V. Lower Bound Evaluation

Evaluation of the lower bound formulas developed above involves three computational problems. First is the evaluation of the expectations in (7) and (11). In general, this computation must be done using Monte Carlo simulation techniques [17], [18], although in some simple cases an analytical evaluation may be possible. Second is the computation of matrix square roots. Algorithms for this purpose are available [19]. Finally, the Riccati equation for the linear filtering problem must be solved. This subject is treated in [3] and [9].

V. Conclusions

A lower bound based on the Cramér–Rao theory has been given for the evaluation of nonlinear filtering and smoothing mean-square error. The algorithm presented is a generalization of [7] to the multidimensional case and produces a tighter filtering lower bound. The procedure is used to obtain results for the important singular plant noise covariance case. Resulting formulas can be conveniently evaluated by using Monte Carlo simulation techniques.

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Comrade Matrix and Systems Excited by Colored Noise

BRIAN D. O. ANDERSON

***Abstract*—Stable linear systems described by state variable equations are considered where the input process is stationary and, in general, nonwhite. We show that there exists a natural coordinate basis for the system, the main properties for which imply that the state covariance matrix is a multiple of the identity, the system matrix itself is a comrade matrix, and the orthonormal polynomial set used to define the comrade matrix is obtained from the input spectrum and the system characteristic polynomial.**

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The author is with the Department of Electrical Engineering, University of Newcastle, Newcastle, N.S.W., Australia.

Consider the system

$$\dot{x} = Ax + bu \tag{1}$$

in which x is an n-vector, A is a companion matrix with all roots of $|sI-A|$ in $\mathrm{Re}[s]<0$, b is $[0\ 0\ \cdots\ 0\ 1]'$, and u is a scalar stationary random process with spectral density $\Phi_u(j\omega)$ (which is even in ω). We consider the following task: find a new coordinate basis with associated equation

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \tag{2}$$

such that in the new coordinate basis, $E[\bar{x}\bar{x}']$ is a multiple of the identity. (It is assumed that (2) has reached stationarity.)

This problem was motivated by several papers [1]–[3] defining and exploring the use of the comrade matrix in system theory: the comrade matrix, inter alia, allows association of a set of orthogonal polynomials with a linear system representation.

One solution to the problem posed above is obtained as follows. Let $p_i(s)$ be the real ith degree polynomial for $i=0,1,\cdots,n-1$ in an orthonormal set, defined by

$$\int_{-\infty}^{+\infty} p_i(j\omega)\frac{\Phi_u(j\omega)}{|\det(j\omega I-A)|^2}p_k(-j\omega)\,d\omega=\delta_{ik} \tag{3}$$

and the requirement that the coefficient of s^i in $p_i(s)$ be positive. Let

$$p_i(s)=\sum_{j=0}^{i} p_{ij}s^j \tag{4}$$

and

$$S=\begin{bmatrix} p_{00} & 0 & 0 & \cdots & 0 \\ p_{10} & p_{11} & 0 & \cdots & 0 \\ p_{20} & p_{21} & p_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1,0} & p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,n-1} \end{bmatrix}. \tag{5}$$

Then we claim the definitions

$$\bar{A}=SAS^{-1} \quad \bar{b}=Sb \tag{6}$$

ensure that $E[\bar{x}\bar{x}']$ is a multiple of the identity.

To see this, observe first that

$$\begin{aligned}(sI-\bar{A})^{-1}\bar{b} &= S(sI-A)^{-1}b \\ &= \frac{1}{|sI-A|}S\begin{bmatrix}1\\ s\\ \vdots\\ s^{n-1}\end{bmatrix} \\ &= \frac{1}{|sI-A|}\begin{bmatrix}p_0(s)\\ p_1(s)\\ \vdots\\ p_{n-1}(s)\end{bmatrix}.\end{aligned} \tag{7}$$

Then observe that, with $\Phi_{\bar{x}}(j\omega)$ the spectral matrix of $\bar{x}(\cdot)$,

$$\begin{aligned}E[\bar{x}\bar{x}'] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi_{\bar{x}}(j\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}(j\omega I-\bar{A})^{-1}\bar{b}\Phi_u(j\omega)\bar{b}'(-j\omega I-\bar{A}')^{-1}d\omega\end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \begin{bmatrix} p_0(j\omega) \\ p_1(j\omega) \\ \vdots \\ p_{n-1}(j\omega) \end{bmatrix} \frac{\Phi_u(j\omega)}{|\det(j\omega I - A)|^2}$$

$$\cdot [p_0(-j\omega) \quad p_1(-j\omega) \quad \cdots \quad p_{n-1}(-j\omega)]\, d\omega$$

$$= \frac{1}{2\pi} I. \tag{8}$$

[The first equality is standard, the second follows from (2), the third follows from (7), and the fourth follows from (3).]

Remark 1: The matrix $\bar{A}$ is termed by Barnett a comrade matrix. It has a simple expression in terms of the coefficients appearing in recursive formulas for the $p_i(s)$ and, assuming $p_n(s)$ can be formed, in terms of the coefficients expressing $sI - A$ as a linear combination of $p_0(s), \cdots, p_{n-1}(s)$. The matrix $\bar{A}$, apart from having a last row which in general has all nonzero entries, has a tridiagonal structure.

Remark 2: Any orthogonal basis change applied to (2) will leave invariant the value of $E[\bar{x}\bar{x}']$. However, the particular basis we have constructed, besides allowing simple expression of $\bar{A}$, is one in which the entries of $\bar{x}$ have the same differentiability properties as those of x and is the only such basis (to within a trivial change of sign of one or more entries of $\bar{x}$). Thus, if $\Phi_u(j\omega)$ is a constant, and $u(\cdot)$ is white noise, $\bar{x}_1$ is not mean-square differentiable, and $\bar{x}_i$ is $(i-1)$ times mean-square differentiable for $i \geq 2$.

Remark 3: We can do the same analysis for

$$x_{k+1} = Ax_k + bu_k \tag{9}$$

where A, b are as before, save that $|zI - A|$ has all roots in $|z| < 1$, and (3) is replaced by

$$\frac{1}{2\pi_j} \oint p_i(z) \frac{\Phi_u(z)}{\det(zI - A)\det(z^{-1}I - A)} p_k(z^{-1}) z^{-1}\, dz = \delta_{ik}. \tag{10}$$

Remark 4: An interesting special case occurs when $\Phi_u(z) = 1$ is taken, i.e., $\{u_k\}$ is a white noise sequence. For then (to within a positive scaling constant) $p_n(z)$, the degree n orthonormal polynomial, becomes $\det|zI - A|$,

$$p_n(z) = \alpha |zI - A|. \tag{11}$$

In this case, the matrix $\bar{A}$ becomes tridiagonal. The corresponding continuous-time result at first glance is not available, since with $\Phi_u(j\omega) = 1$ and $i = k = n$ in (3), the integral is not well defined. Nevertheless, $p_0(s), \cdots, p_{n-1}(s)$ are well defined, and so is $\bar{A}$, which can be checked to have the following structure:

$$\bar{A} = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & & & \\ -\alpha_1 & 0 & \alpha_2 & 0 & & & \\ 0 & -\alpha_2 & 0 & \alpha_3 & & & \\ 0 & 0 & -\alpha_3 & 0 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \alpha_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & -\alpha_{n-1} & -\beta \end{bmatrix}.$$

Here the α_i and β are all positive constants. This is because 1) apart from having a last row with all entries possibly nonzero, A is tridiagonal; 2) $\bar{P} = E[\bar{x}\bar{x}'] = 1/2\pi I$, and $\bar{P}\bar{A}' + \bar{A}\bar{P} = -\bar{b}\bar{b}'$, which establishes that $\bar{A}$ is tridiagonal, that $\bar{a}_{ii} = 0$ for $i \neq n$, and that $a_{i,i+1} = -a_{i+1,i}$, and 3) by equating entries in the first row and columns 2 through n of $\bar{A}S$ and SA, one finds $\alpha_i = p_{i-1,i-1}/p_{ii}$ and the p_{ii} are positive by construction.

Conclusions

It would be interesting to find an application of the special system representation constructed to a filtering problem, for example, involving estimation of some functional of the $u(\cdot)$ process.

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On the Length of Inputs Necessary in Order to Identify a Deterministic Linear System

EDUARDO D. SONTAG

***Abstract*—The family of m-input, n-dimensional linear systems can be globally identified with a generic input sequence of length $2mn$. This bound is the best possible. A best bound is provided also for a corresponding local identification problem.**

I. Introduction

A result of the author, valid for a large class of discrete-time systems ([4]; see also [5] for the continuous-time analog) states that, given a parameterized family of systems, a generic long-enough input sequence is sufficient for I/O behavior identification, the length depending on the general form of the family. Although fairly constructive, this general result does not give useful bounds for the lengths of the needed sequences. For linear systems, we give such bounds in this note. The techniques used here are completely different from those used in order to prove the abstract result.

The results presented below give the smallest lengths needed for identification experiments. These results complement those presented by other authors—see, for example, [1]–[3]—who construct minimal realizations once the corresponding experiment has been performed.

II. Definitions

An (n, m, p)-*system* S (over the real numbers) is an n-dimensional, m-input, p-output, discrete-time, constant linear system; S is uniquely determined by a triple (F, G, H) of matrices (F is n by n, G is n by m, H is p by n). The input sequence $w = u_1 \cdots u_r$ (u_i in R^m) *identifies* (n, m, p)-*systems* iff the following property holds for any two such systems S_i.

1) If S_1 and S_2 give the same zero-state output sequence $y_1, \cdots, y_r$ when w is applied, then S_1 and S_2 have the same zero-state I/O behavior.

When 1) is valid for any two *canonical* (=reachable and observable)

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The author is with the Department of Mathematics, Rutgers University, New Brunswick, NJ 08903.

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