Forwards, Backwards, and Dynamically Reversible Markovian Models of Second-Order Processes

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Abstract—Related causal and anticausal models of second-order processes are studied; new properties are characterized in terms of associated matrix fraction descriptions. Situations in which related causal and anticausal models have the same transfer function matrix are examined in detail, and the models are shown to possess internal and external time-reversibility properties. Connections with ideas of statistical mechanics and passive networks are also indicated.

I. Introduction

It is now commonplace to study finite-dimensional models of second-order processes. More explicitly, we often model processes with a given covariance as the output of a finite-dimensional linear system in state-space form with initial state $x(t_0)$ uncorrelated with its white noise input for $t>t_0$. These are often known as *forwards* or *causal* Markovian models of the process. In several applications, e.g., in smoothing problems, it is useful to consider *backwards* or *anticausal* models in which the excitation starts at some time t_1 and runs "backwards" in time. Just reversing time in the forwards model will destroy the Markovianness and, therefore, a different construction has to be used. The appropriate backwards Markovian model was first described in [1] (see also [4], [5]).

In this paper, we make a further clarification of the relations between the forward and backward models and we study more fully the case of *stationary processes* corresponding to time-invariant models that are stable and commence at $t_0=-\infty$, running forward, or at $t_1=+\infty$, running backward. This specialization both lends greater intuitive content to some of the results, and allows more results to be obtained. (Moreover, time-varying versions of a number of results could also be derived.)

In Section II, we analyze the scheme in [1]–[5] for passing from a forward model to a backward model in transfer function terms. It turns out that for a single-input single-output systems, the backward model has the same zeros as the forward model, with poles that are reflections through the $j\omega$-axis of the forward model poles. In Section III, we describe the appropriate generalization of this notion with the aid of matrix fraction descriptions [6] of the forward and backward models.

In Sections IV and V we study situations in which the forward and associated backward models have the same transfer function matrix, to within sign change of the independent variable. As explained later, certain physical situations may lead to this happening. It turns out that such "self-dual" situations imply that the model output process has a statistical *time-reversibility* property, which is equivalent to the power spectrum being a symmetric matrix. Furthermore, we show that in the right coordinate basis, processes internal to the model (e.g., the state or partial state processes) also have a generalized (so-called *dynamic*) reversibility property when considered along with the output process. The dynamic reversibility property of the state process is also examined in [21].

In Section VI we discuss briefly the determination of self-dual models. The results in Sections IV–VI are closely related to ideas in the theory of passive networks, especially as regards the relations between external and internal reciprocity (see [7]–[10]). These connections will be made more explicit in the paper; we might mention that they have also led us to some new views of the network synthesis problem [11], [12].

II. Forward and Backward Factors

Let $\Phi(s)$ be an $m\times m$ square matrix of real rational functions of s satisfying the following conditions:

$$\Phi(s) \text{ is analytic for } s=j\omega,\ \omega \text{ real, including } \omega=\infty \tag{2.1}$$

$$\Phi(s)=\Phi'(-s) \tag{2.2}$$

$$\Phi(j\omega) \text{ is nonnegative definite Hermitian for all real } \omega. \tag{2.3}$$

Then there exists a second-order process $y(\cdot)$ whose power spectral density matrix is $\Phi(j\omega)$ (or equivalently such that $\Phi(j\omega)$ is the Fourier transform of the covariance function of $y(\cdot)$) [9]. It is also well known that there are many such processes $y(\cdot)$. One class of such processes can be obtained by passing white-noise $u(\cdot)$ through a linear time-invariant filter with transfer function $W(s)$, where

$$\Phi(s)=W(s)W'(-s) \tag{2.4}$$

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$W(s)$ is analytic for $\mathrm{Re}\, s \geq 0$ (including $s=\infty$).

For rational $\Phi(s)$, $W(s)$ can also be chosen to be rational and therefore can also be described in state-space terms. For example, we can find matrices $\{F,G,H,J\}$ such that

$$W(s)=J+H(sI-F)^{-1}G \tag{2.5}$$

where F is a stable matrix (i.e., its eigenvalues have negative real parts). Now suppose the corresponding state-space realization $\{F,G,H,J\}$ is driven by a white-noise process $u(\cdot)$, and a random initial condition $x(t_0)=x_0$ that is uncorrelated with $u(\cdot)$, i.e.,

$$E[u(t)u'(s)]=I\delta(t-s),$$

$$Eu(t)x_0'\equiv 0, \qquad t>t_0. \tag{2.6a}$$

Then provided the initial variance[1]

$$Ex_0x_0' \triangleq \Pi \tag{2.6b}$$

is the unique nonnegative definite solution of the (Lyapunov) equation

$$F\Pi+\Pi F'+GG'=0 \tag{2.6c}$$

it can be checked that the process $y(\cdot)$ described by

$$y(t)=Hx(t)+Ju(t), \qquad t\geq t_0$$

$$\dot{x}(t)=Fx(t)+Gu(t), \qquad x(t_0)=x_0 \tag{2.7}$$

has power spectral density matrix

$$\Phi(j\omega)=W(j\omega)W'(-j\omega),$$

$$W(j\omega)=J+H(j\omega I-F)^{-1}G. \tag{2.8}$$

It can also be checked that (as expected by stationarity)

$$Ex(t)x'(t)=\Pi, \qquad \text{for all } t\geq t_0\geq -\infty.$$

Some calculation shows that $\Phi(\cdot)$ can itself be written in "rational form" as

$$\Phi(s)=JJ'+H(sI-F)^{-1}K+K'(-sI-F')^{-1}H' \tag{2.9a}$$

where

$$K=\Pi H'+GJ'. \tag{2.9b}$$

It will simplify various calculations if we assume that the state process $x(\cdot)$ is nondegenerate (purely nondeterministic) in the sense that the state-variance

$$\Pi \text{ is positive definite.} \tag{2.10a}$$

It is known (see, e.g., [9, p. 131]) that this is ensured by the assumption that

$$\{F,G\} \text{ is controllable.} \tag{2.10b}$$

Now by reversing the direction of time in the above equation (2.7), we shall obviously get a "backwards" model for the process $y(\cdot)$ (and by an abuse of notation, for the spectral matrix $\Phi(\cdot)$). However, an important property of the "forwards" model (2.6) and (2.7) will be lost. Thus note that, because $u(\cdot)$ is white (Gaussian) *and* uncorrelated with x_0, (2.6) and (2.7) is termed a Markovian model for $y(\cdot)$. Now if we just reverse time in (2.7), the "final" value will be correlated with the (backwards) white noise $u(\cdot)$ and this Markovian property will be lost.

[1] All random variables will be assumed to have zero mean.

However, with a little effort, a backwards Markovian model can indeed be obtained. Thus was shown in [1] (see also [4], [5]) for processes with possibly time-variant state-space models. For the case of interest, the results specialize as follows.

Proposition 1: Given a forwards Markovian model obeying (2.6) and (2.7), a backwards Markovian model

$$-\dot{x}_b(t)=F_bx_b(t)+G_b\tilde{u}(t), \qquad x(t_1)=x_1$$

$$y(t)=H_bx_b(t)+J_b\tilde{u}(t), \qquad t\leq t_1\leq\infty \tag{2.11}$$

with

$$E\tilde{u}(t)\tilde{u}'(s)=I\delta(t-s)$$

$$E\tilde{u}(t)x_1\equiv 0$$

$$Ex_1x_1'=\Pi$$

can be obtained by choosing $\{F_b,G_b,H_b,J_b\}$ such that

$$F_b=-(F+GG'\Pi^{-1})$$

$$G_b=G \quad \Pi H_b'=-\Pi H'-GJ' \quad J_b=J. \tag{2.12}$$

Such choice will ensure that $x_b(\cdot)$ and $x(\cdot)$ have the same covariance

$$E[x_b(\sigma)x_b'(\rho)]=E[x(\sigma)x'(\rho)] \tag{2.13}$$

and of course the same for $y_b(\cdot)$ and $y(\cdot)$,

$$E[y_b(\sigma)y_b'(\rho)]=E[y(\sigma)y'(\rho)]. \tag{2.14}$$

The transfer function of this backwards model is

$$W_b(s)=J+H_b(-sI-F_b)^{-1}G_b \tag{2.15}$$

and it obeys the relation

$$\Phi(s)=W(s)W'(-s)=W_b(s)W_b'(-s). \tag{2.16}$$

Remark 1

In fact one can identify the trajectories of $x(\cdot)$ and $x_b(\cdot)$ (and hence also $y(\cdot)$ and $y_b(\cdot)$) [3].

Remark 2

The formula $F_b=\Pi F'\Pi^{-1}$ shows that F and F_b will have the same eigenvalues (all in the left-half plane). Therefore, $W_b(s)$ will have its poles in the right half-plane, corresponding to an *anticausal* impulse response (zero for $t>0$).

Proposition 2: With quantities as defined above, $W(s)$ and $W_b(s)$ can be related as

$$W_b(s)U(s)=W(s) \tag{2.17}$$

where

$$U(s)\triangleq I-G'\Pi^{-1}(sI-F)^{-1}G \tag{2.18a}$$

and $U(s)$ is "paraunitary" or "allpass" in the sense that

$$U(s)U'(-s)=I=U'(-s)U(s). \tag{2.18b}$$

Proof: By verification. ∎

In contrast to the $W_b(s)$ of the backwards Markovian model, we may note that the backwards model obtained by just reversing the direction of time in the forwards model will have transfer function

$$W(-s)=J+H(-sI-F)^{-1}G \triangleq W_B(s). \quad (2.19)$$

The Scalar Case

It is interesting to compare $W_B(s)$ and $W_b(s)$ in the scalar case. If $W(s)=n(s)/d(s)$ for polynomials with $d(s)=|sI-F|$, then

$$W_B(s)=n(-s)/d(-s).$$

Therefore, $W_B(s)$ is obtained from $W(s)$ by reflecting its poles and zeros across the $j\omega$-axis. However for $W_b(s)$ we can see from (2.18) that $U(s)$ must have the same pole, as $W(s)$, so that if it is also to be unitary, we must have

$$U(s)=d(-s)/d(s)$$

and, therefore,

$$W_b(s)=n(s)/d(-s).$$

That is, only the poles are reflected to yield $W_b(s)$, while the zeros are not affected. (Unlike $W_B(s)$, different $W_b(s)$ will result from minimal and nonminimal realizations of $W(s)$. It turns out that this point is of significance in some network-theory applications of these ideas [11], [12]).

The poles have to be reflected for all backward models in order to have them purely noncausal (anticausal), but there is no such constraint on the zeros. Different anticausal backwards models can be obtained by flipping various combinations of zeros across the $j\omega$-axis and the backwards Markovian model is the one corresponding to no zeros being flipped.

That the zeros will remain invariant may also be seen from the fact that the expression $F_b=-(F+GG'\Pi^{-1})$ corresponds to using state-feedback,

$$u(\cdot)=\tilde{u}(\cdot)-G'\Pi^{-1}x(\cdot)$$

in the original forwards realization (2.7). For controllable scalar systems it is well known and easy to prove that state-feedback only moves the poles and does not directly affect the zeros.

To obtain the corresponding results for multivariable systems, we would use the so-called matrix-fraction description (MFD's), as will be explained in the next section.

III. Matrix-Fraction Description of Backwards Markovian Models—Dual Matrix Polynomials

It is a standard result [6] that one may associate with a completely controllable pair $\{F,G\}$ a square-polynomial matrix $D(s)$ such that

$$\det D(s)=\det(sI-F) \quad (3.1)$$

and

$$W(s)=J+H(sI-F)^{-1}G=N(s)D^{-1}(s) \quad (3.2)$$

for some polynomial matrix $N(s)$. The pair $\{N(s),D(s)\}$, or more loosely the rational function $N(s)D^{-1}(s)$, is called a (right) *matrix-fraction description* of the matrix transfer function $W(s)$.

Now it can be shown that the effect of the state-variable feedback on a realization $\{F,G,H,J\}$ is mirrored in the associated MFD by the fact that the new transfer function will have the same "numerator polynomial," but a different denominator polynomial. In our problem, the state-feedback interpretation of (2.12) will show that (see, e.g., [6, p. 229])

$$W_b(s)=N(s)D_b^{-1}(s) \quad (3.3)$$

where

$$D_b(s)=\left[I-G'\Pi^{-1}(sI-F)^{-1}G\right]D(s)=U(s)D(s) \quad (3.4)$$

with $U(s)$ as defined in (2.18). Moreover because $W(s)$ is the transfer function of a causal system,

$$\det D(s) \text{ has all its zeros in the LHP.} \quad (3.5a)$$

while because $W_b(s)$ is the transfer function of a purely anticausal system,

$$\det D_b(s) \text{ has } all \text{ its zeros in the RHP.} \quad (3.5b)$$

In the case of scalar or diagonal $D(s)$, the zeros of $\det D(s)$ and $D(s)$ coincide and then (3.5) and the fact that zeros of $D(s)$ and $D_b(s)$ are zeros of the spectrum, completely specify $D_b(s)$. In the general case, knowledge of the zeros alone does not completely specify a matrix polynomial and we have to capture the key properties of $D_b(s)$ in a less direct way than by (3.5). For this we first note that an equivalent way of saying that the zeros of de-t$D_b(s)$ are the reflections of those of $\det D(s)$ is that

$$\det D_b(s)=\pm\det D(-s). \quad (3.6)$$

Similarly (3.4) and (2.18) can be combined as

$$D_b'(-s)D_b(s)=D'(-s)D(s). \quad (3.7)$$

We shall show in Theorem 1 below that these two properties essentially suffice to determine $D_b(s)$ uniquely.

We may note that matrix pairs obeying (3.6) and (3.7) (but with s and $-s$ replaced by z and z^{-1}) arise as the forward and backward predictors in the so-called LWR (Levinson–Whittle–Wiggins–Robinson) algorithm for predicting stationary sequences, and in the closely related theory of matrix orthogonal polynomials (cf., [13] and the references therein). Such pairs, without the restriction (3.5a) that $D(s)$ be stable also arise in a stability test for matrix polynomials [14]. Motivated by these facts, we shall say that two matrix polynomials $\{D(s),D_b(s)\}$, $D(s)$ not necessarily stable, are *dual* if they are related as in (3.6) and (3.7).

The following theorem, proved in Appendix I, demonstrates the existence of $D_b(s)$. A constructive procedure may be found in the proof of the theorem, and it is closely tied to the construction in Proposition 1.

Theorem 1

Let $D(s)$ be an $n\times n$ nonsingular matrix polynomial. Then there exists a "dual" matrix polynomial $D_b(s)$ obeying (3.6) and (3.7). Further, if $\det D(s)$ and $\det D(-s)$ are coprime, $D_b(s)$ is unique to within left multiplication by a constant orthogonal matrix, and there is one and only

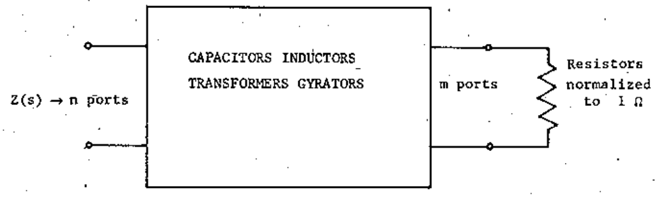

Fig. 1.

$D_b(s)$ with the property that

$$\lim_{s\to\infty} D(s)D_b^{-1}(s)=I. \tag{3.8}$$

Several remarks can be made.

1) The theorem is more general than needed for this paper, where for stationarity of the process $y(\cdot)$ we shall require that $\det D(s)$ have all its zeros in $\mathrm{Re}[s]<0$. Then the coprimeness condition guaranteeing uniqueness of a $D_b(s)$ satisfying (3.8) is automatically fulfilled.

2) If $D(s)$ is diagonal, $D_b(s)=D(-s)$ satisfies (3.6) and (3.7) and, to within sign change of the diagonal entries, (3.8). Generally, though, construction of $D_b(s)$ from $D(s)$ is not straightforward.

3) If $D_b(s)$ is the dual of $D(s)$, then $D(s)$ is the dual of $D_b(s)$. An absence of a transpose caused a very similar definition in [14] to lack this property.

4) Note that by our earlier discussions, we have an explicit statespace construction, at least when (3.5a) holds (or more generally when $\det D(s)$ and $\det D(-s)$ are coprime) for going from a polynomial matrix to its dual. From $D(s)$, set up $[F,G]$, then Π, and finally use (3.4).

Network Interpretations

Several physical systems provide examples of the simultaneous occurrence of spectral factors and their duals. The following ideas are developed at greater length elsewhere, see [11] and [12]. Consider an n-port network N comprised of a finite number of passive capacitors, inductors, transformers, gyrators and resistors and suppose it is drawn as in Fig. 1, as a lossless $(n+m)$ port terminated at m ports in unit resistors. (Transformer normalization will take care of unit resistors.) Suppose the network possesses an impedance matrix $Z(s)$. Let $W(s)$ be the transfer function matrix linking current sources in parallel with the resistors to the voltage vector at the n input ports of the network. Then it can be shown that $W(s)$ is a (forward) spectral factor for $\Phi(s)=Z(s)+Z'(-s)$ (which is a power spectrum matrix); in fact, by Nyquist's theorem (see, e.g., [15]) the thermal noise in the resistors of the network induces a random process at the ports of N which has power spectrum $\Phi(s)$ to within a scaling constant. In a physical sense, the dual of $W(s)$ can be obtained in two ways. First it can be shown that $W_b'(-s)^2$ is the transfer function matrix linking current generators at the n ports of N to the voltage across the resistors. Second, if the direction of all gyrator polarities is reversed, it can be shown that $W_b(-s)$ is the transfer function matrix linking current sources in parallel with the resistors to the voltage vector at the n ports. The state space description or matrix fraction description used in forming the dual is one induced by a corresponding description for the lossless $(n+m)$ port embedded in N.

[2]We have $-s$ here because in the network all transfer functions are causal.

In case N is reciprocal, i.e., contains no gyrators, we shall have $W(s)=W_b(-s)$, a situation which we shall describe as *self-dual*. In the next section, we delve more deeply into the self-duality property.

IV. Self-Dual Models and State Reversibility

In this next section, we shall consider the question of when a forward model and associated backward model are related by the self-duality equation $W(s)=W_b(-s)$. The main conclusions are that the process of which $W(s)W'(-s)$ is the power spectrum matrix must have a reversibility property, and that in the right coordinate basis, the state (or partial state) of the realization of $W(s)$ will have a generalized (so-called dynamic) reversibility property. This section considers state reversibility, while the next section considers partial state reversibility.

A necessary condition for self-duality is easily obtained.

Theorem 2

Let $W(s)$ be a spectral factor of a prescribed rational power spectrum matrix, and let $W_b(s)$ be the dual spectral factor, obtained from a state-variable or matrix fraction description of $W(s)$ via the procedure of Sections II or III. If $W_b(-s)Q=W(s)$ for some orthogonal Q, then $\Phi(s)=\Phi'(s)$. However the converse is not true. Thus all scalar $\Phi(s)$ are symmetric, but the self-duality property could only follow for a scalar transfer function $W(s)$ in case the zero pattern of $W(s)$ was symmetric with respect to the $j\omega$-axis.

Note too that, as suggested by the statement of the above theorem, it proves convenient to expand the definition of self-duality to permit an orthogonal "normalizing" matrix (see also Theorem 1).

Some elementary concepts from statistical thermodynamics [16] provide helpful insights into the symmetry of $\Phi(\cdot)$ and possible self-duality of a spectral factor. Let $y(\cdot)$ be a stationary vector random process. Then $y(\cdot)$ has a second-order reversibility property if statistics computed running forwards in time are the same as those running backwards in time. (Thus a tape recording of $y(\cdot)$ would be indistinguishable as far as statistics from the same tape recording played backwards.) What is the condition for $y(\cdot)$ to have the reversibility property? Clearly, for all t and τ,

$$E[y(t)y'(t+\tau)]=E[y(t)y'(t-\tau)]. \tag{4.1}$$

Using stationarity, we have

$$E[y(t)y'(t-\tau)]=E[y(t+\tau)y'(t)].$$

Thus (4.1) implies that the covariance of $y(\cdot)$ is symmetric, and thus so is its power spectrum matrix. The converse is easy. In summary we have the following proposition.

Proposition 3: A stationary process $y(\cdot)$ has the second-order reversibility property if and only if its power spectral matrix is symmetric.

Note that every scalar process is reversible. Theorem 2 states that reversible models always yield reversible processes; we shall show below that reversible processes have some models that are reversible, at least in a certain extended sense.

For this analysis, we develop state-variable interpretations in Proposition 4 below of the symmetry of $\Phi(s)$ and the self-duality or model-reversibility of a spectral factor in state-variable form. The calculations are very similar to some that have been used in studying reciprocal networks, [9], [10]. Thus let us suppose that the model $W(s)$ is defined by a quadruple $\{F,G,H,J\}$ and that the associated power spectrum is as given in (2.9), which we repeat as

$$\Phi(s)=JJ'+H(sI-F)^{-1}K+K'(sI-F')^{-1}H'. \quad (4.2)$$

The next proposition, which slightly extends a result of Youla and Tissi [8], sets up the existence of a certain matrix which will be used in Theorem 3 to define a new coordinate basis displaying internal reversibility.

Proposition 4: Let $W(s)$ with minimal state variable realization $\{F,G,H,J\}$ be a spectral factor of $\Phi(s)$ such that $\mathrm{Re}\lambda_i(F)<0$. Suppose that

$$W(s)=W_b(-s)Q, \quad \text{some orthogonal } Q \quad (4.3a)$$

i.e.,

$$J+H(sI-F)^{-1}G=JQ-(H'+\Pi^{-1}GJ')'\cdot(sI-\Pi F'\Pi^{-1})^{-1}GQ \quad (4.3b)$$

where Π satisfies $\Pi F'+F\Pi=-GG'$. Then there exists a unique nonsingular symmetric T such that

$$K'T=H \quad T^{-1}F'T=F$$
$$-T^{-1}\Pi^{-1}GQ=G \quad T\Pi T=\Pi^{-1} \quad (4.4)$$

with (cf. (2.9))

$$K=\Pi H'+GJ'.$$

Before proving the proposition, we remark that if $\Phi(s)$ is symmetric, (4.2) implies $H(sI-F)^{-1}K=K'(sI-F')^{-1}H'$. Then if $\{F,K,H\}$ is a minimal triple, the existence of a symmetric T satisfying the first two equalities in (4.4) follows. In the proposition, however, we make a different, and weaker, minimality assumption. (The minimality of $\{F,G,H\}$ need not imply the minimality of $\{F,K,H\}$, though the converse is true.)

Proof: (See [8] or [9, p. 321] and [21]).

First note that (4.3) implies

$$H(sI-F)^{-1}G=-K'\Pi^{-1}(sI-\Pi F'\Pi^{-1})^{-1}GQ$$
$$=K'(sI-F')^{-1}\Pi^{-1}GQ.$$

Next we recall a basic state-space theorem which says that any two minimal realizations must be related by a unique similarity transformation. In our problem this theorem shows that there is a unique nonsingular T such that the first three equations of (4.4) hold.

Now let us check that T must be symmetric. We shall establish that the first three equations of (4.4) hold with T' replacing T; the uniqueness of T then implies $T=T'$. Now because $W(s)=W_b(-s)Q$, $\Phi(s)$ is symmetric and then so is $H(sI-F)^{-1}K$. Therefore,

$$H(sI-F)^{-1}K=K'(sI-F)^{-1}H'$$
$$=K'T(sI-T^{-1}F'T)^{-1}T^{-1}H'$$
$$=H(sI-F)^{-1}T^{-1}H'.$$

(The third equality follows from the first two equations of (4.4).) Now because $[F,H]$ is completely observable, $K=T^{-1}H'$ or $K'T'=H$. The second equation of (4.4) yields $T^{-T}F'T'=F$ on transposition, so that

$$K'(sI-F')^{-1}\Pi^{-1}GQ$$
$$=K'T'[sI-T^{-T}F'T']^{-1}T^{-T}\Pi^{-1}GQ$$
$$=H(sI-F')^{-1}T^{-T}\Pi^{-1}GQ.$$

However, this expression is also, as noted above, $-H(sI-F)^{-1}G$, so that $G=T^{-T}\Pi^{-1}GQ$. Therefore, T' satisfies the first three equations of (4.4), and so $T=T'$.

The last equation of (4.4) is derived as follows. Since $\Pi F'+F\Pi=-GG'$, we have

$$T^{-1}\Pi^{-1}\Pi F'\Pi^{-1}T^{-1}+T^{-1}\Pi^{-1}F\Pi\Pi^{-1}T^{-1}$$
$$=T^{-1}\Pi^{-1}GQQ'G'\Pi^{-1}T^{-1}$$

or

$$FT^{-1}\Pi^{-1}T^{-1}+T^{-1}\Pi^{-1}T^{-1}F'=-GG'.$$

Since Π is a unique solution of $FX+XF'=-GG'$, $\Pi=T^{-1}\Pi^{-1}T^{-1}$. ∎

Our task is now to exhibit a reversibility property of the state process of a reversible or self-dual model. We shall show that in a suitable coordinate basis, the state vector, call it $\bar{x}$, has the property

$$\Sigma E[\bar{x}(t)\bar{x}'(t-T)]=E[\bar{x}(t)\bar{x}'(t+T)]\Sigma \quad (4.5)$$

where Σ is a diagonal (so-called signature) matrix with $+1,-1$ elements on the diagonal. By arranging the vector entries so that $\Sigma=I_p\dotplus -I_q$, (4.5) says that

$$E[\bar{x}_p(t)\bar{x}_p'(t-T)]=E[\bar{x}_p(t)\bar{x}_p'(t+T)]$$
$$E[\bar{x}_q(t)\bar{x}_q'(t-T)]=E[\bar{x}_q(t)\bar{x}_q'(t+T)]$$
$$E[\bar{x}_p(t)\bar{x}_q'(t-T)]=-E[\bar{x}_p(t)\bar{x}_q'(t+T)].$$

Thus $\bar{x}_p(\cdot)$ and $\bar{x}_q(\cdot)$ are separately reversible, but jointly are not. However, modulo a sign change there is a reversibility. This type of generalized reversibility of a process is a standard concept in the thermodynamics of irreversible processes, see [16, ch. XI] and also arises in network theory (see, e.g., [8]–[10]). Motivated by examples from physics, Whittle [14] has suggested the name *dynamic reversibility* for this property, and we shall use this designation.

The following theorem is related to one in network theory (cf. [9, p. 324]).

Theorem 3

Let $W(s)$ with minimal state variable realization $\{F,G,H,J\}$ be a spectral factor of a symmetric rational power spectrum matrix $\Phi(s)$ such that $\mathrm{Re}\lambda_i(F)<0$. With notation as earlier, suppose that $W(s)=W_b(-s)Q$, i.e., (4.3) holds where Π solves $\Pi F'+F\Pi=-GG'$. Let T be the unique symmetric nonsingular matrix satisfying (4.4) with $K=\Pi H'+GJ'$. Then there exists an *orthogonal* matrix V such that

$$\Pi^{1/2}T\Pi^{1/2}=V'\Sigma V \tag{4.6}$$

where $\Pi^{1/2}$ is the positive definite square root of Π and Σ is a diagonal matrix with $+1$ and -1 elements on the diagonal. Define a state-space transformation

$$\bar{x}=V\Pi^{-1/2}x \tag{4.7}$$

where x is the state-variable in the noise model (2.9) defined by $W(s)$. Then $\bar{x}$ has the generalized reversibility property of (4.5).

Before proving the result, we note that (4.6) implies $T=\Pi^{-1/2}V'\Sigma V\Pi^{-1/2}$, and in view of the appearance of $V\Pi^{-1/2}$ in (4.7), it might be thought that any decomposition of T as $W'\Sigma W$ might definite a suitable transformation matrix W. This is actually not the case: the orthogonality of V is critical. Note also that there exist decompositions of the form (4.6) in which V is not necessarily orthogonal. *For example, if* $\Sigma=\mathrm{diag}[1,-1]$ *and* V *is orthogonal, then*

$$V'\Sigma V=V'\begin{bmatrix}\cosh x & \sinh x\\ \sinh x & \cosh x\end{bmatrix}\Sigma\begin{bmatrix}\cosh x & \sinh x\\ \sinh x & \cosh x\end{bmatrix}V$$

and

$$\begin{bmatrix}\cosh x & \sinh x\\ \sinh x & \cosh x\end{bmatrix}V$$

is clearly not orthogonal.

Proof of Theorem 3: We first exhibit the existence of V. From the fourth equation of (4.4) we have $(\Pi^{1/2}T\Pi^{1/2})(\Pi^{1/2}T\Pi^{1/2})=I$. Therefore, $\Pi^{1/2}T\Pi^{1/2}$ is a symmetric matrix with eigenvalues ± 1. By a standard theorem of linear algebra, an orthogonal V exists satisfying (4.6).

Now we turn to the reversibility property. Let $\bar{F}=V\Pi^{-1/2}F\Pi^{-1/2}V'$. Observe that $E[\bar{x}(t)\bar{x}'(t)]=V\Pi^{-1/2}E[\bar{x}(t)\bar{x}'(t)]\Pi^{-1/2}V'=I$. Therefore, (4.5) is equivalent to

$$\Sigma e^{\bar{F}T}=e^{\bar{F}'T}\Sigma$$

for all T, which will hold if and only if $\Sigma\bar{F}=\bar{F}'\Sigma$ or

$$\Sigma V\Pi^{-1/2}F\Pi^{-1/2}V'=V\Pi^{-1/2}F'\Pi^{-1/2}V'\Sigma.$$

However, this is a direct consequence of the fact that $TF=F'T$. ∎

Several comments are appropriate.

1) One has $E[\bar{x}(t)\bar{x}'(t)]=I$. Moreover the construction of the transformation leading to $\bar{x}$ can be obtained in two steps if desired, the first step to a state $\hat{x}(t)$ for which $E[\hat{x}(t)\hat{x}'(t)]=I$, this step being achievable by a coordinate basis transformation $\hat{x}=\Pi^{1/2}x$. The second step takes $\hat{x}$ to $\bar{x}$, preserving the covariance of $\bar{x}$. If Π is replaced by I in Proposition 4, one sees that $T^2=I$ and $T=T'$, while in Theorem 4, $T=V'\Sigma V$.

2) It is important to note that the joint process $[\bar{x}'(t)y'(t)]'$ possesses a dynamic reversibility property. One can check that

$$\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix}E\left\{\begin{bmatrix}\bar{x}(t)\\ y(t)\end{bmatrix}\left[\bar{x}'(t-T)y'(t-T)\right]\right\}=E\left\{\begin{bmatrix}\bar{x}(t)\\ y(t)\end{bmatrix}\left[\bar{x}'(t+T)y'(t+T)\right]\right\}\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix}. \tag{4.8}$$

Given any spectral factor in state-variable form with state process $x(t)$ and dynamic reversibility of $[\bar{x}'(t)y'(t)]$, it does not follow that $E[x(t)x'(t)]=I$, but it does follow that the self-duality property (4.3) holds, and in this sense we have a converse of Theorem 3. To check this, observe that the dynamic reversibility property is equivalent to

$$\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}e^{FT}\Pi & e^{FT}K\\ He^{FT}\Pi & He^{FT}K\end{bmatrix}=\begin{bmatrix}\Pi e^{F'T} & \Pi e^{F'T}H'\\ K'e^{F'T} & K'e^{F'T}H'\end{bmatrix}\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix}$$

from which $\Sigma\Pi=\Pi\Sigma$, $\Sigma F\Pi=\Pi F'\Sigma$, $\Sigma K=\Pi H'$. These equations imply that

$$\begin{aligned}J+H(sI-F)^{-1}G&=J+K'\Pi^{-1}\Sigma(sI-\Sigma\Pi F'\Pi^{-1}\Sigma)^{-1}G\\&=J+K'\Pi^{-1}(sI-\Pi F'\Pi^{-1})^{-1}\Sigma G.\end{aligned}$$

Also, pre- and post-multiplying $\Pi F'+F\Pi=-GG'$ by Σ yields $\Sigma GG'\Sigma=GG'$ or $\Sigma G=GQ$ for some orthogonal Q, i.e., (4.3) holds.

4) If we set $\bar{H}'=\Pi^{-1/2}V'H'$, one can check that $\Phi(s)=JJ'+\bar{H}(sI-\bar{F})^{-1}\Sigma\bar{H}'+\bar{H}\Sigma(sI-\bar{F})^{-T}H'$, where, we recall from the proof of Theorem 3, $\Sigma\bar{F}=\bar{F}'\Sigma$. It follows that if $\Sigma=I$,

$$\Phi(s)=JJ'+\sum_{i=1}\frac{A_i}{(s+\lambda_i)}+\sum_{i=1}\frac{A_i}{(-s+\lambda_i)} \tag{4.9}$$

where $A_i=A_i'$ and is nonnegative definite, and the λ_i are distinct positive real constraints. In order, therefore, to have strict reversibility of the process $[\bar{x}'(t)y'(t)]'$, $\Phi(s)$ must be the spectral density of a vector of correlated RC-noise (or stationary Markov) processes.

Further, if $\Phi(s)$ has the form (4.9), it is always possible to obtain a self-dual model for $\Phi(s)$. One possibility (but not the only one) is provided by

$$\begin{aligned}F&=\mathrm{diag}\left[-\lambda_1 I,\cdots,-\lambda_k,I\right]\\G&=\left[\mathrm{diag}\left[\sqrt{2\lambda_1},I,\cdots,\sqrt{2\lambda_k},I\right]\;\vdots\;0\right]\\H'&=\left[A_1^{T/2},A_2^{T/2},\cdots,A_k^{T/2}\right]\\J&=\left[0\;\vdots\;\Phi^{1/2}(\cdot\,\cdot)\right].\end{aligned} \tag{4.10}$$

Actually, one can also construct a irreducible finite-state Markov model with reversible state process and with output process possessing the spectrum (4.8), [18].

V. Self-Dual Models and Partial State Reversibility

In this section, we consider the situation in which an intermediate variable $z(\cdot)$, the "partial state" associated with a matrix fraction description of a spectral factor matrix, links the input $u(\cdot)$ and output $z(\cdot)$:

$$D\left(\frac{d}{dt}\right)z(t)=u(t) \quad y(t)=N\left(\frac{d}{dt}\right)z(t). \tag{5.1}$$

We aim to discover a reversibility property of $z(\cdot)$ given the self-duality of $W(s)=N(s)D^{-1}(s)$. However, it is preferable to consider $z(\cdot)$ as part of a larger process $[z'(\cdot)y'(\cdot)]'$, and look for a reversibility property of this process. For otherwise, we could conceive of a nonself-dual scalar spectral factor with $\{D(s),N(s)\}$ scalar; then because $z(\cdot)$ is a scalar process, it automatically is reversible. By considering the larger process $[z'(\cdot)y'(\cdot)]'$, we shall be able to draw the conclusion that in the right coordinate basis, this process has dynamic reversibility if and only if $W(s)=N(s)D^{-1}(s)$ has the self-duality property.

To discover a reversibility property for $z(\cdot)$ given the self-duality of $W(s)=N(s)D^{-1}(s)$, the general approach is to obtain a unimodular matrix that transforms D and N so that the resulting new $z(\cdot)$, call it $\tilde{z}(\cdot)$, has with $y(\cdot)$ the same sort of generalized reversibility property as did the transformed state-variable $\bar{x}(\cdot)$ with $y(\cdot)$ in Theorem 3. We begin with a proposition that will allow construction of the transforming unimodular matrix.

Proposition 5: Let $W(s)$ with matrix fraction description $N(s)D^{-1}(s)$, where $N(s)$ and $D(s)$ are relatively right prime, be a spectral factor of a power spectrum matrix $\Phi(s)$ with all zeros of $\det D(s)$ in $\mathrm{Re}[s]<0$. Let $D_b(s)$ be the unique matrix polynomial dual to $D(s)$ and suppose that

$$N(-s)D_b^{-1}(-s)Q=N(s)D^{-1}(s)$$

for some orthogonal Q. Then for some orthogonal T and diagonal signature matrix Σ,

$$N(s)D^{-1}(s)=N(-s)D_b^{-1}(-s)T\Sigma T' \tag{5.2}$$

and there exists a unimodular matrix $V(s)$ such that

$$\begin{aligned} N(s)V(s)&=N(-s),\\ D(s)V(s)&=T\Sigma T'D_b(-s),\\ V(s)V(-s)&=I. \end{aligned} \tag{5.3}$$

Remark: The effect of (5.2) is to restrict the orthogonal matrices Q that can arise in the equality $W(s)=W_b(-s)Q$. Rewriting (5.2) as

$$N(s)D^{-1}(s)T=N(-s)D_b^{-1}(-s)T\Sigma$$

and redefining $W(s)$ as

$$N(s)[T'D(s)]^{-1}$$

it is immediate the $W(s)=W_b(-s)\Sigma$.

Proof: By the right coprimeness of $[N(s),D(s)]$ and the equality of (5.2) there exists [6] a unimodular $U(s)$ such that

$$N(s)U(s)=N(-s) \quad D(s)U(s)=Q'D_b(-s)$$

and then

$$\begin{aligned} W(s)&=N(-s)U(-s)[Q'D_b(-s)U(-s)]^{-1}\\ &=N(s)[Q'D_b(-s)U(-s)]^{-1}\\ &=N(s)\tilde{D}^{-1}(s) \end{aligned}$$

where $\tilde{D}(s)=Q'D_b(-s)U(-s)$. We can easily check that $|\tilde{D}|=\pm|D|$ and $\tilde{D}_*\tilde{D}=D_*D$. Hence $\tilde{D}(s)=KD(s)$ for some orthogonal K, which we can identify as follows.

$$\begin{aligned} K&=\tilde{D}(s)D^{-1}(s)\\ &=Q'D_b(-s)U(-s)D^{-1}(s)\\ &=Q'D_b(-s)D^{-1}(-s)Q'D_b(s)D^{-1}(s). \end{aligned}$$

Letting $s\to\infty$ yields $K=Q'^2$, so that

$$W(s)=N(s)D^{-1}(s)=N(s)D^{-1}(s)Q^2. \tag{5.4}$$

Now let T be an orthogonal matrix such that

$$T'QT=\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Delta\end{bmatrix}$$

with Σ_1 a signature matrix and $I-\Delta^2$ a $q\times q$ nonsingular matrix for some q. Then (5.4) yields

$$N(s)D^{-1}(s)T\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}=N(s)D^{-1}(s)T\begin{bmatrix}I & 0\\ 0 & \Delta^2\end{bmatrix}$$

from which it is clear that the last q columns of $N(s)D^{-1}(s)T$ are zero. Then

$$\begin{aligned} N(s)D^{-1}(s)T&=N(-s)D_b^{-1}(-s)T\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Delta\end{bmatrix}\\ &=N(-s)D_b^{-1}(-s)T\begin{bmatrix}\Sigma_1 & 0\\ 0 & I\end{bmatrix} \end{aligned}$$

and with $\Sigma=\mathrm{diag}[\Sigma_1,I]$, (5.2) follows.

Now repeat the proof of the proposition with Q replaced by $T\Sigma T'$ and $U(s)$ by $V(s)$. The first two equations of (5.3) are immediate. We also obtain $\tilde{D}(s)=D(s)$, since $K=Q^2$ is replaced by $K=I$. Thus

$$T\Sigma T'D_b(-s)V(-s)=D(s)$$

or

$$D(s)V(s)V(-s)=D(s)$$

using the second equation of (5.3). The third equation of (5.3) is immediate.

The unimodular nature of $V(s)$ and the fact that it satisfies the third equation in (5.3) allows us to obtain a certain decomposition, proved in [16].

Proposition 5: Let $V(s)$ be a unimodular matrix such that $V(s)V(-s)=I$. Then there exists a unimodular $M(s)$ and a diagonal signature matrix $\tilde{\Sigma}$ such that

$$V(s)=M(s)\tilde{\Sigma}M^{-1}(-s). \tag{5.5}$$

The matrix $M(s)$ is now used to set up a new matrix fraction description of $W(s)$, as

$$\tilde{D}(s)=D(s)M(s) \quad \tilde{N}(s)=N(s)M(s).$$

The dual polynomial of $\tilde{D}(s)$, which we shall denote by $\tilde{D}_b(s)$, is immediately formed as

$$\tilde{D}_b(s)=D_b(s)M(s).$$

Corresponding to the first two equations of (5.3), we have

$$\tilde{N}(s)\tilde{\Sigma}=\tilde{N}(-s) \tag{5.6a}$$

$$\tilde{D}(s)\tilde{\Sigma}=T\Sigma T'\tilde{D}_b(-s). \tag{5.6b}$$

(The third equation is simply $\tilde{\Sigma}^2=I$.) This new matrix fraction description $\tilde{N}(s)\tilde{D}^{-1}(s)$ of $W(s)$ gives us the dynamic reversibility property we are seeking.

Theorem 4

With the hypotheses of Proposition 5, let $\tilde{N}(s)\tilde{D}^{-1}(s)=N(s)M(s)[D(s)M(s)]^{-1}$ be another right matrix fraction decomposition of $W(s)$ with $M(s)$ constructed as described in Propositions 4 and 5, and let $\tilde{z}(\cdot)$ be defined by $\tilde{D}(d/dt)\tilde{z}(t)=u(t), y(t)=\tilde{N}(d/dt)\tilde{z}(t)$ with $u(\cdot)$ a vector of unit intensity independent white noise sources. Then

$$\begin{bmatrix}\tilde{\Sigma} & 0\\ 0 & I\end{bmatrix}E\left\{\begin{bmatrix}z(t)\\ y(t)\end{bmatrix}[z'(t-T)y'(t-T)]\right\} = E\left\{\begin{bmatrix}z(t)\\ y(t)\end{bmatrix}[z'(t+T)y'(t+T)]\right\}\begin{bmatrix}\tilde{\Sigma} & 0\\ 0 & I\end{bmatrix}. \tag{5.7}$$

Proof: The power spectrum of the process $[z'(t)y'(t)]'$ is

$$\Psi(s)=\begin{bmatrix}(\tilde{D}_*\tilde{D})^{-1} & (\tilde{D}_*\tilde{D})^{-1}\tilde{N}_*\\ \tilde{N}(\tilde{D}_*\tilde{D})^{-1} & \tilde{N}(\tilde{D}_*\tilde{D})^{-1}\tilde{N}_*\end{bmatrix}$$

$$=\begin{bmatrix}(\tilde{D}_b\tilde{D}_{b*})^{-1} & (\tilde{D}_b\tilde{D}_{b*})^{-1}\tilde{N}_*\\ \tilde{N}(\tilde{D}_b\tilde{D}_{b*})^{-1} & \tilde{N}(\tilde{D}_b\tilde{D}_{b*})^{-1}\tilde{N}_*\end{bmatrix} \quad \text{by duality}$$

$$=\begin{bmatrix}\tilde{\Sigma}(\tilde{D}_{b*}\tilde{D}_b)^{-T}\tilde{\Sigma} & \tilde{\Sigma}(\tilde{D}_{b*}\tilde{D}_b)^{-T}\tilde{\Sigma}\tilde{N}_*\\ \tilde{N}\tilde{\Sigma}(\tilde{D}_{b*}\tilde{D}_b)^{-T}\tilde{\Sigma} & \tilde{N}\tilde{\Sigma}(\tilde{D}_{b*}\tilde{D}_b)^{-T}\tilde{\Sigma}\tilde{N}_*\end{bmatrix} \quad \text{by (5.6a)}$$

$$=\begin{bmatrix}\tilde{\Sigma} & 0\\ 0 & I\end{bmatrix}\Psi'(s)\begin{bmatrix}\tilde{\Sigma} & 0\\ 0 & I\end{bmatrix} \quad \text{by (5.6b)}.$$

This is equivalent to (5.7).

It is easy to check the converse result, that if (5.7) holds, then we must necessarily have $\tilde{N}(s)\tilde{D}^{-1}(s)$ defining a self-dual realization. We shall omit the proof. ∎

VI. Self-Dual Model Construction

Statistical mechanics suggests that many physically based random processes have an internal dynamic reversibility property [10], [16]. This suggests that we should be concerned with the construction and even identification of self-dual models for symmetric power spectrum matrices. In this section, we refer briefly to some constructions for self-dual models. These constructions are essentially tied to reciprocal network synthesis procedures for passive symmetric impedance or scattering matrices, [7]–[9].

If ND^{-1} is a spectral factor and D_b is the dual of D then

$$W(s)=\frac{1}{\sqrt{2}}[N(s)N(-s)]\begin{bmatrix}D(s) & 0\\ 0 & D_b(-s)\end{bmatrix}^{-1}$$

turns out to be self-dual if $\Phi(s)=\Phi'(s)$. For a state-variable version considered in network-theoretic terms see [9, sec. 9.4].

The Darlington network synthesis procedure takes a rational scalar spectral factor $p(s)/q(s)$ and obtains from it a spectral factor for the same spectrum of the form $m(s)/n(s)$ with $m(s)$ possessing a zero pattern symmetric with respect to the $j\omega$-axis. A multivariable version—the reciprocal Bayard synthesis—can be found in [7], with a state-space description of the procedure in [9].

Finally, given a state-variable realization of a spectral factor $W(s)$, viz., $\dot{x}=Fx+Gu$, $y=Hx+Ju$, with the McMillan degree of $W(s)$ one-half the McMillan degree of $W(s)W'(-s)$ one can find a second, self-dual spectral factor with the same F,H but different $\bar{G},\bar{J}$ by either of two equivalent procedures described in [9] and [10]. A casting of this idea into the framework of reversible processes can be found in [21].

VII. Conclusions

The ideas of this paper have flowed along two streams. First, we have analyzed the relationship between a forward spectral factor of a prescribed power spectrum, and an associated backward spectral factor. Further, we have summarized some physical significance which can be given to the relationship via network theory. Secondly, we have analyzed the special case of symmetric power spectrum matrices. Here there is the possiblity of self-dual spectral factor existence. We also have connection to ideas of statistical thermodynamics: symmetric power spectrum matrices correspond to reversible processes, and self-dual spectral factors to internally reversible models of such processes. In view of the fact that many physical processes are internally reversible (witness the extensive applicability of the Onsager reciprocal relations [10], [16]) such models perhaps deserve more prominence than they have hitherto been given in the system theory (as opposed to the network theory) literature.

There are still important questions to answer concerning such models. In general, a minimum phase spectral factor (expressed as a matrix fraction of two coprime matrices) is not self-dual. There arises the question of whether, among the class of self-dual models of a prescribed symmetric power spectrum, there is one possessing even some of the special properties of the minimum phase spectral factor. Secondly, it has proven possible to establish certain partitions of the class of nonself-dual models of minimal state-variable dimension of a prescribed power spectrum [20]. Equally, one can ask how self-dual models of a symmetric spectrum might be grouped.

APPENDIX I
PROOF OF THEOREM I

Most of Theorem 1 is proved in [14], the proof in [14] containing a constructive procedure. Here, we give an alternative proof.

Suppose first that $D(s)$ and $D(-s)$ have coprime determinants.

Let $V(s)$ be a unimodular matrix such that $D(s)V(s)=\bar{D}(s)$ is column proper (see [6] for a constructive procedure). We shall find a dual $\bar{D}_b(s)$ for $\bar{D}(s)$. It is easily checked that if $D(s)$ and $D_b(s)$ satisfy any of (3.4) through (3.6), $\bar{D}(s)$ and $\bar{D}_b(s)=D_b(s)V^{-1}(-s)$ satisfy the same equations.

With $\bar{D}(s)$ column proper, it is known [6] that there exists a controllable pair $[F,G]$ such that any proper transfer function $A(s)\bar{D}^{-1}(s)$ can be expressed as $J_A+H_A(sI-F)^{-1}G$ for some J_A, H_A and conversely. Moreover if $\bar{D}(s)$ and $\bar{D}(-s)$ have coprime determinants, no two eigenvalues of F sum to zero and there exists a unique Π such that

$$\Pi F'+F\Pi=-GG'. \tag{A.1}$$

Further, we can argue that Π is nonsingular; for suppose $\alpha'\Pi=0$ for same $\alpha\neq 0$. Then $\alpha'GG'\alpha=0$ whence $\alpha'G=0$. Then $\alpha'F\Pi=0$. Thus the left nullspace of Π is F-invariant, which means it contains a nonzero element β' such that $\beta'F=\lambda\beta'$ for some λ. Also $\beta'G=0$. This contradicts the complete controllability of $[F,G]$.

Now define $\bar{D}_b(s)$ by

$$\bar{D}_b(s)\bar{D}^{-1}(s)=I-G'\Pi^{-1}(sI-F)^{-1}G. \tag{A.2}$$

Using (A.1), one readily verifies that

$$\bar{D}'(-s)\bar{D}(s)=\bar{D}_b'(-s)\bar{D}_b(s) \tag{A.3}$$

while obviously

$$\lim_{s\to\infty}\bar{D}(s)\bar{D}_b^{-1}(s)=I. \tag{A.4}$$

Also,

$$\begin{aligned}|\bar{D}_b(-s)|&=|sI-F-GG'\Pi^{-1}|\\&=|sI+\Pi F'\Pi^{-1}|=(-1)^n|\bar{D}(s)|.\end{aligned} \tag{A.5}$$

Next, suppose that in addition to $\bar{D}_b(s)$, there exists $\tilde{D}_b(s)$ with

$$\bar{D}'(-s)\bar{D}(s)=\tilde{D}_b'(-s)\tilde{D}_b(s) \tag{A.6}$$

$$|\bar{D}(s)|=\pm|\tilde{D}_b(s)|. \tag{A.7}$$

Then,

$$\tilde{D}_b^{-T}(s)\bar{D}_b'(s)=\tilde{D}_b(-s)\bar{D}_b^{-1}(-s)\triangleq R(s). \tag{A.8}$$

Comparison of the poles of both sides of (A.8) and use of the coprimeness of $|\bar{D}(s)|$ and $|\bar{D}(-s)|$ shows that $R(s)$ must be polynomial. Then (A.6) and (A.3) imply $R(s)R'(-s)=I$. By considering diagonal entries with $s=j\omega$, we see that $R(s)$ must have constant entries, i.e., must be orthogonal; thus $\bar{D}_b(s)=R'\tilde{D}_b(s)$, as claimed in the theorem statement. The uniqueness of the $\bar{D}_b(s)$ satisfying (A.3)–(A.5) is also immediate.

Finally, let us suppose that $D(s)$ and $D(-s)$ do not have coprime determinants. The following constructive proof was suggested by a referee, and replaces an existing proof in an earlier draft of the paper. By use of a Smith decomposition of $D(s)$, we may write it as $A(s)B(s)$, where $|B(s)|=\pm|B(-s)|$ and $|A(s)|$ and $|A(-s)|$ are coprime. Then we define $D_b(s)=A_b(s)B(s)$, with $A_b(s)$ defined from $A(s)$ using the procedure above.

Clearly (3.6)–(3.8) all hold. The uniqueness results no longer hold however. Consider, for example,

$$D(s)=\begin{bmatrix}s-1&0\\0&s+1\end{bmatrix}.$$

Then,

$$D_{b_1}(s)=\begin{bmatrix}s+1&0\\0&s-1\end{bmatrix}$$

$$D_{b_2}(s)=\begin{bmatrix}s-1&0\\0&s+1\end{bmatrix}$$

both satisfy (3.6)–(3.8), but clearly do not differ through left multiplication by an arbitrary orthogonal matrix.

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