A Dynamical Interpretation of Column and Row Properness of Matrix Fraction Descriptions

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Abstract—**Column or row properness of matrix fraction descriptions (MFD) may be interpreted as coprimeness at infinity. A bilinear mapping of the complex plane is used to discern a structure at infinity and the results are established by considering the dynamic behavior of the system described by the MFD. A criterion for the equivalence of MFD's at all points including infinity is presented also.**

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I. Introduction

Left, right, and mixed matrix fraction descriptions (MFD) of linear systems, i.e., representation of transfer function matrices in the form $A^{-1}B, DC^{-1}, WV^{-1}U$ for matrix polynomials A, B, etc., are commonly encountered. Two structural properties of such matrix fractions are often of interest—coprimeness (which is an idea closely linked to that of minimality of a state variable description), and column or row properness. Material explaining these ideas and their application can be found in, e.g., [1]–[3].

Our aim here is to tie these ideas together. More specifically, we argue below for right MFD's that, whereas coprimeness of C, D in DC^{-1} amounts to an absence of pole-zero cancelation or decoupling zeros for all finite s, column properness of $[C'D']'$ amounts to absence of a decoupling zero at $s=\infty$. (Of course, the sense in which one could have a decoupling zero at $s=\infty$ needs definition, and is possibly arguable.) Results also follow for left MFD's and mixed MFD's.

The fact that coprimeness and column or row properness are in some way specializations of the one idea is discussed in an abstract way using valuation theory in [3]. This note can then be thought of as giving a more concrete basis to this idea. Further references considering related questions are [4], [5].

II. Effect of s-Plane Mappings

The first result shows that if the point $s=\infty$ is mapped to $s=-\frac{1}{\epsilon}$ for some generic ϵ, in the process changing DC^{-1} to $D_\epsilon C_\epsilon^{-1}$, then the pair $[D_\epsilon, C_\epsilon]$ acquires one or more decoupling zeros at $s=-\frac{1}{\epsilon}$, if and only if the original pair was not column proper.

Theorem: Consider a full column rank polynomial $p\times m$ matrix $P(s)=[C'(s)D'(s)]'$ with column degrees ∂_i, $i=1,2,\cdots,m$ and with the greatest degree of any $m\times m$ minor of $P(s)$ being $n_{\min}$. Define for generic ϵ another polynomial matrix

$$P_\epsilon(s)=P\left(\frac{s}{1+\epsilon s}\right)\operatorname{diag}\left[(1+\epsilon s)^{\partial_i}\right]. \tag{1}$$

Then: the ith column of P_ϵ has degree ∂_i, P_ϵ is column proper, $\lim_{\epsilon\to 0} P_\epsilon(s)=P(s)$ and there is a greatest common right divisor of $P_\epsilon(s)$ whose determinant has a factor $(1+\epsilon s)$ of multiplicity precisely $\Sigma\partial_i-n_{\min}$, the *excess column complexity* of $P(s)$.

Proof: The construction procedure guarantees P_ϵ and P have the same column degrees and that $\lim_{\epsilon\to 0} P_\epsilon(s)=P(s)$. The highest order column coefficient matrix of $P_\epsilon(s)$ is $P(\epsilon^{-1})\operatorname{diag}[\epsilon^{\partial_i}]$, which has full rank by the generic nature of ϵ. To establish the final claim, let $V(s)$ be unimodular with $\bar{P}(s)=P(s)V^{-1}(s)$ column proper with column indices $\bar{\partial}_i$. Note that $\Sigma\bar{\partial}_i=n_{\min}$. Then

$$\begin{aligned}P_\epsilon(s)&=\bar{P}\left(\frac{s}{1+\epsilon s}\right)V\left(\frac{s}{1+\epsilon s}\right)\operatorname{diag}\left[(1+\epsilon s)^{\partial_i}\right]\\&=\bar{P}\left(\frac{s}{1+\epsilon s}\right)\operatorname{diag}\left[(1+\epsilon s)^{\bar{\partial}_i}\right]\operatorname{diag}\left[(1+\epsilon s)^{-\bar{\partial}_i}\right]\\&\quad\cdot V\left(\frac{s}{1+\epsilon s}\right)\operatorname{diag}\left[(1+\epsilon s)^{\partial_i}\right]\\&=\bar{P}_\epsilon(s)U_\epsilon(s).\end{aligned} \tag{2}$$

Clearly, $\bar{P}_\epsilon$ is polynomial. We show that $U_\epsilon(s)$ is also, by showing it has no poles for finite s. Clearly, the only possible pole for $U_\epsilon(s)$ is at $s=-\frac{1}{\epsilon}$. However, $P_\epsilon(-\epsilon^{-1})$ is finite and $\bar{P}_\epsilon(-\epsilon^{-1})$ has full rank, being the highest order column coefficient matrix of column proper $\bar{P}$, multiplied by $\operatorname{diag}[(-\epsilon)^{-\bar{\partial}_i}]$. Hence, $U(s)$ has no finite poles. Because $V(s)$ is unimodular, $\det U_\epsilon(s)=k(1+\epsilon s)^{\Sigma\partial_i-n_{\min}}$ for some nonzero constant k. Since $\bar{P}_\epsilon(\epsilon^{-1})$ has full rank, no other right divisor of $P_\epsilon(s)$ can have a

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determinant divisible by $(1+\epsilon s)^{\beta}$ for $\beta > \Sigma\partial_i - n_{\min}$ and the theorem is proved.

The theorem shows that there is an infinitesimal perturbation of $P(s)$, such as to preserve column degrees, which introduces a decoupling zero arbitrarily close to $s=\infty$ of order equal to the excess column complexity. For this reason, we say that $P(s)$ has decoupling zeros at $s=\infty$. In physical terms, one could say that given a network implementation of $D(s)C^{-1}(s)$ addition of a series resistor of value ϵC_i^{-1} to each capacitor C_i and a shunt resistor $(\epsilon L)^{-1}$ to each inductor L_i would cause $D(s)C^{-1}(s)$ to be replaced by $D_\epsilon(s)C_\epsilon^{-1}(s)$

III. Behavior of Partial States

With notation as above, consider the two equation pairs

$$C\left(\frac{d}{dt}\right)\xi(t)=u(t) \quad D\left(\frac{d}{dt}\right)\xi(t)=y(t) \tag{3a}$$

$$C_\epsilon\left(\frac{d}{dt}\right)\xi_\epsilon(t)=u(t) \quad D_\epsilon\left(\frac{d}{dt}\right)\xi_\epsilon(t)=y_\epsilon(t). \tag{3b}$$

Suppose further that $u(\cdot)$ is smooth and that initial conditions at $t=0$ appropriate to the equations in (3) are matched, i.e., ξ and ξ_ϵ and certain derivatives take the same value at $t=0$. We claim that as $\epsilon\downarrow 0$, $\xi_\epsilon(t)\to\xi(t)$ in a distributional sense, and thus, $y_\epsilon(t)\to y(t)$. This fact supports the reasonableness of the perturbation introduced earlier. While we omit a full proof, we explain the key issues. It is clear that $C_\epsilon(s)\to C(s)$ as $\epsilon\to 0$, and so the proof hinges on showing that, as $\epsilon\to 0$, $\mathcal{L}^{-1}[C_\epsilon^{-1}(s)E_\epsilon(s)]\to \mathcal{L}^{-1}[C^{-1}(s)E(s)]$ in a distributional sense for some transforms $E_\epsilon(s)$, $E(s)$ which are well behaved at $s=-\frac{1}{\epsilon}$ and $s=\infty$, which are such that $\lim_{\epsilon\to 0} E_\epsilon(s)=E(s)$, and which take care of nonzero initial conditions and the input. Convergence of these inverse transforms turns out in turn to hinge on the question of whether $\mathcal{L}^{-1}\left[\frac{s^j}{(1+\epsilon s)^i}\right]\to\mathcal{L}^{-1}[s^j]$ as $\epsilon\to 0$. This can be readily checked by direct calculation to be true in case $\epsilon\downarrow 0$, but not $\epsilon\uparrow 0$. The fact that $\epsilon>0$ is required for (3a) and (3b) to be similar is consistent with the network interpretation given earlier—negative ϵ would yield negative resistors and, presumably, introduce instability.

IV. Equivalence

It is clearly of interest to consider when two matrices C_1, C_2 have the same decoupling zeros, both finite and infinite. We know [1], [2] that to have the same finite decoupling zeros, we need $C_1U=C_2$ for U unimodular. It is logical then to demand $C_{1\epsilon}U_\epsilon=C_{2\epsilon}$ for some unimodular U_ϵ as the further condition for C_1 and C_2 to have the same finite and infinite decoupling zeros. We characterize this extension as follows:

Theorem: Two $p\times m$ full rank polynomial matrices C_1, C_2 with column indices $\{\partial^i_{C_1}\}$ and $\{\partial^i_{C_2}\}$ have the same decoupling zeros, finite and infinite, if and only if for some unimodular $U(s)$, $C_1U=C_2$ and

$$\text{degree}\, u_{ij}(s) < \partial^j_{C_2} - \partial^i_{C_1} \qquad \text{for all } i,j \tag{4}$$

and

$$\sum \partial^i_{C_1} = \sum \partial^i_{C_2}. \tag{5}$$

Proof: Suppose C_1, C_2 have the same decoupling zeros. There must exist $U(s)$ such that $C_1U=C_2$. Since they have the same infinite decoupling zeros, there must exist a unimodular U_ϵ with $C_{1\epsilon}U_\epsilon=C_{2\epsilon}$. However, $C_1U=C_2$ leads to

$$C_{1\epsilon}(s)\,\text{diag}\left[(1+s\epsilon)^{-\partial^i_{C_1}}\right]U\left(\frac{s}{1+s\epsilon}\right)\text{diag}\left[(1+s\epsilon)^{\partial^i_{C_2}}\right]=C_{2\epsilon}(s) \tag{6}$$

whence we must have

$$U_\epsilon(s)=\text{diag}\left[(1+s\epsilon)^{-\partial^i_{C_1}}\right]U\left(\frac{s}{1+s\epsilon}\right)\text{diag}\left[(1+s\epsilon)^{\partial^i_{C_2}}\right]. \tag{7}$$

From the fact that $U_\epsilon(s)$ is polynomial and unimodular, the conclusions of the theorem follow. The argument is easily reversed to yield the converse.

It is in fact not hard to conclude that one must have $\{\partial^i_{C_1}\}=\{\partial^i_{C_2}\}$. This alone is however insufficient to guarantee the equivalence of C_1, C_2 described in the theorem. This would indicate that there is a structure associated with zeros at infinity, which is evidently reflected by the Smith forms of $C_{1\epsilon}$ and $C_{2\epsilon}$, and in particular by the powers of $(s+\epsilon^{-1})$ which appear in each diagonal entry of the two forms.

V. Left and Mixed MFD's

The results for left MFD's are obvious. For mixed MFD's, we shall illustrate the style of result by adopting several simplifications in the theorem statement.

Theorem: Consider the MFD $WV^{-1}U$ in which $[W, V]$ is right coprime, $[V, U]$ is left coprime, V has column and row degrees α_i and β_i, respectively, with ith column degree of W not greater than α_i and ith row degrees of U not greater than β_i. Let

$$\begin{bmatrix} V_\epsilon & U_\epsilon \\ W_\epsilon & 0 \end{bmatrix} = \text{diag}\left[(1+\epsilon s)^{\beta_i}\right]\oplus I \cdot \begin{bmatrix} V\left(\frac{s}{1+\epsilon s}\right) & U\left(\frac{s}{1+\epsilon s}\right) \\ W\left(\frac{s}{1+\epsilon s}\right) & 0 \end{bmatrix} \text{diag}\left[(1+\epsilon s)^{\alpha_i}\right]\oplus I. \tag{8}$$

Let ν_i, ν_o, ν_{io} denote the number of input, output and input–output decoupling zeros at $s=-\epsilon^{-1}$ of the matrix on the left side of (8). Then

$$\nu_i+\nu_o-\nu_{io}=\deg[\det V]+\text{excess column complexity of } V + \text{excess row complexity of } V \tag{9}$$

Proof: The McMillan degrees of $WV^{-1}U$ and $W_\epsilon V_\epsilon^{-1}U_\epsilon$ are known to be equal, since one transfer function matrix is obtained from the other by a bilinear transformation of the independent variable [6]. Accordingly, in view of the coprimeness of $[W, V]$ and $[V, U]$, the McMillan degree of $WV^{-1}U=\deg\det V$, see [6]. Also, for the same reason, all decoupling zeros of the matrix on the left side of (8) are at $-\epsilon^{-1}$, and $W_\epsilon V_\epsilon^{-1}U_\epsilon$ has McMillan degree equal to (see [1])

$$\deg\det V_\epsilon - [\nu_i+\nu_o-\nu_{io}].$$

Now $\deg\det V_\epsilon=\Sigma\alpha_i+\Sigma\beta_i$, as inspection of (8) shows, so that

$$\begin{aligned}\deg\det V &= \sum\alpha_i+\sum\beta_i-[\nu_i+\nu_o-\nu_{io}] \\ &= (\deg\det V+\text{excess column complexity}) \\ &\quad +(\deg\det V+\text{excess row complexity}) \\ &\quad -[\nu_i+\nu_o-\nu_{io}]\end{aligned}$$

and the result is immediate.

The above result is not as appealing as that for right, or left, MFD's. Decoupling zeros are introduced almost as unwanted artifacts by the perturbation, giving rise to the deg[det V] term on the right side of (9). Also, there is no separate accounting of ν_i and ν_o.

VI. Conclusions

From several points of view, it makes sense to conceive of lack of column or row properness in matrix fraction decompositions as corresponding to a lack of coprimeness at $s=\infty$.

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