Triangularization Technique for the Design of Multivariable Control Systems

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Abstract—**This paper presents a novel technique for the design of multivariable control systems. A stable and proper precompensator is to be determined for a multivariable plant such that the compensated plant transfer function matrix is triangular and diagonally dominant in a nonstandard way. As a consequence of the triangular-diagonal-dominance property, only the diagonal elements need to be considered in an overall closed-loop design. In effect, the technique provides a systematic procedure to reduce a multivariable design problem to independent scalar design problems.**

I. Introduction

Inspired from the broad philosophy of the British School (Rosenbrock, MacFarlane, Mayne, etc.), this paper is concerned with the problem of reducing a multivariable feedback controller design problem to scalar design problems. A number of multivariable techniques are currently available, as for example, those known as the inverse Nyquist array [1], [2], the characteristic locus [3]–[5], and the sequential return difference [6]. The existence of more than one technique underlines the fact that no one technique is universally more attractive than the others. Also the fact that each requires a trial and error procedure in order to decompose the multivariable problem into a series of scalar design problems indicates that a certain prerequisite design experience is helpful and may be necessary for the methods to be successfully applied. It would be clearly desirable to develop a multivariable design approach which is simple, systematic, and not requiring any prior design experience other than for scalar design and which can be implemented on a broader class of problems than is covered by the existing techniques. In this paper we work towards this end.

In order to work systematically with multivariable system transfer matrices, we exploit the concept of triangularization. Let us denote $G(s)$ as the open-loop plant transfer matrix, and $U(s)$ as a compensator transfer matrix. The triangularization approach seeks a realizable, stable, and preferably minimum phase precompensator $U(s)$ such that $G(s)U(s)$ is triangular. Then one seeks to apply classical design techniques in a sequence of scalar designs to obtain a further diagonal precompensator $K(s)$. Unity negative feedback is then applied. However, without additional constraints on the triangular transfer function matrix, it is not clear how $K(s)$ should be found. In order to avoid this difficulty, we impose the constraint that $G(s)U(s)$ be not only triangular but also diagonal dominant in a nonstandard sense: off-diagonal elements of $G(s)U(s)$ must be such that, when ignored in applying the classical designs, allowing the classical designs to be implemented in any sequence and independently of one another, the design is almost as acceptable as when there is an attempt to take into account the off-diagonal terms. It should be noted that the motivation for the introduction and definition of the diagonal dominance concept here is somewhat different than in [1]. For this reason, we shall use the terminology triangular-diagonal-dominant.

In any multivariable design procedure, it is, of course, crucial that the precompensator $U(s)$ be realizable, and it is highly desirable that it be stable and minimum phase. Thus, we seek a mechanism to design $U(s)$ to satisfy these constraints as well as achieve triangularization and triangular-diagonal-dominance of $G(s)U(s)$. To do this, we follow an idea exploited in Anderson and Hung [7]: the set of proper, stable, real rational transfer functions forms a principal ideal domain [8], [9]—actually a Euclidean domain. (Since all transfer functions encountered in the paper will be real rational, we shall henceforth omit this qualification.) Let us denote this special Euclidean domain by $\mathcal{P}_S$. A unimodular matrix over $\mathcal{P}_S$ (i.e., a matrix whose inverse also belongs to $\mathcal{P}_S$) is realizable, stable, and minimum phase. Thus, we initially search for suitable $U(s)$ in the class of unimodular matrices over $\mathcal{P}_S$, and mathematical tools allow us to calculate $U(s)$ by direct manipulation, so as to achieve as far as possible the design objectives. Diagonal compensators are then used to further the achieving of the design objectives.

Since the design methods of this paper are related to Rosenbrock's techniques, more specific comparisons are now made. Rosenbrock has demonstrated in numerous industrial applications the power of his diagonal dominance techniques [10]. Roughly, he seeks a precompensator $C(s)$ such that both $G(s)C(s)$ and the unity negative feedback closed-loop matrix $R(s)$ (or $R(s)$ alone) are diagonal dominant so as to allow the multivariable problem to be decomposed into a set of classical design problems which ignore the off-diagonal elements of $G(s)C(s)$. There is no *a priori* guarantee that such a precompensator $C(s)$ exists nor that a trial and error procedure for selecting an appropriate $C(s)$ will succeed, although when trial and error procedures have been applied, they seem to have been virtually universally successful. A comparison with our techniques shows that we are able, by introducing the triangularization constraint, to relax the diagonal dominance definition and eliminate the diagonal dominance restriction on $R(s)$ to ensure that a direct procedure to achieve a suitable precompensator $C(s)$ can be found. On the other hand, it should be conceded that in simple examples, the precompensator complexity achieved by Rosenbrock is simpler than achieved here, although there may be a consequent degradation of performance.

Anderson and Hung [7] have set the multivariable design problem into the framework of finding canonical forms of matrices over Euclidean domains. The Smith canonical form led us to give serious thought to the possibility of developing diagonalization procedures [8], [11] based on it, but it turns out that a postcompensator is often needed as well as a precompensator and thus is clearly a disadvantage. Moreover, the resulting designs often appear to be quite complex. Eliminating the notion of using a postcompensator suggests that one bases a design on the Hermite form [7]. Again, in design terms, the technique can still be complicated and unnecessarily so. Consequently, we develop our special triangularization technique which gives a simpler compensator and allows more design freedom. It should be noted that, as for the Smith or Hermite form techniques, our triangularization technique works in any Euclidean domain.

The structure of this paper is as follows. In the next section, we present in detail the multivariable control system design technique. Section III is concerned with worked examples in order to compare the new technique against those developed earlier. Finally, we include concluding remarks and several open questions for future research.

An Appendix is included which explains the necessary background facts relating to Euclidean domains. It also describes an aspect of the compensator design procedure that is related to the Hermite form concept.

II. Triangularization Design Technique

We present in this section a procedure for triangularizing a proper square $n \times n$ plant $G(s)$ via precompensation. We aim for the resulting triangular matrix to have an additional property. If off-diagonal terms are neglected and a stable closed-loop design is then developed using a second diagonal precompensator and unity negative feedback, the set-up with off-diagonal terms present remains stable.

Manuscript received April 26, 1978; revised November 13, 1978. Paper recommended by M. K. Sain, Past Chairman of the Linear Systems Committee. This work was supported by the Australian Research Grants Committee.

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0018-9286/79/0600-0455$00.75 ©1979 IEEE

Phase 1: Triangularization

If $G(s)$ is stable,[1] using the procedure of the Appendix we can find a precompensator $U(s)$ with $U(s)$ and $U^{-1}(s)$ proper and stable and such that

$$P(s)=G(s)U(s) \tag{2.1}$$

is triangular. If $G(s)$ is not stable, the following device is used. Find $L(s)=\operatorname{diag}[l_i(s)/m_i(s)]$ such that $L(s)G(s)$ is proper, and after cancellations, stable. Find $U(s)$ such that LGU is triangular; because L is diagonal, GU is triangular.

Phase 2: Triangular-Diagonal-Dominance and Its Checking

For any proper rational transfer function matrix $X(s)$, let $X_+(s)$ denote the sum of those strictly proper terms in a partial fraction expansion of $X(s)$ with poles in $\operatorname{Re}[s]\geq 0$. Let $\delta[X(s)]$ denote the McMillan degree of $X(s)$ (= dimension of a minimal state variable realization = degree of determinant of denominator in a coprime matrix fraction description [12], [13] = degree of the least common denominator monic polynomial of all minors of X [14]).

For any square transfer function matrix $X(s)$, let

$$\tilde{X}(s)=\operatorname{diag}[x_{11}(s),\cdots,x_{nn}(s)].$$

We say that *a triangular matrix $P(s)$ is triangular-diagonal-dominant* (TDD) if

$$\delta[P_+(s)]=\delta[\tilde{P}_+(s)]. \tag{2.2}$$

This says, roughly, *any instability of the diagonal of $P(s)$ must also show up on the diagonal.*

Example of a non-TDD matrix:

$$P(s)=\begin{bmatrix}\dfrac{1}{(s-1)(s+1)} & 0 & 0\\ \dfrac{1}{s-2} & \dfrac{1}{s-2} & 0\\ \dfrac{1}{s-1} & \dfrac{1}{s-2} & \dfrac{1}{s-3}\end{bmatrix},$$

$$P_+(s)=\begin{bmatrix}\dfrac{1}{2(s-1)} & 0 & 0\\ \dfrac{1}{s-2} & \dfrac{1}{s-2} & 0\\ \dfrac{1}{s-1} & \dfrac{1}{s-2} & \dfrac{1}{s-3}\end{bmatrix},$$

$$\tilde{P}_+(s)=\operatorname{diag}\left[\tfrac{1}{2}(s-1)^{-1},(s-2)^{-1},(s-3)^{-1}\right],$$

$$\delta[P_+]=4\neq\delta(\tilde{P}_+)=3.$$

Using properties of the McMillan degree, it is not hard to show that

$$\delta[P_+(s)]=\delta[G_+(s)] \tag{2.3}$$

and that $\delta[\tilde{P}_+(s)]$ is the same for all $U(s)$ with the properties noted above which triangularize $G(s)$. Consequently, we have the following.

Lemma 1: With $G(s)$ a square transfer function matrix, either all $P(s)$ or no $P(s)$ obtained by the Phase 1 procedure are TDD.

In Phase 3, we show how removal of the diagonal instability of a TDD matrix implies removal of the off-diagonal instability.

Phase 3: Closed-Loop Design

Suppose $G(s)$ can be precompensated to give a TDD $P(s)$. (The contrary case will be considered later.) Then neglecting the off-diagonal terms of $P(s)$, each loop[2] is closed by a scalar compensator to give a good closed-loop characteristic for the scalar plant defined by $p_{ii}(s)$. It turns out that *the resulting diagonal compensator also stabilizes $P(s)$.* A proof is given below, and examples which indicate in particular cases the loss of quality of closed-loop performance—principally due to interactions—are given in the next section.

Validity of the Phase 3 Claim

The claim above rests on the following theorem.

Theorem 1: With quantities as defined above, suppose $G(s)$ can be precompensated to TDD form $P(s)$, and let $K(s)=\operatorname{diag}[k_i(s)]$ be such that the compensator $K(s)$ stabilizes $\tilde{P}(s)$, so that $p_{ii}k_i(1+p_{ii}k_i)^{-1}$ is stable for each i, and there are no unstable pole-zero cancellations between p_{ii} and k_i. Then $K(s)$ stabilizes $P(s)$.

The proof of the theorem uses several lemmas.

Lemma 2: Let $X(s)$ be a lower triangular proper transfer function matrix. There exist right matrix fraction descriptions[3] (MFD's) $N_X(s)D_X^{-1}(s)$ with N_X, D_X lower triangular.

Proof: By polynomial unimodular transformation, if necessary, any MFD of $X(s)$ can be constructed to have lower triangular denominator $D_X(s)$ [13]. Then $N_X(s)=X(s)D_X(s)$ is lower triangular.

Lemma 3: Let $X(s)$ be a lower triangular proper transfer function. Then $\delta[X(s)]\geq\delta[\tilde{X}(s)]$.

Proof: Let N_X, D_X be as in Lemma 1 and also define a minimal MFD. Then $\tilde{X}=\tilde{N}_X\tilde{D}_X^{-1}$, and $\delta[\tilde{X}]\leq\deg\det[\tilde{D}_X]=\deg\det[D_X]=\delta[X]$.

Lemma 4: Let $X(s), Y(s)$ be proper transfer function matrices for which the product $X(s)Y(s)$ can be formed. Then $\delta[(XY)_+]\leq\delta[X_+]+\delta[Y_+]$.

Proof: Let $X=BA^{-1}$, $Y=C^{-1}D$ be minimal MFD's. Then $XY=N(CA)^{-1}D$ defines a possibly nonminimal mixed MFD, and if $XY=FE^{-1}$ is a minimal MFD, $|E|$ divides $|CA|$. The result follows from the easily checked fact that $\delta[X_+]=$ number of zeros of $|A|$ in $\operatorname{Re}[\delta]\geq 0$, and similarly for $\delta[Y_+], \delta[(XY)_+]$.

Lemma 5: With $P=(p_{ij})$ being TDD and $K=\operatorname{diag}(k_i)$ with no unstable pole zero cancellations between p_{ii} and k_i, then PK is TDD. In addition, there are no unstable pole-zero cancellations between P and K, i.e., $\delta[(PK)_+]=\delta[P_+]+\delta[K_+]$.

Proof:

$$\begin{aligned}\delta[(PK)_+]&\leq\delta(P_+)+\delta(K_+) && \text{(Lemma 4)}\\ &=\delta(\tilde{P}_+)+\delta(K_+) && \text{(TDD)}\\ &=\delta[(\tilde{P}K)_+] && \text{(diagonal structure and no unstable pole-zero cancellations)}\\ &\leq\delta[(PK)_+] && \text{(Lemma 3)}\end{aligned}$$

whence $\delta[(PK)_+]=\delta[(\tilde{P}K)_+]=\delta[(\tilde{P}K)_+]$ implying PK is TDD. Note that the proof also shows that $\delta[(PK)_+]=\delta(P_+)+\delta(K_+)$.

Proof of Theorem: Because of Lemma 5, we can without loss of generality assume that $K=I$. Let $P(s)=N_P(s)D_P^{-1}(s)$ be a minimal MFD with lower triangular $N_P(s)$ and $D_P(s)$. Then $\tilde{P}(s)=\tilde{N}_P(s)\tilde{D}_P^{-1}(s)$ is a possibly nonminimal MFD. Now $|D_P|=|\tilde{D}_P|$, and $\delta[P_+]=$ number of zeros of $|D_P|$ in $\operatorname{Re}[s]\geq 0$. By TDD then, $\delta[\tilde{P}_+]=$ number of zeros of $|\tilde{D}_P|$ in $\operatorname{Re}[s]\geq 0$, and so any cancellations between $\tilde{N}_P$ and $\tilde{D}_P$ can only be of factors with zeros in $\operatorname{Re}[s]<0$. Since $\tilde{P}[I+\tilde{P}]^{-1}$ is stable, $\tilde{N}_P[\tilde{N}_P+\tilde{D}_P]^{-1}$ in cancelled form is stable and so $|\tilde{N}_P+\tilde{D}_P|$ is Hurwitz, since all cancellations between $\tilde{N}_P$ and $\tilde{N}_P+\tilde{D}_P$ involve factors with zeros in $\operatorname{Re}[s]<0$. However, $|\tilde{N}_P+\tilde{D}_P|=|N_P+D_P|$ in view of the lower triangular structure of N_P and D_P, so $P(I+P)^{-1}=N_P(N_P+D_P)^{-1}$ is stable, proving the theorem.

Provided that we can achieve TDD, the design procedure involves obtaining two compensators, the unimodular $U(s)$ and the diagonal $K(s)$. No unstable pole-zero cancellations occur in the product GUK. The natural question arises as to when TDD will be achieved, and what to do when it is not achieved.

[1] A stable transfer function matrix is one for which the denominator of every entry of the matrix, after cancellations, is Hurwitz, i.e., has all zeros in $\operatorname{Re}[s]<0$.

[2] The ith loop consists of the transmission path from input i of $P(s)$ to output i of $P(s)$, feedback to a differencing element, and then the transmission path through the ith diagonal entry of a precompensator to $P(s)$ to the ith input of $P(s)$.

[3] A right matrix fraction description of a rational transfer function matrix $W(s)$ is a writing of $W(s)$ as $A(s)B^{-1}(s)$ for some polynomial $A(s), B(s)$; see, e.g., [13].

When is TDD Guaranteed?

Sufficient conditions guaranteeing the TDD property are contained in the following lemma.

Lemma 6: With notation as above, $G(s)$ can be precompensated to TDD form if any of the following conditions hold:

1) $G(s)$ is stable.

2) The off-diagonal part of $P(s)=G(s)U(s)$ is stable.

3) Any unstable pole in the off-diagonal part of $P(s)$ occurs in only one off-diagonal entry and then also occurs in the diagonal element in either the same row or same column, with at least the same multiplicity.

4) $\delta[|G(s)|_+]=\delta[G_+(s)]$. (2.4)

Proof: Conditions 1)–3) are trivial to prove. We prove only condition 4). It is easily seen that $\delta[G_+(s)]=\delta[P_+(s)]$ and $\delta[|G(s)|_+]=\delta[|P(s)|_+]=\delta[|\tilde{P}(s)|_+]\leq\delta[|\tilde{P}_+(s)|]$. Now by Lemma 3, $\delta[\tilde{P}_+(s)]\leq\delta[P_+(s)]$, and also, $\delta[|\tilde{P}_+(s)|]\leq\delta[\tilde{P}_+(s)]$. So (2.4) implies $\delta[\tilde{P}_+(s)]=\delta[P_+(s)]$.

Example:

$$G(s)=\begin{bmatrix}\dfrac{1}{s+1} & \dfrac{1}{s-2}\\[2mm] \dfrac{1}{s+1} & \dfrac{1}{s+1}\end{bmatrix},$$

$$|G(s)|=-\frac{3}{(s+1)^2(s-2)},\qquad \delta[|G(s)|_+]=1$$

$$G_+(s)=\begin{bmatrix}0 & \dfrac{1}{s-2}\\ 0 & 0\end{bmatrix},\qquad \delta[G_+(s)]=1.$$

Hence, $G(s)$ can be precompensated to TDD form.

Equation (2.4) suggests, and experience bears out, that for examples of $G(s)$ selected from real life, the TDD property can be expected. However, one can readily contrive examples where it does not hold, e.g.,

$$G(s)=\begin{bmatrix}\dfrac{1}{s-1} & 0\\[2mm] \dfrac{1}{s-2} & \dfrac{1}{s-3}\end{bmatrix}. \tag{2.5}$$

For the sake of completeness, we outline an approach applicable here.

Obtaining TDD

Failure of TDD is associated with modes which are both unstable and not on the diagonal. It follows that they should be forced onto the diagonal. The following procedure has been found to work empirically. Premultiply $G(s)$ by a constant nonsingular matrix V to form $\hat{G}(s)=VG(s)$. Then $\hat{G}(s)$ can be precompensated to TDD form. Once compensators $U(s),K(s)$ have been found for $\hat{G}(s)$, the compensator $U(s)K(s)V$ is then used for $G(s)$. This corresponds to temporarily introducing a postcompensator V and then at the end of the design, moving it back round the feedback loop. This idea (with a unimodular, rather than constant, V) is used in [7] for a design procedure based on nonstandard Smith and Smith–McMillan forms.

Miscellaneous Points

1) One of the objectives of a multivariable plant design procedure is often to get low transient interaction and zero steady-state interaction. To assist the second objective, one could include in the construction of the compensator $U(s)$—perhaps at the last step—a constant triangular matrix which diagonalizes the plant at $s=0$. It is clear that the triangular-diagonal-dominance property is still preserved. However, compromise of the first objective, low transient interactions, may result.

2) There can be certain numerical dangers in triangularization procedures. Consider a 2×2 real matrix, and suppose that a postmultiplying matrix yields a product

$$\begin{bmatrix}1 & 0\\ 10^6 & 1\end{bmatrix}.$$

Then it is conceivable that a small adjustment to the original matrix could have yielded instead

$$\begin{bmatrix}1+0(\epsilon) & 0(\epsilon)\\ 10^6+0(\epsilon) & 1+0(\epsilon)\end{bmatrix}.$$

If $0(\epsilon)$ is 10^{-6} or thereabouts, it is clear that the allegedly triangular matrix is far from triangular in one sense. Its diagonal elements are both $1+0(\epsilon)$ yet it has an eigenvalue that is approximately zero.

One can conceive of potential difficulties arising in triangularizing transfer function matrices which are extensions of the above sort of difficulty. The remedy would seem to rest in executing a sensitivity analysis if such difficulties are anticipated.

3) Throughout this section, we have concentrated on securing stability, rather than, say, securing all poles to lie in a certain cone, or other restricted region of $\mathrm{Re}[s]<0$. As the Appendix makes clear, there is in principle no difficulty in restricting pole positions more closely. The penalty is likely to be more complicated compensators. The transient response may be improved by such a restriction, however. Also, by choosing a bounded region for permissible poles, the unimodular compensator poles and zeros become confined within this region, and this suggests that the phase shift introduced by the compensator will not be unrealistically high. (The situation is analogous to the scalar design notion that the pole and zero of a lead or lag compensator should not vary in magnitude by a factor of more than approximately 10.)

4) Interaction is caused by signals from one loop entering another loop; accordingly, *high-loop gains, which serve to reject unwanted disturbances, must reduce interaction.* Thus, consider the 2×2 case

$$P(s)=\begin{bmatrix}p_{11}(s) & 0\\ p_{21}(s) & p_{22}(s)\end{bmatrix}$$

$$P(s)K(s)[I+P(s)K(s)]^{-1}=\begin{bmatrix}\dfrac{k_1p_{11}}{1+k_1p_{11}} & 0\\[2mm] \dfrac{k_1p_{21}}{(1+k_1p_{11})(1+k_2p_{22})} & \dfrac{k_2p_{22}}{1+k_2p_{22}}\end{bmatrix}.$$

The transfer functions

$$\frac{p_{21}}{p_{22}}\quad\text{and}\quad\frac{k_1p_{21}}{(1+k_1p_{11})k_2p_{22}}$$

evaluated on the $j\omega$-axis are measures of interaction before and after feedback, and they illustrate the point that having large k_2, rather than large k_1, helps reduce interaction. High k_2, of course, cuts down the effect of unwatned signals in loop 2, the unwanted signal here being coupling from loop 1.

5) In case $G(s)$ is a minimum phase plant, [7] shows that there exists a $V(s)$ with all poles ane zeros in $\mathrm{Re}[s]<0$ such that $G(s)V(s)$ is diagonal. Thus interaction can be eliminated, but this may be at the cost of the compensation being rather complex.

6) There is quite clearly no getting around the difficulties of nonminimum phase plants. The plant zeros in $\mathrm{Re}[s]\geq0$ will be zeros of the diagonal entries of $P(s)$, i.e., the entries of $P(s)$ for which scalar compensators are to be formed, and will clearly make the control design task harder.

III. Design Examples

We first illustrate the triangularization technique in comparison with the diagonal dominance technique (inverse Nyquist array) developed by Rosenbrock, using one of his examples. The problem is to search for a compensator so that the closed-loop system is stable and has a good tracking property for step inputs. In addition, interactions are kept as low as possible and some security against loop failure is inherently achieved with the technique as classical feedback designs are done for each loop independently.

Example 1: The open-loop system has the following transfer matrix [1]:

$$G(s)=\begin{bmatrix}\dfrac{1-s}{(1+s)^2} & \dfrac{2-s}{(1+s)^2}\\ \dfrac{1-3s}{3(1+s)^2} & \dfrac{1-s}{(1+s)^2}\end{bmatrix}.$$

Using the inverse Nyquist array and diagonal dominance technique, Rosenbrock selects the following precompensator:

$$C=\begin{bmatrix}-100 & -300\\ 75 & 150\end{bmatrix}.$$

The closed-loop transfer matrix is

$$R(s)=\frac{1}{\rho_c(s)}\begin{bmatrix}25(102+3s+s^2) & 150(s+s^2)\\ \frac{50}{6}(5+8s+3s^2) & 50(51+4s+3s^2)\end{bmatrix}$$

$$\rho_c(s)=2601+278s+178s^2+s^3.$$

Now using the TDD technique, the following steps are made in order to construct a desirable compensator.

Step 1: In the first row, the element $(1,1)$ already has lowest degree $(\delta=2)$.

Step 2: Subtract a multiple of the first column from the second to ensure $\delta(g_{12})<\delta(g_{11})$

$$\frac{2-s}{(1+s)^2}=a\frac{(1-s)}{(1+s)^2}+\frac{b}{(1+s)}$$

where

$$a=\frac{3}{2},\qquad b=\frac{1}{2}.$$

Thus,

$$G(s)\begin{bmatrix}1 & -\frac{3}{2}\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\dfrac{1-s}{(1+s)^2} & \dfrac{1}{2(1+s)}\\ \dfrac{1-3s}{3(1+s)^2} & \dfrac{1}{2(1+s)}\end{bmatrix}.$$

Step 3: As $g_{12}\neq 0$, return to Step 1.

Step 1: Again, move to $(1,1)$ position the element with lowest degree $(\delta=1)$

$$G(s)\begin{bmatrix}1 & -\frac{3}{2}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}\dfrac{1}{2(1+s)} & \dfrac{1-s}{(1+s)^2}\\ \dfrac{1}{2(1+s)} & \dfrac{1-3s}{3(1+s)^2}\end{bmatrix}.$$

Step 2:

$$\frac{1-s}{(1+s)^2}=a\frac{1}{2(1+s)}+b$$

where

$$a=\frac{2(1-s)}{1+s}\quad\text{and}\quad b=0$$

$$G(s)U(s)=G(s)\begin{bmatrix}1 & -\frac{3}{2}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & \frac{2(1-s)}{1+s}\\ 0 & -1\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{1}{2(1+s)} & 0\\ \dfrac{1}{2(1+s)} & \dfrac{2}{3(1+s)^2}\end{bmatrix}.$$

The matrix is triangular-diagonal-dominant. Closed-loop design on the diagonal elements shows that the system is stable for any constant feedback loop gains (i.e., $k_1>0$, $k_2>0$).

The following choice of feedback loop gains resulting from classical designs ensures a good input following property and small steady-state error in the first loop as well as a limit on the overshoot in the second loop to about 40 percent. At the same time, by introducing some phase shift in the second loop, interaction can be kept reasonably low, the plant inputs have approximately the same maxima as with the INA compensator, and the steady-state error is less than 3 percent:

$$K(s)=\begin{bmatrix}70 & 0\\ 0 & \dfrac{7.2(s+1)}{(s+0.15)}\end{bmatrix}.$$

Consequently, the following precompensator is chosen:

$$C(s)=U(s)K(s)=\begin{bmatrix}-105 & \dfrac{14.4(s-2)}{s+0.15}\\ 70 & \dfrac{14.4(1-s)}{s+0.15}\end{bmatrix}.$$

The resulting closed-loop transfer matrix is

$$R(s)=\begin{bmatrix}\dfrac{35}{36+s} & 0\\ \dfrac{35(s^2+1.15s+0.15)}{s^3+37.15s^2+46.35s+178.2} & \dfrac{4.8}{s^2+1.15s+4.95}\end{bmatrix}.$$

Fig. 1 shows the output step responses of the two systems and Fig. 2 the corresponding plant inputs. It is clear that both systems perform comparably. Rosenbrock's compensator is, of course, simpler, but it is worth remembering that via the new technique, the precompensator is obtained systematically.

Space limitations preclude the development at length of two remaining examples. We simply summarize the highlights. Example 2 was studied in [15] using the sequential return difference technique [6], and Example 3 in [16] using the characteristic locus method.

Example 2 (Unstable Chemical Reactor): The open-loop transfer matrix is

$$G(s)=\frac{\begin{bmatrix}-(2.63+11.11s+2.11s^2) & 233.5+29.2s\\ 1.515+0.943s & -96.5-58.25s+43.83s^2+5.68s^3\end{bmatrix}}{s^4+11.87s^3+18.47s^2-79.43s+7.55}$$

$$\text{Sequential return difference compensator}=\begin{bmatrix}\dfrac{-50(s+0.4)}{s} & 0\\ 0 & \dfrac{10(s+1.2)}{s}\end{bmatrix}$$

$$\text{TDD compensator}=\begin{bmatrix}-70 & \dfrac{138.4(s+8)}{(s+5.02)(s+0.248)}\\ 0 & 10\end{bmatrix}.$$

(The TDD property turns out to be satisfied for this unstable plant.) Rise times, overshoot in the 1-1 term and coupling from input 2 to input 1 are better for the TDD compensator, but plant input magnitudes for step reference inputs are greater by a factor of approximately 2. Steady-state errors for the first compensator are zero because of the integral feedback; for the second, the error is 3.8 percent for the first loop, 2.0 percent for the second.

Example 3 (Pressurized Flow-Box): The open-loop transfer matrix for a pressurized flow-box is

$$G(s)=\begin{bmatrix}\dfrac{0.0336}{s+0.395} & \dfrac{1.03s}{\alpha(s)}\\ \dfrac{(9.66s+0.117)10^{-4}}{\alpha(s)} & \dfrac{-0.0114}{\alpha(s)}\end{bmatrix}$$

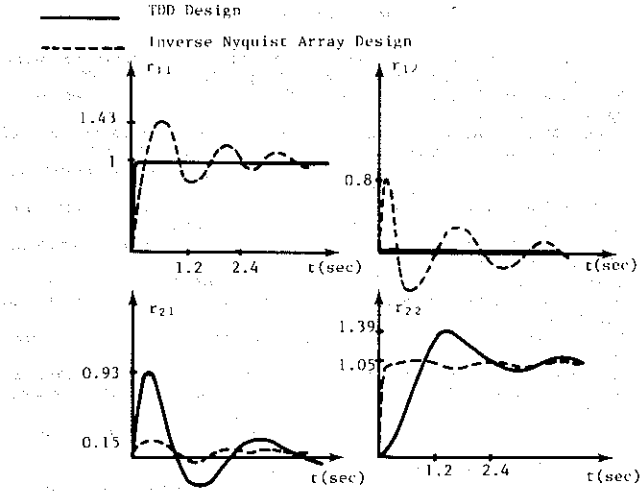

Fig. 1. Output step responses (r_{ij}: plant output i for step on reference input j).

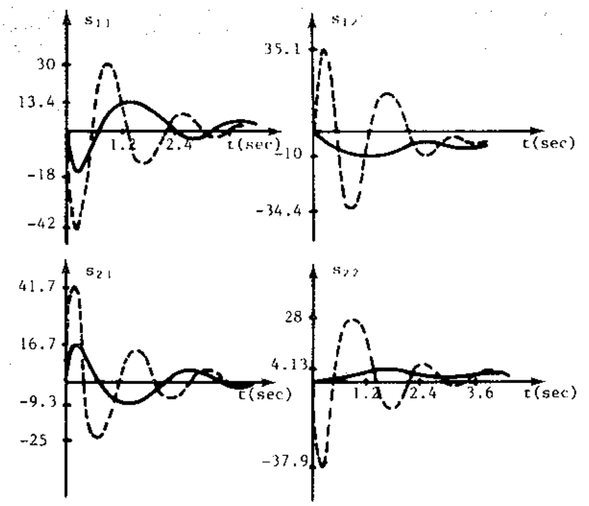

Fig. 2. Plant inputs (s_{ij}: plant input i for step on reference input j).

where

$$\alpha(s) = s^2 + 0.395s + 1.26 \times 10^{-4}.$$

The characteristic locus compensator and the TDD compensator[4] are, respectively,

$$C(s) = \begin{bmatrix} \dfrac{117.2}{s} & 1030 \\ 99.6 + \dfrac{0.12}{s} & -33.6 - \dfrac{1.1}{s} \end{bmatrix} \text{ and } \begin{bmatrix} 1000 & 3064 \\ 0 & -100 \end{bmatrix}.$$

Interaction for both compensators is very small and overshoot is nonexistent. The rise time for the 1-1 term is significantly better for the TDD compensator, and for the 2-2 term significantly better for the characteristic locus compensator. Plant input magnitudes for step reference inputs are less for the TDD compensator, and steady-state errors are comparable and very small (< 1 percent) for both.

IV. Conclusions

We have presented a method of reducing a multivariable design problem to a collection of independent scalar design problems, thereby allowing closed-loop design. This method can be compared against other methods using at least three different criteria: ease of use, complexity of controller, and quality of performance. By eliminating some of the trial and error in existing frequency-domain approaches, the method scores well on ease of use. For the three examples chosen, the method performs acceptably in relation to complexity of controller and quality of performance. Controller complexities are a good deal less than those which would result from state variable methods.

One difficulty with the present method relates to plants which cannot be made triangular-diagonal-dominant. Other factors which demand further consideration are inaccuracies and associated sensitivity problems, nonsquare plants, and the use of compensators with greater restrictions on pole and zero positions.

Last, we should stress that the three examples neither prove nor disprove the usefulness of the design approach, although it is fair to claim they provide a prima facie case for not dismissing it. Further evaluation by examples is needed.

[4]The TDD compensator is actually an obvious approximation to that obtained by the TDD method, the approximation allowing degree reduction.

Appendix
Transfer Function Triangularization

In this Appendix, we first review and develop a number of algebra ideas that are relevant in studying the design techniques of the major part of the paper. The most important points below stem from the fact that the class of stable, proper rational transfer functions forms a Euclidean domain. References dealing with the algebraic ideas of this Appendix include [11], [17], [18] while their application to transfer functions has been described in [8], [9].

Euclidean Domain

Recall first the concept of a *field* as a collection of objects which can be added, subtracted, multiplied, and divided (except by zero) with the usual associative, distributive, and cummutative rules.

A *commutative ring* is a collection of objects with all the field properties except division, and an *integral domain* is a commutative ring possessing a 1 (i.e., $1a = a$) and no zero divisors (i.e., $ab = 0$ implies $a = 0$ or $b = 0$). An equivalent requirement to the last property is that a cancellation law holds, i.e., $ab = ac$ implies $b = c$ if $a \neq 0$.

Let $\mathfrak{R}$ be an integral domain and suppose that with every nonzero $a \in \mathfrak{R}$ there is associated a nonnegative integer $\delta(a)$ with the following property: for each $a, b \in \mathfrak{R}$ with $a \neq 0$ there exist $q, r \in \mathfrak{R}$ such that

$$b = aq + r \text{ and } r = 0 \text{ or } \delta(r) < \delta(a). \tag{A1}$$

Then $\mathfrak{R}$ is termed a *Euclidean domain*. The set of real polynomials in a single variable forms a Euclidean domain with $\delta(a)$ being the degree of a.

There are a number of standard properties of Euclidean domains [17]. Define a *unit* to be any element of a ring whose inverse is in the ring, and a prime to be any element of a ring that has no factors but itself, and units. Then in a Euclidean domain, every element can be factored into a product of a finite number of primes which are unique up to multiplication by units. Any set of elements in a Euclidean domain has a *greatest common divisor* (gcd) which is the unique (up to units) greatest product of primes dividing all members of the set. Such a gcd is also expressible as a linear combination of members of the set.

A matrix over a Euclidean domain with inverse also in the domain is termed unimodular. A unimodular matrix over a Euclidean domain is always expressible as a product of matrices representing the elementary transformations of column interchange, multiplication of a column by a unit, and addition of a multiple of one column to another column, [11, p. 34].

There are other special types of rings, including the class of principal ideal domains. We shall not bother with the definition here, but note simply that any Euclidean domain is automatically a principal ideal domain, and so any theorem dealing with matrices over a principal ideal domain (such as that concerned with the existence of the Hermite form) applies to matrices over a Euclidean domain.

Euclidean Domains of Transfer Functions

Let $\mathbb{S}$ be a region in the complex plane symmetrically located with respect to the real axis and including at least one point on the real axis.

(Thus $\mathbb{S}$ might be the open left half-plane, or the region $\mathrm{Re}[s]<-1$, or $(\mathrm{Re}[s]<0)\cap(|\pi-\arg(s)|<\pi/3)$. For applications, $\mathbb{S}$ will correspond to a region in which we desire closed-loop system poles to be, and for the bulk of the paper, $\mathbb{S}$ is the left half-plane $\mathrm{Re}[s]<0$).

Consider the set of real rational transfer functions $\mathcal{P}_{\mathbb{S}}=\{\alpha/\beta$ with α,β real polynomials, α/β proper and all zeros of β lying in $\mathbb{S}\}$.

(In applications, $\mathcal{P}_{\mathbb{S}}$ will correspond to the set of desirable transfer functions.) Then it is an important result that $\mathcal{P}_{\mathbb{S}}$ is a Euclidean domain. (The principal ideal domain property was shown in [9].) The fact that $\mathcal{P}_{\mathbb{S}}$ is an integral domain is easy to check. We shall review however the verification of the "degree" property. Observe first that the units of $\mathcal{P}_{\mathbb{S}}$ are precisely those transfer functions α/β for which $\deg\alpha=\deg\beta$ and all zeros of both α and β lie in $\mathbb{S}$. (For $\mathbb{S}=\{s|\mathrm{Re}[s]<0\}$, minimum phase transfer functions which are nonzero at $s=\infty$ are the units.) Second, observe that if π is a degree 1 polynomial with one zero in $\mathbb{S}$, we have

$$\frac{\alpha}{\beta}=e\frac{\gamma}{\pi^n} \tag{A2}$$

for some n, some unit e, and some γ whose zeros are disjoint from $\mathbb{S}$.

We shall establish the division part of the Euclidean domain property by taking $\delta(\alpha/\beta)=n$. Thus, let $b=\alpha_b/\beta_b$ with α_b,β_b coprime and $a=\alpha_a/\beta_a=e_a\gamma_a/\pi^{n_a}$. We seek then a proper $q=\alpha_q/\beta_q$ and proper $r=e_r\gamma_r/\pi^{n_r}$ with $n_r<n_a$ and γ_r possessing all zeros disjoint from $\mathbb{S}$ such that

$$\frac{\alpha_b}{\beta_b}=\frac{\alpha_q}{\beta_q}e_a\frac{\gamma_a}{\pi^{n_a}}+e_r\frac{\gamma_r}{\pi^{n_r}}. \tag{A3}$$

As shown in [9], the construction is achievable as follows. Find polynomials ρ and θ with $\deg\theta<\deg\beta_b$ such that

$$\alpha_b\pi^{n_a-1}=\rho\beta_b+\gamma_a\theta. \tag{A4}$$

(This is possible because γ_a and β_b must be coprime as polynomials.) Then

$$\frac{\alpha_q}{\beta_q}=\frac{1}{e_a}\frac{\pi\theta}{\beta_b}\qquad e_r\frac{\gamma_r}{\pi^{n_r}}=\frac{\rho}{\pi^{n_a-1}}. \tag{A5}$$

Triangularization via Postmultiplication by a Unimodular Matrix

Consider an $n\times n$ matrix $[g_{ij}(s)]$ over $\mathcal{P}_{\mathbb{S}}$. Then triangularization is possible via a series of elementary column transformations (this is equivalent to postmultiplication by a unimodular matrix). The idea is as follows.

1) Move the lowest degree element in the first row to the 1-1 position.

2) Subtract a multiple of the first column from the second, third, $\cdots$ column to ensure $\delta(g_{1i})<\delta(g_{11})$, $i>1$.

3) If one or more of g_{1i} for $i>1$ is nonzero, return to 1. Otherwise proceed to step 4).

4) Temporarily delete the first row and column.

5) Repeat the procedure of steps 1) to 4) on the remaining matrix. This leaves the temporarily deleted rows and columns unaltered.

If at the end of this procedure, one subtracts a multiple of column 2 from column 1, multiples of column 3 from columns 2 and 1, and so on, ensuring that $\delta[g_{ii}]>\delta[g_{ij}]$, $j<i$, we obtain (to within element multiplication by units) the Hermite form of $[g_{ij}(s)]$. Since this involves further manipulations, the resulting unimodular matrix is in a rough sense more complex.

ACKNOWLEDGMENT

The authors are indebted to Prof. J. B. Moore for his comments during the course of writing this paper, to Prof. G. C. Goodwin for the use of the computer-aided classical design package, and to Mr. S. Salehi for comments on an early draft of the paper.

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