Single-Channel Control of a Two-Channel System

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Abstract—Using elementary systems concepts, a new derivation is given of a recent result [1] characterizing those two-channel systems which can be made single-channel controllable with output feedback.

In this note we give a direct proof of the following theorem. The result is of importance to the synthesis of decentralized feedback laws for assigning the closed-loop spectrum of a multichannel linear system [2].

Theorem: Let $A_{n 	imes n}$, $B_{n 	imes m}$, $C_{p 	imes n}$, and $D_{n 	imes q}$ be fixed, real-valued matrices with $C(J-A)^{-1}B 
eq 0$. There exists a real-valued feedback matrix $F_{A}$ such that $(A + DF_{A}, B)$ is controllable if and only if

$$\text{rank} \begin{bmatrix} A - B C \end{bmatrix} > n, \quad \lambda \in \sigma(A)$$

(1)

and

$$\text{rank} \begin{bmatrix} A - B D \end{bmatrix} = n, \quad \lambda \in \sigma(A)$$

(2)

where $\sigma(A)$ is the spectrum of $A$.

Although this theorem is actually a simple consequence of certain more general results derived in [1], in comparison with [1], the direct proof given here has the virtue of being shorter, less technical, and more transparent.

The theorem's proof relies on two lemmas.

Lemma 1: If (1) and (2) are true, then the set

$$\mathbf{E} = \{ F : \text{rank} \begin{bmatrix} A - B C \end{bmatrix} = n, \lambda \in \sigma(A), F \in \mathbb{R}^{q 	imes p} \}$$

is robust \(^{1}\) in $\mathbb{R}^{q 	imes p}$.

Proof: Fix $\lambda \in \sigma(A)$. It will first be shown that there exists a matrix $F_{A}$ such that $\text{rank} \begin{bmatrix} A - B C \end{bmatrix} = n$. For this define

$$(A, B, C) = \begin{bmatrix} A - B C \end{bmatrix}, \quad (\lambda - A, B, C) = \begin{bmatrix} \lambda - A & B \end{bmatrix}$$

(3)

and let $\lambda B, \lambda C, \lambda F_{A}$ denote their respective column spans over $C$. By definition

$$\lambda B \subset \lambda C \subset \lambda F_{A}$$

(4)

and $X = X + \mathcal{E}$, respectively. The latter can be written as

$$\mathcal{E} \subset X \subset \lambda F_{A}$$

(5)

where $\mathcal{E}$ is a completion from $\mathcal{E} \cap \mathcal{X}$. To $\mathcal{E}$.

From (4) and (5) we note that $\text{dim}(\mathcal{E}) < \text{dim}(\mathcal{X}) < \text{dim}(X)$; this and (3) imply that $\text{dim}(\mathcal{E}) < \text{dim}(\mathcal{X}) < \text{dim}(X)$ and thus that $\text{dim}(\mathcal{E}) < \text{dim}(\mathcal{X})$. It follows that there exists a matrix $G$ satisfying

$$LG(\mathcal{X} \cap \mathcal{Y}) = \mathcal{E}$$

(6)

Since for any completion $\mathcal{E}$ from $\mathcal{X} \cap \mathcal{Y}$ to $\mathcal{E}$, we can write $\mathcal{X} \cap \mathcal{Y} = \mathcal{X} \subset \mathcal{X} \cap \mathcal{Y} \subset \mathcal{Y}$, $G$ can be chosen so that, in addition

$$G(\mathcal{X} \cap \mathcal{Y}) = 0$$

(7)

From (7), $G \mathcal{X} = 0$ implying that $G$ is of the form $[0_{q 	imes n}, G]$. Hence, if we define $F_{A} = -G$

$$R = \begin{bmatrix} A - B C & B \end{bmatrix}$$

(8)

then $R = [X + L G] M$; thus the column span of $R$, written $\mathcal{X}$, must satisfy $\mathcal{X} = [X + L G]. \mathcal{X}$. Observe that $XX' (\mathcal{X} \cap \mathcal{Y}) = 0$, that $\mathcal{X} = \mathcal{X} \cap \mathcal{Y} \subset \mathcal{X} \cap \mathcal{Y}$ and thus from (7) that $\mathcal{X} = L G (\mathcal{X} \cap \mathcal{Y}) = 0$. Hence, from the identity $XX' M = N$ and (6) we see that $\mathcal{X} = \mathcal{X} \cap \mathcal{Y}$. This and (5) imply that $\mathcal{X} = \mathcal{X}$, which in view of the definitions of $R$ and $X_{s}$ shows that $[X - A - B C] \cap \mathcal{X} = n$.

The preceding implies that for each fixed $\lambda$ the vector $z_{(\lambda)}(F)$ of all nth order minors of $(A - B C, B)$ must be nonzero at $F = F_{A}$. Since, in addition, the elements of $z_{(\lambda)}(F)$ are polynomial functions of the elements of $F$, $z_{(\lambda)}(F)$ is not identically zero. This proves that $z_{(\lambda)}(F) = (F \cap [X - A - B C])$ is robust in $\mathbb{R}^{q 	imes p}$ as is $\mathcal{E} = \cap_{\lambda \in \sigma(A) \in \mathcal{X}}$. Hence, if we define $F$ to be the largest subset of $\mathcal{E}$ which is a robust subset of $\mathbb{R}^{q 	imes p}$, then $F$ has the required property.

Lemma 2: Let $A, B, C, D$ be as in the preceding theorem with $(A, B, D)$ controllable. Write $\lambda_{F}$ for the uncontrollable spectrum of $(A + B D, C)$ and let $\mathcal{X} \subset \mathcal{Y}$ denote the robust subset of $\mathbb{R}^{q 	imes p}$ consisting of all real-valued matrices which maximize the dimension of the controllable space of $(A + B D, C)$. There exists a unique subset $\mathcal{X} \subset \sigma(A)$ and $\lambda_{F} = \lambda_{X}$ and $\lambda_{F} = \lambda_{Y}$ for all $F \in \mathcal{X} \subset \mathcal{Y}$.

Proof: Set $\mathcal{X} = \mathcal{X} \subset \mathcal{Y}$ and view $A, B, C, D$ as maps of real number spaces of appropriate dimensions. With $\mathcal{X} \subset \mathcal{Y}$, the observability space of $(C, A)$, and $\mathcal{X} \subset \mathcal{Y}$, the restriction of $A$ to $\mathcal{X} \subset \mathcal{Y}$, let $F : \mathcal{X} \subset \mathcal{Y}, \mathcal{X} \subset \mathcal{Y}$ denote the canonical projection, $A$ the map induced by $A$ in $\mathcal{X}$, $b_{A}$, $b_{B}$, $b_{D}$, and write $C$ for the unique solution to $C = b_{C}$. The construction implies that $A, C, D$ are observable, that $(A, b_{B}, b_{D})$ is controllable (since $\lambda \in \sigma(A)$), and that $b_{B} = 0$ (since $C(J - A)^{-1} = b_{B}$). Thus, by the main result of [3], there exists a map $F$ such that $(A + B D, C)$ is controllable. It follows if $F$ is the set of all real-valued $F$ with this property, then $F$ is robust in $\mathbb{R}^{q 	imes p}$. Since for all $F \in \mathbb{R}^{q 	imes p}$, $\sigma(A + B D, C) = \sigma(A, b_{B}, b_{D}) \cap \sigma(A \in C, A)$, it must be true that $\lambda_{F} \cap \sigma(A \in C, A)$ for all $F \in \mathcal{X} \subset \mathcal{Y}$. Noting that $\mathcal{X} \subset \mathcal{Y} \subset \sigma(A)$ and that $\mathcal{E} \cap \mathcal{F} = \emptyset$ is robust and thus nonempty, we can write

$$\lambda_{F} \subset \sigma(A), \quad F \in \mathcal{X} \subset \mathcal{Y} \subset \mathcal{E}$$

(8)

Since the number of elements of $\lambda_{F}$ is the same for all $F \in \mathcal{X}$, $\lambda_{F}$ must vary continuously on $\mathcal{Y}$ and thus on $\mathcal{X} \subset \mathcal{Y}$. This together with (8) and the finite cardinality of $\sigma(A)$ imply that $\lambda_{F}$ is independent of $F \in \mathcal{X} \subset \mathcal{Y}$. Since $\mathcal{X} \subset \mathcal{Y}$ is dense in $\mathcal{Y}$ and $\lambda_{F}$ is continuous on $\mathcal{Y}$, it must be true that $\lambda_{F} \subset \sigma(A \in C, A)$ for all $\mathcal{Y}$ in $\mathcal{X} \subset \mathcal{Y}$. Therefore, if for fixed $F \in \mathcal{X}$, we define $\mathcal{E} = \mathcal{X} \subset \mathcal{F}$ then $\lambda_{F}$ has the required property.

In the proof which follows we make repeated use of the well-known fact that for any matrix pair $(A_{n 	imes n}, B_{n 	imes m})$, rank $[A - B, B]$ is less than $n$ only for those values of $\lambda$ in the uncontrollable spectrum of $(A, B)$ [4].
Proof of Theorem—Necessity: If \((A + DFC, B)\) is controllable, then by [4], rank \(|\lambda I - A - DFC, B|\) = \(n\) for all \(\lambda\). From this, the matrix identities

\[
\begin{align*}
|\lambda I - A - DFC, B| &= |\lambda I - A, B, D| \\
&= \begin{bmatrix}
I & 0 \\
0 & I \\
-DFC & 0
\end{bmatrix}
\end{align*}
\]

and Sylvester's inequality, it follows that both (1) and (2) are true.

Sufficiency: Suppose (1) and (2) hold, thereby ensuring that \(A, B, C,\) and \(D\) satisfy the hypotheses of Lemmas 1 and 2. With \(\mathcal{F}\) and \(\mathcal{F}_*\) the robust subsets referred to in the statements of Lemmas 1 and 2, respectively, fix \(F_0 \in \mathcal{F} \cap \mathcal{F}_*\). If \((A + DFC, B)\) is not controllable, then for some uncontrollable eigenvalue \(\lambda_0\), rank \(|\lambda_0 I - A - DFC, B|\) < \(n\) (cf. [4]); but this leads to a contradiction since, by Lemma 2, \(\lambda_0 \in \sigma(A)\) and by Lemma 1, rank \(|\lambda I - A - DFC, B|\) = \(n\), \(\lambda \in \sigma(A)\). It follows that \((A + DFC, B)\) is controllable.

Another way of studying the problem discussed in this note is available in [5] which adopts an approach using matrix polynomials.

REFERENCES