The Time-Variable Lattice and Nonreciprocal RLC Networks

Because the lattice network is of fundamental importance in classical network design [1], page 251, [2], page 338, one expects it should also be useful for time-variable synthesis. As this is indeed the case [3], we consider here some useful properties of the lattice which extend to the time-variable case. With a special choice of lattice arms this allows us to extend the examples of [4] and [5] to include a transformerless RLC nonreciprocal network. Likewise this readily allows the synthesis in cascade lattice form of a subclass of passive voltage transfer functions.

Consider the $2n$-port symmetric lattice shown embedded between normalized sources and loads in Fig. 1; here $\mathbf{N}_a$ denotes an n-port with time-variable scattering matrix [6] $\mathbf{s}_a(t, \tau)$ and impedance matrix $\mathbf{z}_a(t, \tau)$, similarly for $\mathbf{N}_b$; $\mathbf{1}_n$ denotes n uncoupled unit resistors, or, in matrix terms, the $n \times n$ identity. By direct calculation one

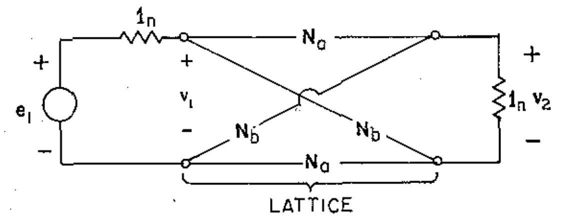

Fig. 1. $2n$-port lattice between sources and loads.

finds the lattice described by the $2n \times 2n$ impedance matrix $\mathbf{z} = \mathbf{z}(t, \tau)$

$$\mathbf{z} = \frac{1}{2}\begin{bmatrix} \mathbf{z}_b + \mathbf{z}_a & \mathbf{z}_b - \mathbf{z}_a \\ \mathbf{z}_b - \mathbf{z}_a & \mathbf{z}_b + \mathbf{z}_a \end{bmatrix} \tag{1a}$$

$$= \left\{\frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{1}_n \end{bmatrix}\right\}\begin{bmatrix} \mathbf{z}_a & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{z}_b \end{bmatrix}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{1}_n \end{bmatrix}\right\} \tag{1b}$$

where $\mathbf{0}_n$ is the $n \times n$ zero matrix. Since $\mathbf{z}$ is of the form $\mathbf{z} = \tilde{\mathbf{T}}\mathbf{z}_l\mathbf{T}$, with $\tilde{}$ denoting the transpose and $\mathbf{T}$ orthogonal, $\mathbf{z}$ can be also realized by terminating an orthogonal transformer of turns ratio matrix $\mathbf{T}$ by a load of impedance matrix $\mathbf{z}_l$, where

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{1}_n \end{bmatrix}, \qquad \mathbf{z}_l = \begin{bmatrix} \mathbf{z}_a & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{z}_b \end{bmatrix}. \tag{1c}$$

As a consequence [7], page 11, $\mathbf{s} = \tilde{\mathbf{T}}\mathbf{s}_l\mathbf{T}$ is an expression for the scattering matrix of the lattice, which must then take the same form as $\mathbf{z}$

$$\mathbf{s} = \frac{1}{2}\begin{bmatrix} \mathbf{s}_b + \mathbf{s}_a & \mathbf{s}_b - \mathbf{s}_a \\ \mathbf{s}_b - \mathbf{s}_a & \mathbf{s}_b + \mathbf{s}_a \end{bmatrix}. \tag{2}$$

If the lattice arms are dual, that is, $\mathbf{s}_a = -\mathbf{s}_b$, then

$$\mathbf{s} = \begin{bmatrix} \mathbf{0}_n & \mathbf{s}_b \\ \mathbf{s}_b & \mathbf{0}_n \end{bmatrix}. \tag{3}$$

Using $\cdot$ to denote composition [8], section 2, we directly calculate for Fig. 1

$$2\mathbf{v}_2(t) = \mathbf{s}_{21}\cdot\mathbf{e}_1 = \int_{-\infty}^{\infty} \mathbf{s}_{21}(t, \tau)\mathbf{e}_1(\tau)\, d\tau \tag{4a}$$

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where $\mathbf{s}_{21}$ is the (2, 1) $n \times n$ submatrix of the scattering matrix for any $2n$-port in the lattice position. In particular, for the lattice with dual arms we find, from (3) and (4a),

$$\mathbf{v}_2 = \tfrac{1}{2}\mathbf{s}_b\cdot\mathbf{e}_1 = \mathbf{s}_b\cdot\mathbf{v}_1 \tag{4b}$$

or, the *voltage transfer matrix* $\frac{1}{2}\mathbf{s}_{21}$ is one-half the scattering matrix of the cross arms, $\frac{1}{2}\mathbf{s}_b$ in this case.

If we now consider the cascade connection, shown in Fig. 2, of lattices with dual arms (denoted by a superscript d), then applying source and load at either end and using (3) and (4) (which show a match at all ports) gives for the network of Fig. 2,

$$\mathbf{s} = \begin{bmatrix} \mathbf{0}_n & \mathbf{s}_1 o \mathbf{s}_2 \\ \mathbf{s}_2 o \mathbf{s}_1 & \mathbf{0}_n \end{bmatrix} \tag{5a}$$

where o denotes Volterra composition [8], section 2, and

$$\mathbf{s}_1 o \mathbf{s}_2 = \int_{-\infty}^{\infty} \mathbf{s}_1(t, \lambda)\mathbf{s}_2(\lambda, \tau)\, d\lambda. \tag{5b}$$

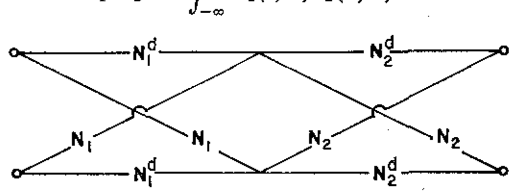

Fig. 2. Cascade of lattices with dual arms.

We conclude that, in contrast to the scalar time-invariant case, the cascade of two or more $2n$-port time-variable lattices is generally no longer another lattice. Nevertheless, (5) shows how certain time-variable voltage transfer matrices can be synthesized.

From the above results and the present development of time-variable network synthesis we can give a synthesis of voltage transfer matrices if $\mathbf{s}_{21}$, specified by (4a), is passive [8], section 4, and quasi-lossless, that is, $\mathbf{s}_{21}$ satisfies [3]

$$\mathbf{s}_{21a}^{a} o \mathbf{s}_{21} = \delta\mathbf{1}_n \tag{6a}$$

where $\delta = \delta(t - \tau)$ is the unit impulse and

$$\mathbf{s}_{21}(t, \tau) = \mathbf{A}(t)\,\delta(t - \tau) + \mathbf{\Phi}(t)\tilde{\mathbf{\Psi}}(\tau)u(t - \tau) \tag{6b}$$

$$\mathbf{s}_{21a}^{a}(t, \tau) = \tilde{\mathbf{A}}(t)\,\delta(t - \tau) - \mathbf{\Psi}(t)\tilde{\mathbf{\Phi}}(\tau)u(t - \tau) \tag{6c}$$

with u the unit step function and $\mathbf{A}$, $\mathbf{\Phi}$, and $\mathbf{\Psi}$ $n \times n$, $n \times r$, and $n \times r$ matrices, respectively, of infinitely differentiable entries. Equation (6b) states that the network is described by differential equations, while (6a) states that the network, being passive, is nondissipative. When $\mathbf{s}_{21}$ is passive and quasi-lossless, it can be factored into the product of (real) first-degree quasi-lossless passive $\mathbf{s}_i$,

$$\mathbf{s}_{21} = \mathbf{s}_m o \mathbf{s}_{m-1} o \cdots o \mathbf{s}_2 o \mathbf{s}_1 \tag{7}$$

for which each $\mathbf{s}_i$ can be realized using one unit inductor or capacitor and a time-variable transformer bank [3]. Noting the (2, 1) term of (5a) shows that this factorization gives a realization in cascade lattice form of a voltage transfer matrix which is passive and quasi-lossless (after multiplication by 2). Such a synthesis is a complete generalization of time-invariant all-pass lattice synthesis to the time-variable case.

With $n = 1$, now consider Fig. 2, redrawn as Fig. 3, with N_1 and N_2 inductors of inductances $l_1(t)$ and $l_2(t)$, respectively. Then, s_1 and s_2 are known as [6]

$$s_i(t, \tau) = \delta(t - \tau) - \frac{2}{l_i(t)} \exp\left[-\int_{\tau}^{t} \frac{d\lambda}{l_i(\lambda)}\right]u(t - \tau) \tag{8a}$$

and we calculate from (5b)

$s_1 o s_2 - s_2 o s_1$

$$= \frac{4}{l_1(t)} \int_\tau^t \exp\left[-\int_\sigma^t \frac{d\lambda}{l_1(\lambda)}\right] \frac{1}{l_2(\sigma)} \exp\left[-\int_\tau^\sigma \frac{d\lambda}{l_2(\lambda)}\right] d\sigma$$
$$- \frac{4}{l_2(t)} \int_\tau^t \exp\left[-\int_\sigma^t \frac{d\lambda}{l_2(\lambda)}\right] \frac{1}{l_1(\sigma)} \exp\left[-\int_\tau^\sigma \frac{d\lambda}{l_1(\lambda)}\right] d\sigma. \tag{8b}$$

Consequently, for general nonconstant $l_1 \neq l_2$ we have $s_1 o s_2 \neq s_2 o s_1$, and Fig. 2 will have a nonsymmetric scattering matrix (as well as nonsymmetric **z**). Thus, using a cascade of lattices with inductor and capacitor arms we can obtain a *nonreciprocal* 2-port (without transformers and gyrators). Intuitively, such a situation can be justified by considering the lattice as approximating delay. A pulse first sent into the right-hand section of Fig. 3 will be delayed differently than one sent into the left-hand section, due to the time variation of the elements. Consequently, transmission in one direction differs from that in the other, showing nonreciprocity. As suggested by Desoer [9], nonreciprocity (without transformers or gyrators) can be also obtained by modulating a resistance value.

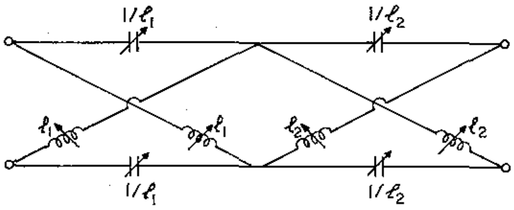

Fig. 3. A generally nonreciprocal 2-port.

We point out that a constant resistance 1-port constructed as a lattice with nonlinear and time-variable arms has been discussed by Desoer and Wong [10], but not with transfer function synthesis in mind. Although the noncommutativity of time-variable systems is known [11], page 395, neither this nor the example of (nonlinear) modulators preceding or following an IF strip is within the framework of RLC interconnections, as discussed here.

In summary, we have shown how some of the useful properties of 2-port lattices extend to the time-variable n-port case. However, not all properties extend and one can obtain nonreciprocity by cascading two time-variable lattices.

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