\partial h_2}{\partial x_0} & \frac{\partial h_2}{\partial x_3}
\end{bmatrix} = \begin{bmatrix}
4/11 & 1/11 & -5/11 \\
1/11 & 3/11 & -4/11 \\
5/11 & 4/11 & 57/11 + 3u^2
\end{bmatrix}
\]

From (5.10), we see that \( T_{34} = 5/11 \) \( 4/11 \), \( T_{31} = i \), \( T_{24} = 0 \) (this identifies \( T \)) and

\[
H^{-1} = \begin{bmatrix}
4/11 & 1/11 & 0 & 0 \\
1/11 & 3/11 & 0 & 0 \\
0 & 0 & 57/11 + 3u^2
\end{bmatrix}
\]

whence the equations of \( R \) become

\[
\begin{align*}
v_0 &= \begin{bmatrix} 3 & -1 & -1 \end{bmatrix} h_3 \\
v_0 &= \frac{57}{11} i_4 + i_4.
\end{align*}
\]
VI. Conclusions

It has been shown how the synthesis problem for special types of nonlinear reciprocal networks such as lossless reciprocal networks or nonlinear reciprocal RCT networks can be reduced to the problem of determining an appropriate stored energy function of the corresponding network.

The problems solved are far less numerous and less important than those remaining. Among the latter, we can note the following: reciprocal synthesis of a LRC network, reciprocal synthesis of a lossless network with state-vector entries not equal to inductor fluxes or capacitor charges, and the developing of a property on the operator mapping the port inputs \( u(t) \) into port outputs \( y(t) \) of a reciprocal network which reflects that reciprocity.

References


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1 An equivalent requirement is that \( T \) has a hybrid matrix description with excitation variables \( v, u, \) and \( u \). Of course, \( T \) always has a hybrid matrix description with some selection of variables \( [18] \), but the choice here is important. The effect of the assumption is to disallow the possibility that \( y \) can depend instantaneously on, inter alia, \( u \) -- a situation that could arise if there was, in effect, a capacitor in parallel with the input. Such a topology could be handled by a new choice of input and output variables, i.e., some or all of the entries of \( v \) become entries of \( y \), and vice versa.

2 The state vector of the network state equations should be distinguished from the state function of reciprocal algebraic multipoles of the previous section.

3 We are not linearizing around a trajectory, so that \( f_p \) and \( f_q \) are not time-varying.