Single-Channel Control of a Two-Channel System

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Abstract—**Using elementary systems concepts, a new derivation is given of a recent result [1] characterizing those two-channel systems which can be made single-channel controllable with output feedback.**

In this note we give a direct proof of the following theorem. The result is of importance to the synthesis of decentralized feedback laws for assigning the closed-loop spectrum of a multichannel linear system [2].

Theorem: Let $A_{n\times n}$, $B_{n\times m}$, $C_{p\times m}$, and $D_{n\times q}$ be fixed, real-valued matrices with $C(\lambda I-A)^{-1}B\neq 0$. There exists a real-valued feedback matrix $F_{q\times p}$ such that $(A+DFC,B)$ is controllable if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda I-A & B\\ C & 0\end{bmatrix}\geq n,\qquad \lambda\in\sigma(A)\tag{1}$$

and

$$\operatorname{rank}[\lambda I-A,B,D]=n,\qquad \lambda\in\sigma(A)\tag{2}$$

where $\sigma(A)$ is the spectrum of A.

Although this theorem is actually a simple consequence of certain more general results derived in [1], in comparison with [1], the direct proof given here has the virtue of being shorter, less technical, and thus more transparent.

The theorem's proof relies on two lemmas.

Lemma 1: If (1) and (2) are true, then the set

$$\mathcal{F}\equiv\{F\colon \operatorname{rank}[\lambda I-A-DFC,B]=n,\lambda\in\sigma(A),F\in\mathbb{R}^{q\times p}\}$$

is robust[1] *in $\mathbb{R}^{q\times p}$.*

Proof: Fix $\lambda\in\sigma(A)$. It will first be shown that there exists a matrix F_λ such that $\operatorname{rank}[\lambda I-A-DF_\lambda C,B]=n$. For this define

$$M=\begin{bmatrix}\lambda I-A & B\\ C & 0\end{bmatrix},\qquad N=\begin{bmatrix}\lambda I-A & B\\ 0 & 0_{p\times m}\end{bmatrix},$$

$$X=\begin{bmatrix}I_{n\times n}\\ 0_{p\times n}\end{bmatrix},\qquad Y=\begin{bmatrix}0_{n\times p}\\ I_{p\times p}\end{bmatrix},\qquad L=\begin{bmatrix}D\\ 0_{p\times q}\end{bmatrix},$$

and let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{L}$ denote their respective column spans over $\mathbb{C}$. By definition

$$\mathcal{N}\oplus\mathcal{Y}=\mathcal{M}+\mathcal{Y}.\tag{3}$$

In addition, (1) and (2) imply that

$$\dim(\mathcal{M})\geq\dim(\mathcal{X})\tag{4}$$

and $\mathcal{X}=\mathcal{N}+\mathcal{L}$, respectively. The latter can be written as

$$\mathcal{X}=\mathcal{N}\oplus\bar{\mathcal{L}}\tag{5}$$

where $\bar{\mathcal{L}}$ is a completion from $\mathcal{L}\cap\mathcal{N}$ to $\mathcal{L}$.

From (4) and (5) we note that $\dim(\bar{\mathcal{L}})\leq\dim(\mathcal{M})\text{-}\dim(\mathcal{N})$; this and (3) imply that $\dim(\bar{\mathcal{L}})\leq\dim(\mathcal{M})+\dim(\mathcal{Y})\text{-}\dim(\mathcal{M}+\mathcal{Y})$ and thus that $\dim(\bar{\mathcal{L}})\leq\dim(\mathcal{M}\cap\mathcal{Y})$. It follows that there exists a matrix G satisfying

$$LG(\mathcal{M}\cap\mathcal{Y})=\bar{\mathcal{L}}.\tag{6}$$

Since for any completion $\bar{\mathcal{Y}}$ from $\mathcal{M}\cap\mathcal{Y}$ to $\mathcal{Y}$, we can write $\mathcal{X}\oplus\mathcal{Y}=\mathcal{X}\oplus\mathcal{M}\cap\mathcal{Y}\oplus\bar{\mathcal{Y}}$, G can be chosen so that, in addition,

$$G(\mathcal{X}\oplus\bar{\mathcal{Y}})=0.\tag{7}$$

From (7), $G\mathcal{X}=0$ implying that G is of the form $[0_{q\times n},\bar{G}]$. Hence, if we define $F_\lambda=-\bar{G}$ and

$$R=\begin{bmatrix}\lambda I-A-DF_\lambda C, & B\\ 0 & 0\end{bmatrix}$$

then $R=[XX'+LG]M$; thus the column span of R, written $\mathcal{R}$, must satisfy $\mathcal{R}=[XX'+LG]\mathcal{M}$. Observe that $XX'(\mathcal{M}\cap\mathcal{Y})=0$, that $\mathcal{M}=\mathcal{M}\cap\mathcal{Y}\oplus\mathcal{M}\cap(\mathcal{X}+\bar{\mathcal{Y}})$ and thus from (7) that $\mathcal{R}=LG(\mathcal{M}\cap\mathcal{Y})+XX'\mathcal{M}$. Hence, from the identity $XX'M=N$ and (6) we see that $\mathcal{R}=\bar{\mathcal{L}}\oplus\mathcal{N}$. This and (5) imply that $\mathcal{R}=\mathcal{X}$, which in view of the definitions of R and X, shows that $\operatorname{rank}[\lambda I-A-DF_\lambda C,B]=n$.

The preceding implies that for each fixed λ the vector $z_\lambda(F)$ of all nth order minors of $[\lambda I-A-DFC,B]$ must be nonzero at $F=F_\lambda$. Since, in addition, the elements of $z_\lambda(F)$ are polynomial functions of the elements of F, $z_\lambda(F)$ is not identically zero. This proves that $\mathcal{F}_\lambda\equiv\{F\colon \operatorname{rank}[\lambda I-A-DFC,B]=n,\ F\in\mathbb{C}^{q\times p}\}$ is robust in $\mathbb{C}^{q\times p}$ as is $\bar{\mathcal{F}}\equiv\cap_{\lambda\in\sigma(A)}\mathcal{F}_\lambda$. Hence, if we define $\mathcal{F}$ to be the largest subset of $\bar{\mathcal{F}}$ which is a robust subset of $\mathbb{R}^{q\times p}$, then $\mathcal{F}$ has the required property. □

Lemma 2: Let (A, B, C, D) be as in the preceding theorem with $(A,[B,D])$ controllable. Write Λ_F for the uncontrollable spectrum of $(A+DFC,B)$ and let $\mathcal{F}^*$ denote the robust subset of $\mathbb{R}^{q\times p}$ consisting of all real-valued matrices which maximize the dimension of the controllable space of $(A+DFC, B)$. There exists a unique subset $\Lambda^*\subset\sigma(A)$ such that $\Lambda_F=\Lambda^*$ for all $F\in\mathcal{F}^*$.

Proof: Set $\mathcal{X}=\mathbb{R}^n$ and view A, B, C, D as maps of real number spaces of appropriate dimensions. With $[C|A]$, the unobservable space of (C,A), and $A|[C|A]$, the restriction of A to $[C|A]$, let $P\colon\mathcal{X}\to\mathcal{X}/[C|A]$ denote the canonical projection, $\bar{A}$ the map induced by A in $\mathcal{X}/[C|A]$, $\bar{B}=PB$, $\bar{D}=PD$, and write $\bar{C}$ for the unique solution to $C=\bar{C}P$. The construction implies that $(\bar{C},\bar{A})$ is observable, that $(\bar{A},[\bar{B},\bar{D}])$ is controllable (since $(A,[B,D])$ is), and that $\bar{B}\neq 0$ (since $C(\lambda I-A)^{-1}B\neq 0$. Thus, by the main result of [3], there exists a map F such that $(\bar{A}+\bar{D}F\bar{C},\bar{B})$ is controllable. It follows that if $\mathcal{F}^\dagger$ is the set of all real-valued F with this property, then $\mathcal{F}^\dagger$ is robust in $\mathbb{R}^{q\times p}$. Since for all $F\in\mathbb{R}^{q\times p}$, $\sigma(A+DFC)=\sigma(\bar{A}+\bar{D}F\bar{C})\cup\sigma(A|[C|A])$, it must be true that $\Lambda_F\subset\sigma(A|[C|A])$ for all $F\in\mathcal{F}^\dagger$. Noting that $\sigma(A|[C|A])\subset\sigma(A)$ and that $\mathcal{F}^\dagger\cap\mathcal{F}^*$ is robust and thus nonempty, we can write

$$\Lambda_F\subset\sigma(A),\qquad F\in\mathcal{F}^\dagger\cap\mathcal{F}^*.\tag{8}$$

Since the number of elements of Λ_F is the same for all $F\in\mathcal{F}^*$, Λ_F must vary continuously on $\mathcal{F}^*$ and thus on $\mathcal{F}^\dagger\cap\mathcal{F}^*$. This together with (8) and the finite cardinality of $\sigma(A)$ imply that Λ_F is independent of $F\in\mathcal{F}^\dagger\cap\mathcal{F}^*$. Since $\mathcal{F}^\dagger\cap\mathcal{F}^*$ is dense in $\mathcal{F}^*$ and Λ_F is continuous on $\mathcal{F}^*$, it must also be true that Λ_F is independent of $F\in\mathcal{F}^*$. Therefore, if for fixed $F_0\in\mathcal{F}^*$, we define $\Lambda^*\equiv\Lambda_{F_0}$, then Λ^* has the required property. □

In the proof which follows we make repeated use of the well-known fact that for any matrix pair $(A_{n\times n},B_{n\times m})$, $\operatorname{rank}[\lambda I-A,B]$ is less than n only for those values of λ in the uncontrollable spectrum of (A,B) [4].

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[1] A subset of $\mathbb{R}^{q\times p}$ (respectively $\mathbb{C}^{q\times p}$) is a *robust* subset (i.e., Zariski open set) of $\mathbb{R}^{q\times p}$ (respectively $\mathbb{C}^{q\times p}$) if it is nonempty and if its complement is the set of solutions in $\mathbb{R}^{q\times p}$ (respectively $\mathbb{C}^{q\times p}$) to a finite set of polynomial equations in the elements of $F_{q\times p}$. Such sets are open and dense in $\mathbb{R}^{q\times p}$ (respectively $\mathbb{C}^{q\times p}$), and each robust subset of $\mathbb{C}^{q\times p}$ contains a largest subset which is a robust subset of $\mathbb{R}^{q\times p}$. The intersection of two robust subsets of $\mathbb{R}^{q\times p}$ (respectively $\mathbb{C}^{q\times p}$) is also robust in $\mathbb{R}^{q\times p}$ (respectively $\mathbb{C}^{q\times p}$).

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Proof of Theorem—Necessity: If $(A+DFC,B)$ is controllable, then by [4], rank $[\lambda I-A-DFC,B]=n$ for all λ. From this, the matrix identities

$$[\lambda I-A-DFC,B]=[\lambda I-A,B,D]\begin{bmatrix} I & 0 \\ 0 & I \\ -FC & 0 \end{bmatrix}$$

$$[\lambda I-A-DFC,B]=[I,-DF]\begin{bmatrix} \lambda I-A & B \\ C & 0 \end{bmatrix}$$

and Sylvester's inequality, it follows that both (1) and (2) are true.

Sufficiency: Suppose (1) and (2) hold, thereby ensuring that A, B, C, and D satisfy the hypotheses of Lemmas 1 and 2. With $\mathfrak{F}$ and $\mathfrak{F}^*$ the robust subsets referred to in the statements of Lemmas 1 and 2, respectively, fix $F_0\in\mathfrak{F}\cap\mathfrak{F}^*$. If $(A+DF_0C,B)$ is not controllable, then for some uncontrollable eigenvalue λ_0, rank $[\lambda_0 I-A-DF_0C,B]<n$ {cf. [4]}; but this leads to a contradiction since, by Lemma 2 $\lambda_0\in\sigma(A)$ and by Lemma 1, rank $[\lambda I-A-DFC,B]=n$, $\lambda\in\sigma(A)$. It follows that $(A+DF_0C,B)$ is controllable. □

Another way of studying the problem discussed in this note is available in [5] which adopts an approach using matrix polynomials.

References

[1] J. P. Corfmat and A. S. Morse, "Control of linear systems through specified input channels," *SIAM J. Contr. Optimiz.*, vol. 14, pp. 163–175, Jan. 1976.

[2] ——, "Decentralized control of linear multivariable systems," *Automatica*, vol. 12, pp. 479–495, Sept. 1976.

[3] C. Y. Ding, F. M. Brasch, Jr., and J. B. Pearson, "On multivariable linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 96–97, Feb. 1970.

[4] M. L. J. Hautus, "Controllability and observability conditions of linear autonomous systems," *Kon. Ned. Akad. Wetensch.*, series A, vol. 72, no. 5, 1969.

[5] D. J. Clements, "Remnant polynomials—A polynomial matrix approach," submitted for publication.