The Time-Variable Lattice and Nonreciprocal RLC Networks

Because the lattice network is of fundamental importance in classical network design [1], page 251, [2], page 338, one expects it also be useful for time-variable synthesis. As this is indeed the case [3], we consider here some useful properties of the lattice which extend to the time-variable case. With a special choice of lattice arms this allows us to extend the examples of [4] and [5] to include a transformless RLC nonreciprocal network. Likewise, this readily allows the synthesis in cascade lattice form of a subclass of passive voltage transfer functions.

Consider the 2n-port symmetric lattice shown embedded between normalized sources and loads in Fig. 1; here \( \mathbf{N}_s \) denotes an \( n \times n \) matrix of passive voltage transfer functions. As a consequence [7], page 11, \( \mathbf{s} = \mathbf{T}\mathbf{s}\mathbf{T} \) is an expression for the scattering matrix of the lattice, which must then take the same form as \( \mathbf{z} \)

\[
\mathbf{z} = \frac{1}{2} \begin{bmatrix}
    z_0 + z_4 & z_3 & z_2 & z_1
    z_3 & z_5 & z_6 & z_7
    z_2 & z_6 & z_8 & z_9
    z_1 & z_7 & z_9 & z_0
\end{bmatrix}
\]

where \( z_0 \) is the \( n \times n \) zero matrix. Since \( \mathbf{z} \) is of the form \( \mathbf{z} = \mathbf{z}_0 \mathbf{T} \), where \( \mathbf{T} \) is the transpose of \( \mathbf{T} \), the lattice described by the \( 2n \times 2n \) impedance matrix \( \mathbf{z}(t, \tau) \).

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\end{bmatrix}
\]

If the lattice arms are dual, that is, \( s_n = -s_{-n} \), then

\[
\mathbf{s} = \frac{1}{2} \begin{bmatrix}
    s_0 & s_1 & s_2 & s_3
    s_1 & s_0 & s_3 & -s_2
    s_2 & s_3 & s_0 & s_1
    s_3 & -s_2 & s_1 & s_0
\end{bmatrix}
\]

Using \( \mathbf{e}_1 \) to denote composition [8], section 2, we directly calculate for Fig. 1

\[
2v(t) = s_{11} \cdot e_1 = \int_{0}^{t} s_{11}(t, \tau) e_1(\tau) \, d\tau
\]

\[
\mathbf{v}_2 = \frac{3}{2} \mathbf{s}_o \cdot \mathbf{e}_1 = \mathbf{s}_o \cdot \mathbf{v}_1
\]

or, the voltage transfer matrix \( \mathbf{s}_{2n} \) is one-half the scattering matrix of the cross arms, \( \mathbf{s}_{2n} \) in this case.

We conclude that, in contrast to the scalar time-invariant case, the cascade of two or more 2n-port time-variable lattices is generally no longer another lattice. Nevertheless, (5) shows how certain time-variable voltage transfer matrices can be synthesized.

From the above results and the present development of time-variable network synthesis we can give a synthesis of voltage transfer matrices if \( \mathbf{s}_{2n} \) specified by (4a), is passive [8], section 2, and quasi-lossless, that is, \( \mathbf{s}_o \) satisfies

\[
\mathbf{s}_{21} \mathbf{s}_{21} = \delta \mathbf{I}_n
\]

where \( \delta = \delta(t - \tau) \) is the unit impulse and

\[
\mathbf{s}_{21}(t, \tau) = \mathbf{A}(t) \delta(t - \tau) - \mathbf{B}(t) \mathbf{B}^*(\tau) u(t - \tau)
\]

with \( \mathbf{u} \) the unit step function and \( \mathbf{A}, \mathbf{B} \), and \( \mathbf{B}^* \) \( n \times n \) matrices, respectively, of infinitely differentiable entries. Equation (6a) states that the network is described by differential equations, while (6a) states that the network, being passive, is nondissipative. When \( \mathbf{s}_o \) is passive and quasi-lossless, it can be factored into the product of (real) first-degree quasi-lossless passive \( \mathbf{s}_o \)

\[
\mathbf{s}_{21} = \mathbf{s}_{o1} \mathbf{s}_{o2} \cdots \mathbf{s}_{on} \mathbf{s}_o
\]

for which each \( \mathbf{s}_o \) can be realized using one unit inductor or capacitor and a time-variable transformer bank [3]. Noting the (2, 1) term of (2a) shows that this factorization gives a realization in cascade lattices form of a voltage transfer matrix which is passive and quasi-lossless (after multiplication by 2). Such a synthesis is a complete generalization of time-invariant all-pass lattice synthesis to the time-variable case.

With \( n = 1 \), now consider Fig. 2, redrawn as Fig. 3, with \( N_2 \) and \( N_3 \) inductors of inductances \( l_2(t) \) and \( l_3(t) \), respectively. Then, \( s_1 \) and \( s_2 \) are known as [6]

\[
s_i(t, \tau) = \delta(t - \tau) - \frac{2}{l_i(t)} \exp \left[ -\int_{0}^{\tau} \frac{dl}{l_i(\xi)} \right] u(t - \tau)
\]

and we calculate from (5b)

\[
\mathbf{s}_o \mathbf{e}_2 = \mathbf{s}_o \mathbf{e}_1
\]
transformers and gyrators). Intuitively, such a situation can be justified by considering the lattice as approximating delay. Consequently, for general nonconstant $L \neq L_0$ we have $s \omega L \neq s \omega L_0$, and Fig. 2 will have a nonsymmetric scattering matrix (as well as nonsymmetric $z$). Thus, using a cascade of lattices with inductor and capacitor arms we can obtain a nonreciprocal 2-port (without transformers or gyrators). Intuitively, such a situation can be justified by considering the lattice as approximating delay. A pulse first sent into the right-hand section of Fig. 3 will be delayed differently than one sent into the left-hand section, due to the time variation of the elements. Consequently, transmission in one direction differs from that in the other, showing nonreciprocity. As suggested by Desoer [9], nonreciprocity (without transformers or gyrators) can be also obtained by modulating a resistance value.

We point out that a constant resistance 1-port constructed as a lattice with nonlinear and time-variable arms has been discussed by Desoer and Wong [10], but not with transfer function synthesis in mind. Although the noncommutativity of time-variable systems is known [11], page 335, neither this nor the example of (nonlinear) modulators preceding or following an IF strip is within the framework of RLC interconnections, as discussed here.

In summary, we have shown how some of the useful properties of 2-port lattices extend to the time-variable n-port case. However, not all properties extend and one can obtain nonreciprocity by cascading two time-variable lattices.

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