Passive Network Synthesis Via Dual Spectral Factorization

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Abstract—**The role of dual transfer function matrices in network analysis and synthesis is described. The transfer function matrices are actually spectral factors of a certain power spectrum matrix. Procedures for constructing dual and self-dual transfer function matrices are presented, from which general synthesis methods are derived for nonreciprocal and reciprocal networks. A number of known synthesis procedures are shown to be special instances of the general methods.**

I. Introduction

THE ROLE OF spectral factorization in passive network synthesis has been recognized for a long time: as any text on the subject shows, e.g., [1]–[4] there are numerous particular synthesis procedures that require the construction of spectral factors, often with special properties. The purpose of this paper is to explore the idea that to a certain extent it is pairs of associated spectral factors that are important in synthesis, rather than single spectral factors.

More precisely, via some network analysis we show that any passive network has associated with it a pair of spectral factors which we term dual. Then we turn this result around to show that knowledge of dual spectral factors leads to impedance or scattering matrix synthesis procedures. Analysis also shows that the two factors of the pair become identical in case the network is reciprocal (we then call the factor self-dual), and so reciprocal synthesis becomes in large measure a problem of constructing self-dual spectral factors.

The idea of dual pairs of spectral factors has proved useful in studying related forward and backward Markovian representations of second-order processes [5] and it was in searching for physical insight into these results that the network results started to become evident. (Of course, any passive lossy network has associated with it a stationary process, generated by thermal motion of the electrons in the resistors of the network.) Self-dual spectral factors from the stochastic process viewpoint turn out to be associated with so-called dynamically reversible processes; such processes are recognized as occurring in any thermodynamic system for which the Onsager relations are valid [6] and reciprocal passive networks constitute one such system.

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The paper is structured in the following way. Section II reviews the definitions of dual transfer function matrices, using both state-variable and matrix factor descriptions. Sections III and IV explore general analysis and synthesis results involving, respectively, impedance and scattering matrix descriptions. Section V discusses self-duality and reciprocal networks. Then in Section VI we discuss how the ideas of dual and self-dual spectral factors arise in many known synthesis procedures.

II. Dual State-Variable and Matrix-Fraction Descriptions

Let $\{F,G,H,J\}$ define a *controllable realization* of a transfer function matrix

$$W(s)=J+H'(sI-F)^{-1}G \tag{1}$$

where also $\mathrm{Re}\lambda_i(F)<0$. Define Π as the unique positive definite solution of

$$\Pi F'+F\Pi=-GG'. \tag{2}$$

Then we define a *dual realization* $\{F_d,G_d,H_d,J_d\}$, with transfer function

$$W_d(s)=J_d+H_d'(sI-F_d)^{-1}G_d \tag{3}$$

as one for which

$$F_d=\Pi F'\Pi^{-1}=-(F+\Pi^{-1}GG') \tag{4}$$

$$G_d=G,\quad H_d=-H-\Pi^{-1}GJ,\quad J_d=J. \tag{5}$$

It is then not hard to establish by direct calculation the following properties, as shown in [5]:

$$W(s)W'(-s)=W_d(-s)W_d'(s) \tag{6}$$

$$W(s)=W_d(-s)U(s) \tag{7}$$

where

$$U(s)=I-G'\Pi^{-1}(sI-F)^{-1}G \tag{8}$$

and

$$U'(-s)U(s)=I. \tag{9}$$

Now it is known (see, e.g., [7]) that associated with any controllable pair (F,G) there is a (nonunique) polynomial $D(s)$ such that $|D(s)|=|sI-F|$ and such that any strictly proper transfer function $H'(sI-F)^{-1}G$ can be expressed as $N(s)D^{-1}(s)$ for some $N(s)$. The converse is also true. Moreover, it is easy to extend the result to connect proper transfer functions expressed as $N(s)D^{-1}(s)$ and $J+H'(sI-F)^{-1}G$. Suppose then that $W(s)$ has a so-called right matrix fraction description

$$W(s)=N(s)D^{-1}(s). \tag{10}$$

Then it was shown in [5] that the transfer function of the dual realization is given by

$$W_d(s)=N(-s)D_d^{-1}(s) \tag{11}$$

where $D_d(s)$ is a unique polynomial matrix, called the "dual" of $D(s)$, and obeying

$$D'(-s)D(s)=D_d'(s)D_d(-s) \tag{12}$$

$$|D(s)|=\pm|D_d(s)| \tag{13}$$

$$\lim_{s\to\infty} D(s)D_d^{-1}(-s)=I. \tag{14}$$

(We actually adopt a slightly different definition to that of [5] in order to suit the application of the idea in this paper.) The dual polynomial $D_d(s)$ can actually be found as

$$D_d(s)=U(-s)D(-s) \tag{15}$$

but there are other ways of obtaining it, [8]. In case $W(s)$ is a scalar, $D_d(s)$ and $D(s)$ are to within a $\pm$ sign identical, and one evidently obtains $W_d(s)$ from $W(s)$ by reflecting zeros across the $j\omega$-axis, and leaving poles unaltered.[1] Equations (7) through (14) express the matrix generalization of this idea.

We shall say that a state-space realization or matrix fraction description is *self-dual* if

$$W(s)=W_d(s)T'\Sigma T$$

for some orthogonal T and signature matrix[2] Σ.

The self-duality is obvious when $T'\Sigma T=I$; the significance of the name in the more general case will become evident in Section V.

III. Impedance Matrices and Duals

In this section, we shall first argue that each network synthesis of a prescribed rational positive real impedance matrix generates a dual pair of transfer function matrices. We shall then prove a type of converse, showing that knowledge of a spectral factor of the parahermitian part of a positive real impedance matrix together with its dual in effect solves a network synthesis problem.

Suppose a network is comprised of a finite number of linear, time-invariant, passive resistors, inductors, capacitors, transformers, and gyrators, and has accessible n-ports or terminal pairs. Suppose further that this n-port

[1]If there is a common factor between numerator and denominator, the numerator zeros are still reflected and the denominator not, so that the common factor is no longer present in $W_d(s)$.

[2]A signature matrix is a diagonal matrix with diagonal entries all $+1$ or -1.

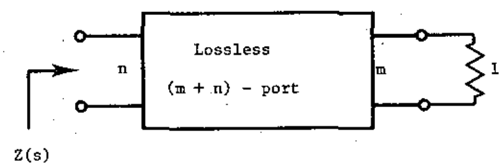

Fig. 1.

network has an impedance matrix $Z(s)$. Then $Z(s)$ has the property known as positive realness. As a consequence of this property:

$$\Phi(s) \triangleq Z(s)+Z'(-s)$$

is a power spectrum matrix. There is in fact a physical interpretation for $\Phi(s)$. Assume that each resistor R in the network has in parallel with it a white noise current generator $i_R(\cdot)$ (to model the thermal noise produced by the resistor). One has $E[i_R(t)]=0$ and $E[i_R(t)i_R(s)]=2kTR^{-1}\delta(t-s)$, with k Boltzmann's constant and T the absolute temperature. Then a stationary noise (voltage) process will appear at the network ports, with spectrum $kT\Phi(s)$. (This result is reasonably well known, at least in its 1-port version; in the course of this section, the n-port version will be derived.)

Our purpose in this section is to show that the network determines a pair of forward and backward spectral factors of $\Phi(s)$ with matrix fraction descriptions as related in the last section. Moreover, given simply a positive real $Z(s)$ and related forward and backward spectral factors of the associated $\Phi(s)$, we can show that a network is thereby defined.

Suppose then that there is available a network N with port impedance $Z(s)$. Then we can conceive of it as drawn in Fig. 1, that is, as an $(m+n)$-port network N_L of capacitors, inductors, transformers, and gyrators terminated at m-ports in resistors. By regarding a resistor of r Ω as a cascade of a transformer of turns ratio $\sqrt{r}:1$ with a 1-Ω resistor, we can assume, using such transformer normalizations if necessary, that all resistors are 1 Ω. This way of looking at a network has long found use in network theory, for example in the classical Darlington synthesis procedure, [1], [3].

It will be useful to introduce the hybrid matrix, say $M(s)$, associated with the $(m+n)$-port lossless network. Assume that the first n-ports are current-excited. Being a hybrid matrix of a lossless network, $M(s)$ is lossless positive real; necessary and sufficient conditions for this are that

$$M(s)+M'(-s)=0 \tag{16}$$

$$\det[I+M(s)]\neq 0, \qquad \mathrm{Re}[s]\geq 0. \tag{17}$$

Moreover, with $M(s)$ partitioned as

$$M(s)=\begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \tag{18}$$

with M_{11} $n\times n$, it follows after easy calculations that the impedance matrix can be obtained as

$$Z(s)=M_{11}-M_{12}(I+M_{22})^{-1}M_{21}. \quad (19)$$

Thus the study of $Z(s)$ can be reduced to that of $M(s)$. Suppose that the excitation variables at the last m-port used in defining M are current for the first m_1 of these ports, and voltages for the remaining $(m-m_1)$. Let

$$\Sigma=I_{m1}\dotplus I_{(m-m_1)}.$$

Now define

$$W(s)=M_{12}(I+M_{22})^{-1} \quad (20)$$

$$\overline{W}'(s)=\Sigma(I+M_{22})^{-1}M_{21}. \quad (21)$$

It is easy to check the following interpretations $W(s)$ and $\overline{W}(s)$. If current generators are inserted in parallel with each of the 1-Ω resistors, $W(s)$ is the transfer function matrix from these generators to the resulting voltage at the n-ports of N. Also, $\overline{W}'(s)$ is both the transfer function matrix linking input currents to voltages appearing across the 1-Ω resistors and the transpose of the matrix replacing $W(s)$ when gyrator polarities are reversed. [This polarity reversal causes M to be replaced by $(I\dotplus\Sigma)M'(I\dotplus\Sigma)$]. One also has

$$\tfrac{1}{2}[Z(s)+Z'(-s)]=W(s)W'(-s) \quad (22)$$

$$=\overline{W}(-s)\overline{W}'(s). \quad (23)$$

These equalities can be verified using (16), but they also have physical significance. Thus if voltage generators in series with the 1-Ω resistors produce white noise, (22) provides a formula for the spectrum of the noise voltage process observed at the network ports, (Nyquist's Theorem). Also, the fact that $\frac{1}{2}I'^{*}[Z(j\omega)+Z'(-j\omega)]I=I'^{*}\overline{W}(-j\omega)\overline{W}'(j\omega)I$ for any complex n-vector I can readily be shown to reflect the fact that the power entering the left port of N_L with sinusoidal excitation equals the power leaving it.

The network N_L can be described by state-variable equations, inducing a state-variable realization for $W(s)$ and $\overline{W}(s)$. However the transfer function matrix $M(s)$ can also be described via a matrix fraction description, and this induces a matrix fraction description of $W(s)$ and $\overline{W}(s)$. The matrix fraction descriptions seem to be more convenient and lead to the following result.

Theorem 1

With definition as above, suppose that

$$\begin{bmatrix} M_{12}(s) \\ M_{22}(s) \end{bmatrix}=\begin{bmatrix} A_{12}(s) \\ A_{22}(s) \end{bmatrix}B^{-1}(s) \quad (24)$$

with the matrix fraction description coprime. This will yield fraction descriptions of $W(s)$ and $\overline{W}(s)$ such that the denominator matrix of $W(s)$, say $D(s)$, has all zeros of $|D(s)|$ in $\mathrm{Re}[s]<0$, and $\overline{W}(s)=W_d(s)Q$, with W_d the dual of W and Q an orthogonal matrix.

Proof: Define $D(s)=A_{22}(s)+B(s)$. We shall show first that all zeros of $|D(s)|$ lie in $\mathrm{Re}[s]<0$. If $A_{22}(s)$ and $B(s)$ are coprime, then $D(s)$ and $B(s)$ are coprime; from the fact that $0\neq\det[I+M_{22}(s)]=\det[D(s)A_{22}^{-1}(s)]$ for $\mathrm{Re}[s]\geq 0$, we get $\det D(s)\neq 0$ for $\mathrm{Re}[s]\geq 0$. Suppose then that $A_{22}(s)$, $B(s)$ have a right greatest common divisor $K(s)$ with determinant nonconstant. We shall deduce a contradiction.

Since (24) defines a coprime matrix fraction description and all poles of $M(s)$ lie on $\mathrm{Re}[s]=0$, all zeros of $K(s)$ must be there too. Now $K(s)$ is a right divisor of $D(s)$, and (20) yields

$$W(s)=A_{12}(s)K^{-1}(s)D_1^{-1}(s)$$

for some D_1. Since $Z(j\omega)+Z'(-j\omega)$ is finite for all ω [2], (22) shows that $W(j\omega)$ is finite for all real ω, and therefore $W(j\omega)D_1(j\omega)=A_{12}(j\omega)K^{-1}(j\omega)$ has this property. Since all zeros of $|K(s)|$ lie on $s=j\omega$, the finiteness can only hold if $K(s)$ is a right divisor of $A_{12}(s)$; but then it is a right divisor of $A_{12}(s)$, $A_{22}(s)$, and $B(s)$, which is a contradiction.

For the remainder of the proof, note that $W(s)=A_{12}(s)D^{-1}(s)$, while

$$\overline{W}(s)=A_{12}(-s)[A_{22}(-s)-B(-s)]^{-1}\Sigma.$$

This follows from (21) and the facts that $M'_{21}(-s)=M_{12}(s)$ and $M'_{22}(-s)=M_{22}(s)$, these in turn flowing from losslessness.

Comparing the matrix fraction description for $W(s)$ and $\overline{W}(s)$, we see that the result $\overline{W}(s)=W_d(s)Q$ will follow if and only if $\overline{D}(s)=\Sigma[A_{22}(-s)-B(-s)]=Q'D_d(s)$ for some orthogonal Q.

Now $M'_{22}(-s)+M_{22}(s)=0$ implies $B'(-s)A'_{22}(s)+A'_{22}(-s)B(s)=0$, whence

$$D'(-s)D(s)=\overline{D}'(s)\overline{D}(-s). \quad (25)$$

Further, using the lossless nature of $M_{22}(s)$ and the even or odd nature of $|B(s)|$, we have

$$\begin{aligned}|D(s)|&=|I+M_{22}(s)||B(s)|\\&=\pm|B'(-s)||I-M'_{22}(-s)|\\&=\pm|B'(-s)-A'_{22}(-s)|=\pm|\overline{D}(s)|.\end{aligned}$$

From (25), and the definition of $D_d(s)$ it follows easily that $\overline{D}=Q'D_d$ for some constant orthogonal Q.

To obtain a converse result, it is sufficient to exhibit a lossless positive real hybrid matrix $M(s)$ such that (19) holds.

Theorem 2

Let $Z(s)$ be a prescribed $n\times n$ positive real matrix, and let $W(s)W'(-s)=\frac{1}{2}[Z(s)+Z'(-s)]$ with $W(s)$ analytic in $\mathrm{Re}[s]\geq 0$. Write $W(s)$ in matrix fraction form as $N(s)D^{-1}(s)$ with $|D(s)|$ nonzero in $\mathrm{Re}[s]\geq 0$, and let $D_d(s)$ be the polynomial dual to $D(s)$. Define an $(n+m)\times(n+m)$ matrix $M(s)$ by

$$M_{22}(s)=[D(s)+D_d(-s)][D(s)-D_d(-s)]^{-1} \quad (26)$$

$$M_{12}(s)=N(s)[D(s)-D_d(-s)]^{-1}=-M'_{21}(-s) \quad (27)$$

$$M_{11}(s)=Z(s)-M_{12}(s)[I+M_{22}(s)]^{-1}M_{21}(s). \quad (28)$$

Then with any assignment of the last m excitation variables, the hybrid matrix $M(s)$ is lossless positive real, and if a network synthesising $M(s)$ is terminated at its last m ports in unit resistors, the impedance matrix at the first n ports is $Z(s)$.

Proof: (Outline Only): Check that $M(s)+M'(-s)=0$. Check also that $\det(I+M)=\det(I+M_{22})\det(I+Z)$ and use the stability of $D(s)$ and positive real nature of Z to conclude this quantity is nonzero in $\mathrm{Re}[s]\geq 0$. Use (16) and (17) to check the first claim and (28) to check the second claim of the theorem.

If in (26)–(28) we replace each occurrence of $D_d(-s)$ by $Q'D_d(-s)$ where Q is an arbitrary orthogonal matrix, the conclusion of the theorem is unaltered. So *all network syntheses can be obtained by the above procedure.* Note that there are three points at which nonuniqueness occurs: at the selection of the spectral factor $W(s)$, at the selection of a particular matrix fraction description of $W(s)$, (in particular in the selection of the greatest common divisor of numerator and denominator polynomials[3]), and finally in the selection of Q.

The above ideas are translatable into state-variable terms. Mostly one simply reinterprets results of, e.g., [4] on resistance extraction synthesis.

IV. Scattering Matrices and Duals

Though abstractly the use of scattering matrices for analysis and synthesis is essentially equivalent to the use of impedances, specific procedures, especially synthesis procedures, sometimes look very different. For this reason, it seems worthwhile to briefly examine the ideas of the last section in scattering matrix terms. We first exhibit the duality of matrix fraction descriptions of two transfer function matrices defined by the network. Fig. 1 still applies. The lossless network N_L has lossless bounded real scattering matrix

$$\Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix} \tag{29}$$

while N has scattering matrix Σ_{11}. One obtains

$$W(s)=(I-\Sigma_{11})^{-1}\Sigma_{12} \tag{30}$$

$$\overline{W}'(s)=\Sigma_{21}(I-\Sigma_{11})^{-1} \tag{31}$$

where $W(s)$ and $\overline{W}(s)$ have the same physical significance as in Section III.

The easiest way to capture the duality results is to observe that with

$$V\triangleq\Sigma_{22}+\Sigma_{21}(I-\Sigma_{11})^{-1}\Sigma_{12} \tag{32}$$

one has

$$V'(-s)V(s)=I \tag{33}$$

and

$$W(-s)V'(s)=\overline{W}(s). \tag{34}$$

Both (33) and (34) can be checked via direct calculation. However, (33) also follows from the fact that $V(s)$ is the scattering matrix of a lossless m-port network, obtained by considering N_L with its right-hand ports only, i.e., the first n-ports of N_L are regarded as being terminated in open circuits.

Let $V(s)=E^{-1}(s)D'(-s)$ be a coprime matrix fraction description. The all pass character and stability of $V(s)$ imply that $E(s)E'(-s)=D'(-s)D(s)$ and $|E(s)|=|D(s)|$, whence $E(s)=D_d'(s)Q$ for some orthogonal Q. Set $\overline{D}(s)=Q'D_d(s)$. Let $W(s)=F(s)G^{-1}(s)$ be a coprime matrix fraction description. Then

$$\begin{aligned}\overline{W}(s)&=W(-s)V'(s)\\&=F(-s)G^{-1}(-s)D(-s)E^{-T}(s)\\&=F(-s)G^{-1}(-s)D(-s)D_d^{-1}(s)Q.\end{aligned}$$

Because $\overline{W}(s)$ is stable, and because of the coprimeness of $F(s)$ and $G(s)$, we must have $G^{-1}(-s)D(-s)$ polynomial, $K(-s)$ say, so that $G(s)K(s)=D(s)$. Then $W(s)=[F(s)K(s)][G(s)K(s)]^{-1}=F(s)K(s)D^{-1}(s)$ and $\overline{W}(s)=F(-s)K(-s)D_d^{-1}(s)Q$. In this way we have established the following theorem.

Theorem 3

Let Σ in (29) be a lossless bounded real scattering matrix and define $W(s)$ and $\overline{W}(s)$ by (30) and (31). Then $\Sigma(s)$ induces via the construction outlined above a matrix fraction description of $W(s)$ such that $\overline{W}(s)=W_d(s)Q$, $W_d(s)$ being the dual and Q an orthogonal matrix.

The network synthesis problem in the scattering framework is one of taking a prescribed bounded real $\Sigma_{11}(s)$ and bordering it with matrices Σ_{12}, Σ_{21}, and Σ_{22} to form a lossless bounded real matrix. We shall establish that the following procedure, involving the use of a dual matrix fraction description, achieves this bordering. Assume that $I-\Sigma_{11}(s)$ is nonsingular.[4]

1) Let Σ_{12} be any stable matrix such that

$$I-\Sigma_{11}(s)\Sigma_{11}'(-s)=\Sigma_{12}(s)\Sigma_{12}'(-s). \tag{35}$$

2) Define $W(s)=[I-\Sigma_{11}(s)]^{-1}\Sigma_{12}(s)$ and construct a matrix fraction description $N(s)D^{-1}(s)$ of $W(s)$ with $\det D(s)$ possessing all zeros in $\mathrm{Re}[s]<0$.

3) Define $\overline{W}(s)=N(-s)D_d^{-1}(s)Q$ where Q is an arbitrary orthogonal matrix and $D_d(s)$ is the dual of $D(s)$.

4) Define

$$\Sigma_{21}(s)=\overline{W}'(s)[I-\Sigma_{11}(s)] \tag{36}$$

$$\begin{aligned}\Sigma_{22}(s)=&-\Sigma_{21}(s)[I-\Sigma_{11}(s)]^{-1}\Sigma_{12}(s)\\&+Q'D_d^{-T}(s)D'(-s).\end{aligned} \tag{37}$$

Notice that (36) and (37) are essentially rewritten versions of (31) and (32), the latter because in the course of

[3]Matrix fraction descriptions of $W(s)$ differing only via a unimodular matrix lead to the same $M(s)$ if the same Q is used.

[4]In case $I-\Sigma_{11}(s)$ is singular, a standard preliminary extraction step involving the isolation by transformers of open-circuit ports should be executed [2].

establishing the duality property we obtained

$$V(s)=Q'D_d^{-T}(s)D'(-s). \tag{38}$$

Notice also that as an alternative to steps 1 and 2 one can have

1') Let $W(s)$ be any stable matrix such that

$$[I-\Sigma_{11}(s)]^{-1}[I-\Sigma_{11}(s)\Sigma'_{11}(-s)][I-\Sigma'_{11}(-s)]^{-1} = W(s)W'(-s). \tag{39}$$

Construct a matrix fraction description $N(s)D^{-1}(s)$ of $W(s)$ with $\det D(s)$ possessing all zeros in $\mathrm{Re}[s]<0$.

2') Define

$$\Sigma_{12}(s)=[I-\Sigma_{11}(s)]W(s). \tag{40}$$

The procedure is justified as follows. Because Σ_{11} is bounded real, all poles of $(I-\Sigma_{11})^{-1}$ and, therefore, $W(s)$ lie in $\mathrm{Re}[s]\leqslant 0$. Because $(I-\Sigma_{11})$ is nonsingular, $Z(s)$ exists and accordingly (22) and (23) imply that $W(j\omega)W'(-j\omega)=\frac{1}{2}(Z(j\omega)+Z'(-j\omega))$, from which as before we conclude that $W(s)$ must have all poles in $\mathrm{Re}[s]<0$; so, therefore, has $\overline{W}(s)$ by construction, and thus $\Sigma_{21}(s)$ and $\Sigma_{22}(s)$ certainly have all their poles in $\mathrm{Re}[s]\leqslant 0$. That they have all poles in $\mathrm{Re}[s]<0$ follows by considering on the $j\omega$-axis the identity $\Sigma'(-s)\Sigma(s)=I$ which can be checked by tedious algebra. This justifies the procedure.

A state variable version of these calculations may be found in [4].

V. Reciprocal Networks and Self-Duality

No assumption of reciprocity has been made up to this point. In this section we shall show that such an assumption is equivalent to requiring the matrix fraction description of $W(s)$ to be *self-dual*, in the sense that $W(s)=W_d(s)T'\Sigma T$ for some orthogonal T and signature matrix Σ.

The analysis result is as follows.

Theorem 4

Consider a reciprocal network N, and let $W(s)$, $\overline{W}(s)$ be as defined earlier. Then

$$W(s)=\overline{W}(s) \tag{41}$$

and

$$\overline{W}(s)=W_d(s)T'\Sigma T \tag{42}$$

where $W_d(s)$ is obtained from the matrix fraction description of $W(s)$ induced (in the manner described earlier) by a hybrid or scattering matrix description of N_L, T is orthogonal and Σ is a signature matrix.

Proof: Recalling the physical interpretations of $W(s)$ and $\overline{W}(s)$ as certain current-to-voltage transfer functions, (41) becomes a well-known consequence of reciprocity. It also follows from the formulas (20), (21), (30), and (31) for $W(s)$ and $\overline{W}(s)$.

Equation (42) follows easily either from the hybrid or scattering matrix analysis. Let us work with the scattering matrix analysis. The following matrix was proved to be lossless bounded real, and is symmetric on account of the reciprocity.

$$V(s)=Q'D_d^{-T}(s)D'(-s)=\overline{D}^{-T}(s)D'(-s). \tag{43}$$

Letting $s\to\infty$ and using the dual polynomial definition shows that $V(\infty)=Q'$, whence Q is symmetric. Since Q is also orthogonal, $Q=T'\Sigma T$ for some orthogonal T and signature matrix Σ. Since we earlier showed $\overline{W}(s)=W_d(s)Q$, (42) is then immediate.

Let us turn now to the synthesis problem, to show how knowledge of a self-dual matrix fraction description will define a reciprocal synthesis.

We shall need a preliminary result, contained in Lemma 2 below. Lemma 2 in turn depends on Lemma 1.

Lemma 1

Let $W(s)=N(s)D^{-1}(s)$ be a minimal matrix fraction description with $\det D(s)$ nonzero in $\mathrm{Re}[s]\geqslant 0$. Suppose that

$$N(s)D^{-1}(s)=N(-s)D_d^{-1}(s)T'\Sigma T \tag{44}$$

for some orthogonal T and signature matrix Σ. Then their exists a unimodular $L(s)$ such that

$$\begin{aligned} N(s)L(s)&=N(-s)\\ D(s)L(s)&=T'\Sigma TD_d(s)\\ L(-s)L(s)&=I. \end{aligned} \tag{45}$$

Proof: Because $N(s)D^{-1}(s)$ is minimal and the degrees of $\det D(s)$ and $\det[T'\Sigma TD_d(s)]$ are the same, $N(-s)D_d^{-1}(s)T'\Sigma T$ is minimal. Then there exists a unimodular $L(s)$ satisfying the first two equations of (45).

Now $L(-s)L(s)$ is clearly polynomial. Also

$$\begin{aligned} L(-s)L(s)&=D^{-1}(-s)T'\Sigma TD_d(-s)D^{-1}(s)T'\Sigma TD_d(s)\\ &=D^{-1}(-s)[T'\Sigma TD_d(-s)D^{-1}(s)\\ &\quad\cdot T'\Sigma TD_d(s)D^{-1}(-s)]D(-s). \end{aligned}$$

Let $s\to\infty$ and use the dual definition to conclude that

$$\lim_{s\to\infty} L(-s)L(s)=I.$$

Equation (45) is immediate.

Lemma 2

Let $W(s)=N(s)D^{-1}(s)$ be a matrix fraction description with $D(s)$ nonzero in $\mathrm{Re}[s]\geqslant 0$, and suppose that (44) holds for some orthogonal T and signature matrix Σ. Suppose further that (45) holds for some unimodular $L(s)$, a sufficient condition being that $N(s)$, $D(s)$ be right coprime. Then

$$V(s)=T'\Sigma TD_d^{-T}(s)D'(-s) \tag{46}$$

is symmetric.

Proof: From the second equation of (45) we have

$$V(s)=D^{-T}(s)L^{-T}(s)D'(-s).$$

Using the third equation, we have $V(s)V(-s)=I$, while also it follows from duality that $V(s)V'(-s)=I$, whence the result.

With the aid of the result of Lemma 2, a reciprocal scattering matrix synthesis result is virtually immediate.

Theorem 5

Let $\Sigma_{11}(s)$ be a symmetric bounded real scattering matrix and suppose that a matrix fraction description ND^{-1} of a $W(s)$ satisfying (39) is known such that the hypotheses of Lemma 2 hold. Then the procedure of the last section for scattering matrix synthesis, with $\overline{W}(s)$ identified with $N(-s)D_d(s)T'\Sigma T=W(s)$, leads to

$$\Sigma(s)=\begin{bmatrix}\Sigma_{11}(s) & \Sigma_{12}(s)\\ \Sigma_{21}(s) & \Sigma_{22}(s)\end{bmatrix}$$

being symmetric and lossless bounded real.

Proof: The lossless bounded real property was established in the last section. The fact that $\Sigma_{12}(s)=\Sigma_{21}'(s)$ comes from the requirements of the synthesis procedure that

$$W(s)=[I-\Sigma_{11}(s)]^{-1}\Sigma_{12}(s)$$

$$\Sigma_{21}(s)=\overline{W}'(s)[I-\Sigma_{11}(s)]$$

together with the symmetry of $\Sigma_{11}(s)$. The fact that $\Sigma_{22}(s)=\Sigma_{22}'(s)$ comes from the formula—see (37) and (46)—

$$\Sigma_{22}=-\Sigma_{21}[I-\Sigma_{11}]^{-1}\Sigma_{12}+V$$

and the result of Lemma 2.

The effect of the above theorem is to reduce the lossy reciprocal synthesis problem to a lossless reciprocal synthesis problem, when a self-dual matrix spectral factor is available. Since lossless reciprocal synthesis is comparatively straightforward, the lossy synthesis problem is virtually solved.

Impedance synthesis runs almost as quickly as the scattering synthesis.

Theorem 6

Let $Z(s)$ be a prescribed $n\times n$ positive real symmetric matrix, and let $W(s)W'(-s)=\frac{1}{2}[Z(s)+Z'(-s)]$ with $W(s)$ analytic in $\text{Re}[s]\geq 0$. Suppose that $W(s)$ has a matrix fraction description $N(s)D^{-1}(s)$ such that

$$M_{22}(s)=[TD(s)+TD_d(-s)][TD(s)-TD_d(-s)]^{-1} \tag{47}$$

$$M_{12}(s)=N(s)[TD(s)-TD_d(-s)]^{-1}=-M_{21}'(-s) \tag{48}$$

$$M_{11}(s)=Z(s)-M_{12}(s)[I+M_{22}(s)]^{-1}M_{21}(s). \tag{49}$$

Then $M(s)$ is lossless positive real,

$$(I+\Sigma)M(s)=M'(s)(I+\Sigma) \tag{50}$$

and $M(s)$ possesses a reciprocal synthesis such that termination at the last m-ports of the synthesizing network in unit resistors produces an impedance matrix at the first n-ports of $Z(s)$.

Proof: In view of Theorem 2, and the remark following it indicating the possibility of replacing D and D_d by QD and QD_d for any orthogonal Q, we see that all that need be proved is (50). (From (50) and the lossless positive real property, it of course follows that a lossless reciprocal synthesis exists [34].)

Observe that

$$M_{22}=[TD(s)D_d^{-1}(s)T'+I][TD(s)D_d^{-1}(-s)T'-I]^{-1}$$
$$=[TVT'\Sigma+I][TVT'\Sigma-I]^{-1}$$

with the second equality coming from the symmetry of $V(s)$ in (46). Then

$$M_{22}'=[\Sigma TVT'-I]^{-1}[\Sigma TVT'+I]$$
$$=\Sigma[TVT'\Sigma-I]^{-1}\Sigma[\Sigma TVT'+I]$$
$$=\Sigma[TVT'\Sigma-I]^{-1}[TVT'\Sigma+I]\Sigma$$
$$=\Sigma M_{22}\Sigma.$$

Equally simple calculation yields $\Sigma M_{21}=M_{12}'$; then (49) and the symmetry of $Z(s)$ yield $M_{11}=M_{11}'$, whence (50) holds.

We might summarize the main synthesis ideas of the last two sections as follows. Nonreciprocal synthesis requires spectral factorizations and the construction of a dual. Reciprocal synthesis requires the construction of a self-dual spectral factor. In the next section, we review some known synthesis techniques from the point of view of these ideas, and also indicate systematic procedures for constructing self-dual spectral factors.

VI. Reapproachment with Known Synthesis Techniques

In this section, we study how in a representative number of known network synthesis techniques the required construction of dual or even self-dual spectral factors can be identified. We begin with nonreciprocal synthesis techniques.

A. Bayard Nonreciprocal Impedance Synthesis (see, e.g., [2])

A crucial step in this synthesis is to execute a spectral factorization given a prescribed $Z(s)$ of the form:

$$Z(s)+Z'(-s)=W(s)W'(-s) \tag{51}$$

with $W(s)=N(s)D^{-1}(s)$, and $D(s)$ is diagonal. Since diagonal polynomial matrices have duals $D_d(s)=\Sigma D(s)$, with Σ a signature matrix, the synthesis scheme outlined in Section III follows easily. A version of this synthesis with $D(s)=d(s)I$, $d(s)$ being a scalar polynomial, occurs in [9].

B. Scattering Function Synthesis [9]

Let $\Sigma_{11}(s)=(n(s))/m(s))$ be a bounded real function, and let $k(s)k(-s)=m(s)m(-s)-n(s)n(-s)$. Then $\Sigma_{11}(s)$

can be embedded in

$$\Sigma(s)=\begin{bmatrix} \dfrac{n(s)}{m(s)} & \dfrac{k(s)}{m(s)} \\ \dfrac{k(-s)}{m(s)} & -\dfrac{n(-s)}{m(s)} \end{bmatrix} \tag{52}$$

which is lossless bounded real.

Here, we have $W(s)=\Sigma_{12}(1-\Sigma_{11})^{-1}=k(m-n)^{-1}$, and $\overline{W}(s)=k(-s)[m(s)-n(s)]^{-1}$. Then $\Sigma_{21}=\overline{W}(1-\Sigma_{11})=k(-s)/n(s)$. The formula for Σ_{22} of Section IV also yields the correct expression $-n(-s)/m(s)$.

C. State Variable Synthesis [4]

As noted in Section II, the construction of dual polynomials and spectral factors can be executed in state-variable terms. Likewise, the results of Sections III and IV have their state-variable counterparts; these are effectively the synthesis schemes in [4].

We turn now to reciprocal networks, in which self-duality of the spectral factors is required.

D. Scalar Darlington Synthesis [1], [3]

Suppose that $z(s)+z(-s)=(k(s))/(m(s))\ (k(-s))/(m(-s))$. We choose $k(s)$ such that its zeros either occur in pairs symmetric with respect to the $j\omega$-axis, or in $\mathrm{Re}\, s \geqslant 0$. Then there exists a Hurwitz $l(s)$ such that $k(s)l(s)$ is even or odd; the transfer function kl/ml (viewed as an uncancelled fraction) is then self-dual, and the standard Darlington synthesis proceeds using this fact.

E. Reciprocal Scattering Function Synthesis [9]

Let $\Sigma_{11}(s)=n/m$ and let $k(s)$ have $k(s)k(-s)=m(s)n(-s)-n(s)n(-s)$ with $k(s)$ possessing all zeros in $\mathrm{Re}[s] \geqslant 0$. Then

$$\Sigma(s)=\begin{bmatrix} \dfrac{n(s)}{m(s)} & \dfrac{k(s)}{m(s)} \\ \dfrac{k(s)}{m(s)} & -\dfrac{k(s)n(-s)}{k(-s)m(s)} \end{bmatrix} \tag{53}$$

is lossless bounded real symmetric. This $\Sigma(s)$ can be thought of as arising from

$$W(s)=\Sigma_{12}(1-\Sigma_{11})^{-1}=\frac{k(s)k(-s)}{[m(s)-n(s)]k(-s)} \tag{54}$$

which has the self-dual property. (Note that the numerator is even and the denominator Hurwitz, unless $k(j\omega)=0$ for some real ω; then $j\omega$-axis zeros are cancelled between numerator and denominator.

Multiport reciprocal syntheses are of course harder. Let us review some of the techniques used. A standing assumption in the following is that $Z(s)=Z'(s)$ is a rational positive real matrix.

F. Koga's Impedance Synthesis [5]

The synthesis depends on the construction of a self-dual spectral factor in the following way. Let $Z(s)+Z'(-s)=W(s)W'(-s)$, where $W(s)$ is any spectral factor. Express $W(s)$ as $N(s)D^{-1}(s)$. Then it is readily checked that

$$\frac{1}{\sqrt{2}}\begin{bmatrix} N(s) & N(-s) \end{bmatrix}\begin{bmatrix} D(s) & 0 \\ 0 & D_d(s) \end{bmatrix}^{-1}$$

is also a spectral factor (the symmetry of $Z(s)$ is important), as well as being self-dual.

G. Bayard–Newcomb Reciprocal Impedance Synthesis [2]

The Bayard–Newcomb synthesis relies, as does the Bayard synthesis, on a Gauss factorization of $Z(s)+Z'(-s)$, i.e., a spectral factor construction of the form $W(s)=N(s)D^{-1}(s)$ where $N(s)$ is lower triangular and $D(s)$ is a diagonal matrix of Hurwitz polynomials. Let $d(s)=$least common multiple of the entries of $D(s)$ and let $M(s)=N(s)[d(s)ID^{-1}(s)]$, so that $W(s)=M(s)[d(s)I]^{-1}$. Using the lower triangular nature of $M(s)$ and the symmetry of $Z(s)+Z'(-s)$, we obtain easily $M(s)=M(-s)$, whence $M(s)[d(s)I]^{-1}$ is seen to be self-dual.

H. Self-Dual Construction from a Maximum Phase Spectral Factor

The following construction is different from, but motivated by, a procedure due to Oona and Yasuura described in [2]. Let $W(s)$ be a minimum phase spectral factor of $Z(s)+Z'(-s)$, and let $W_d(s)$ be the dual. Then $Z(s)+Z'(-s)=W_d(-s)W_d'(s)=W_d(s)W_d'(-s)$, the last equation holding on account of the symmetry. It follows by a fundamental result (see, e.g., [10]) that $W_d(s)=W(s)M(s)$ for some all pass $M(s)$. Now let $W(s)=N(s)D^{-1}(s)$, $M(s)=A^{-1}(s)B(s)$, and $W_d(s)=N(-s)D_d^{-1}(s)$ with the first two matrix fractions at least being coprime. Then

$$N(-s)D_d^{-1}(s)=N(s)D^{-1}(s)A^{-1}(s)B(s).$$

Now write $D^{-1}(s)A^{-1}(s)B(s)=E(s)F^{-1}(s)$ for some pair $E(s)$, $F(s)$ with $|F|=AD$; it is possible to show that E, F are coprime and that $N(s)E(s)F^{-1}(s)$ is self-dual. (The calculations are not difficult.) This construction is a matrix generalization of that used in the Darlington synthesis, where the spectral factor with self-dual realization is, when common factors have been excluded, "maximum" phase.

I. State Variable Construction [4]

Constructions of self-dual spectral factors using state-variable techniques are the basis of reciprocal synthesis procedures in [4]. One starts with an arbitrary spectral factor $\hat{W}(s)=\hat{J}+H'(sI-F)^{-1}\hat{G}$ and from it constructs another spectral factor $W(s)=J+H'(sI-F)^{-1}G$, which may have more columns and which is self-dual.

J. Some Remarks on the Various Synthesis Procedures

Of the multiport procedures discussed, Koga's is the simplest. His procedure also requires a self-dual spectral factor with McMillan degree at least twice the minimum, and a number of columns at least twice the minimum. In synthesis terms, at least twice the minimum number of reactive elements and twice the minimum number of resistors will be used. The Bayard–Newcomb procedure and the procedure based on the maximum phase spectral factor result in a (generally) nonminimal reactive element count and a minimal resistive count, and the state variable constructions in a minimal reactive element and a (generally) nonminimal resistive count. It is known [2] that reciprocal syntheses with simultaneously minimal reactive and resistive element counts are generally unobtainable.

VII. Concluding Remarks

The theme of this paper is that while determination of a spectral factor suffices to determine a synthesis of the associated network, knowledge of a pair of "dual" spectral factors serves to make many synthesis procedures more transparent. To put it another way, we have shown that the specific constructions used in several existing network synthesis procedures serve effectively to implicitly construct a dual spectral factor from the given one. This recognition can show how to relax certain constraints (e.g., to diagonal matrices) imposed in some of the algorithms.

While we have captured most known algorithms by our method, certain schemes still seem to elude it. These are scattering matrix syntheses that are readily stated in polynomial terms, but do not fit easily into the framework of this paper. For example, with $\Sigma_{11}(s)$ bounded real, let $\Sigma_{11}(s)=N(s)[d(s)I]^{-1}$, where $d(s)$ is a scalar. Let $K(s)$, $L(s)$ satisfy

$$I-\Sigma_{11}(s)\Sigma'_{11}(-s)=\frac{K(s)}{d(s)}\,\frac{K'(-s)}{d(s)}$$

$$I-\Sigma'_{11}(-s)\Sigma_{11}(s)=\frac{L(s)}{d(s)}\,\frac{L'(-s)}{d(s)}$$

with $K(-s)$ possessing full rank in $\mathrm{Re}[s]\geqslant 0$. Let $K^{-1}(s)$ denote a left inverse for $K(s)$, analytic in $\mathrm{Re}[s]\geqslant 0$. Then an embedding of $\Sigma_{11}(s)$ into a lossless bounded real $\Sigma(s)$ is provided by

$$\Sigma(s)=\begin{bmatrix}\dfrac{N(s)}{d(s)} & \dfrac{K(s)}{d(s)}\\[2ex] \dfrac{L'(s)}{d(s)} & \dfrac{-L'(-s)N'(-s)K^{-T}(-s)}{d(s)}\end{bmatrix}.$$

This result is apparently not easily derivable in the framework established in Section IV.

Nevertheless, we feel it is of interest that we have been able to relate certain stochastic process and thermodynamical ideas (of reversible models) to the classical subject of network synthesis.

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