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Correspondence

Comments on 'On the number of roots of an algebraic equation contained between given limits, by C. Hermite' †‡

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Hermite's paper contains an unproved claim that the signature of a bézoutian-like form formed using derivatives of a homogenized prescribed polynomial is equal to the number of real zeros of the polynomial less one. This note proves the result, correcting in the process what appears to be a sign error in its statement.

The penultimate paragraph of Hermite's paper makes the following claim. Let $u=f(x, y)$ be a homogeneous polynomial of degree n, and construct the expression

$$\bar{B}(x, x_0, y, y_0)=\frac{1}{xy_0-x_0y}\left[\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{x=x_0,\, y=y_0}-\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)_{x=x_0,\, y=y_0}\right] \tag{1}$$

This is a bilinear form in $({x_0}^{n-2}, {x_0}^{n-3}\, y_0, \ldots, x_0{y_0}^{n-3}, {y_0}^{n-2})$ and $(x^{n-2}, x^{n-3}y, \ldots, xy^{n-3}, y^{n-2})$, whose signature is one less than the total number of real roots of $u=0$.

A simple example shows the claim to be somehow in error. Let $u(x, y)=x^2+2\alpha xy+\beta^2y^2$. Then $\bar{B}(x, x_0, y, y_0)$ evaluates as $4(\beta^2-\alpha^2)$, and the claim is seen to be untrue.

As it turns out, if we replace eqn. (1) by

$$B(x, x_0, y, y_0)=\frac{1}{xy_0-x_0y}\left[\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)_{x=x_0,\, y=y_0}-\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{x=x_0,\, y=y_0}\right] \tag{2}$$

the claim proves true, both for the example and in general. Of course, all that is involved is a sign reversal.

To understand this, let us first find an alternative expression for eqn. (2) which is equivalent in the sense that it defines the same bilinear form. This we shall do by forcing y_0 and y to be 1. Let

$$g(x)=f(x, 1) \tag{3}$$

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For an arbitrary nth degree polynomial $p(x)$, let $p^{\mathrm{I}}(x)$ or $[p(x)]^{\mathrm{I}}$ denote the inverse polynomial $x^n p(x^{-1})$. Then it is easy to check that

$$\left(\frac{\partial u}{\partial x}\right)_{y=1} = g'(x) \tag{4}$$

and

$$\left(\frac{\partial u}{\partial y}\right)_{y=1} = \left[\frac{d}{dx}\, g^{\mathrm{I}}(x)\right]^{\mathrm{I}} \tag{5}$$

Let us set

$$h(x) = \left[\frac{d}{dx}\, g^{\mathrm{I}}(x)\right]^{\mathrm{I}} \tag{6}$$

Then

$$\begin{aligned} h(x) &= \left[\frac{d}{dx}\, x^n g(x^{-1})\right]^{\mathrm{I}} \\ &= [n x^{n-1} g(x^{-1}) - x^{n-2} g'(x^{-1})]^{\mathrm{I}} \\ &= n g(x) - x g'(x) \end{aligned} \tag{7}$$

In view of eqns. (4) and (5), we see that eqn. (2) yields

$$B(x, x_0, 1, 1) = \frac{1}{x - x_0}\,[h(x) g'(x_0) - g'(x) h(x_0)] \tag{8}$$

which is a bilinear form in $({x_0}^{n-2}, {x_0}^{n-3}, \ldots, 1)$ and $(x^{n-2}, x^{n-3}, \ldots, 1)$; the matrix defining the form is of course the same as that defining the bilinear form associated with $B(x, x_0, y, y_0)$. Observe that eqn. (8) is precisely the usual Bézoutian form associated with the polynomial pair $[h(x), g'(x)]$. Therefore, to prove Hermite's result in corrected form, we need to show that the signature of the form is one less than the number of real zeros of $g(x)$.

To this end, let us give the definition and several properties of the Cauchy index of a rational function over $(-\infty, \infty)$: for the polynomials $p(x)$ and $q(x)$ (Gantmacher 1959),

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} q/p = (\text{number of jumps of } q/p \text{ from } -\infty \text{ to } +\infty)$$
$$\text{less (number of jumps of } q/p \text{ from } +\infty \text{ to } -\infty)$$
$$\text{as } x \text{ moves from } -\infty \text{ to } +\infty$$

We then have the following three properties:

(1) The number of distinct real roots of a real monic polynomial $g(x)$ is $\mathop{\mathrm{I}}_{-\infty}^{+\infty} g'(x)/g(x)$ (Gantmacher 1959).

(2) $\mathop{\mathrm{I}}_{-\infty}^{+\infty} q/p + \mathop{\mathrm{I}}_{-\infty}^{+\infty} p/q = \tfrac{1}{2}[\operatorname{sign}_{x\to\infty}(p/q) - \operatorname{sign}_{x\to-\infty}(p/q)]$. This formula appears in Gantmacher (1959) with an incorrect sign. It may readily be checked by taking $p = x$ and $q = 1$. Then $\mathop{\mathrm{I}}_{-\infty}^{+\infty} q/p = 1$, $\mathop{\mathrm{I}}_{-\infty}^{+\infty} p/q = 0$, $\operatorname{sign}_{x\to\infty}(p/q) = 1$ and $\operatorname{sign}_{x\to-\infty}(p/q) = -1$.

(3) When $\deg p \geqslant \deg q$, and p is monic, $\overset{+\infty}{\underset{-\infty}{\mathrm{I}}}\; q/p$ becomes the signature of the Bézoutian form $(x-y)^{-1}[p(x)q(y)-p(y)q(x)]$ defined by the pair $[p(x), q(x)]$. This formula can be obtained from Hermite's paper, but is confused by the omission of a scaling term involving $\sqrt{-1}$. It can also be found in Anderson (1972) for the case in which $\deg p > \deg q$, but the extension to cover $\deg p = \deg q$ is trivial.

Now,

$$\begin{aligned}\text{signature of the form of eqn. (2)} &= \text{signature of the form of eqn. (8)} \\ &= -\text{signature of Bézoutian form} \\ &\qquad (x-y)^{-1}[g'(x)h(y)-h(x)g'(y)] \\ &= -\overset{+\infty}{\underset{-\infty}{\mathrm{I}}}\; \frac{h(x)}{g'(x)} \end{aligned} \tag{9}$$

(Note that $\deg h(x) \leqslant n-1$ and $\deg g'(x) = n-1$, so that $h(x)/g'(x)$ is proper. Also, although $g'(x)$ is not monic, its leading coefficient is guaranteed positive and this permits the use of property (3).) Therefore, to say that the signature of eqn. (2) is one less than the number of real roots of $g(x)$ is equivalent, in the light of property (1) and eqns. (7) and (9), to proving that

$$-\overset{+\infty}{\underset{-\infty}{\mathrm{I}}}\; \frac{ng(x)-xg'(x)}{g'(x)} = \overset{+\infty}{\underset{-\infty}{\mathrm{I}}}\; \frac{g'(x)}{g(x)} - 1 \tag{10}$$

Using the definition of the Cauchy index, one can see that the left-hand side is $-\overset{+\infty}{\underset{-\infty}{\mathrm{I}}}\; g(x)/g'(x)$ and the requisite equality then follows from property (2).

It is interesting to consider whether Hermite's thinking ran as above, or whether there is some real advantage in working with the homogeneous form that is lost in the above analysis. Whatever route Hermite's thinking took, however, it is clear that the earlier studies of Sturm and Cauchy provided the foundation on which he developed his work.

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References

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