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Least Order, Stable Solution of the Exact Model Matching Problem*†

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Model matching problems without stability constraints are easily solved, then stable solutions may be found by incorporating stability constraints via a parameterisation procedure using as few parameters as possible.

Key Word Index—Control theory; linear systems; models; transfer functions; stability.

Summary—The set of all solutions of the minimal design problem (MDP) is presented in parametric form. This result is obtained by first deriving a parametric representation of all minimal bases of a particular vector space. From this the set of all solutions of the MDP are obtained. Then conditions are placed on the parameters, which conditions define the set of stable solutions of the MDP.

Solutions to the nonminimal model matching problem are also presented in parametric form, and it is shown how in principle solutions of successively higher degree can be searched for a stable solution, should the MDP have no stable solution, so that a solution to the model matching problem can be found which has the lowest order consistent with a stability constraint.

1. INTRODUCTION

IT HAS been shown how to reformulate such problems as the minimal order system inverse problem[1, 2], the minimal order dynamic observer problem[2, 3] and the model following problem[3–5] as a problem known as the Minimal Design Problem (MDP). The MDP can be stated thus: given a $p\times m$ rational transfer matrix, $T_1(s)$, of rank p and a $p\times q$ rational transfer matrix, $T_2(s)$, find an $m\times q$ proper, rational transfer matrix, $T(s)$, of minimal dynamic order (or McMillan degree) such that

$$T_1(s)T(s) = T_2(s). \qquad (1.1)$$

If $T_1(s)$ has rank less than p, one can of course still study (1.1). Obviously, the row nullspace of $T_1(s)$ would have to be a subspace of the row nullspace of $T_2(s)$, in which case the problem can be reduced to one where rank T_1 is the number of rows of T_1. The problem has been solved in various ways[1–3, 6–8]. It should be noted that if $p \geqq m$ the MDP has either no solution or a unique solution which can easily be found. Normally then $p < m$. The first contribution of this paper is to show how all solutions of the MDP can be obtained and expressed in parametric form.

It is also of interest to solve the MDP with the further constraint that the solution be stable. This extra constraint may be essential for some problems such as the dynamic observer problem or it may merely be a desirable design criterion. Some progress has been made towards solving this stable MDP[3, 7, 9] but the results have been incomplete. This paper indicates how the parameters can be constrained so that from amongst all solutions of the MDP the stable ones can be selected.

If there is no stable solution to the MDP it is desirable to find a stable solution of (1.1) of least order. The question of whether or not there exists a stable solution of (1.1) is readily answered in reference [7]; however, no guide is available as to the least possible order of such a solution. In this paper an algorithm is presented which determines the least order stable solution to (1.1).

In essence, the algorithm constructs a parametric representation of all solutions of a certain degree (in fact, with a certain set of column degrees). Then one seeks to choose the parameters in order to have a stable solution. The problem of finding a proper solution to (1.1) without the constraint on minimality of order is known as the exact model matching problem[5].

The system descriptions used are almost all matrix fraction descriptions. While model matching has been viewed with state-variable concepts[8], questions of minimality and stability have not been well resolved.

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Section 2 discusses a special form of matrix which is used to present, in parametric form, all minimal bases[10] of a vector space. The particular vector space in which we shall be interested is the kernel of $[T_1(s) \vdots -T_2(s)]$, and minimal bases of this vector space may be used to solve the MDP. In Section 3 methods are examined which reduce the number of parameters required to express the solutions and an example of the solution procedure, together with the definition of stable solutions, is presented in Section 4.

Section 5 leads on to a discussion of the least order stable exact model matching problem and the algorithm presented is illustrated by an example in Section 6. Finally, in Section 7, an alternative problem to the exact model matching problem (1.1) is presented in which certain simplifications have been made to facilitate computation. The simplification arises from results of [3].

Perhaps the most important concepts of the paper are the parameterisation procedures for solutions, not just of the minimal design problem, but also of the model matching problem, with the elimination of excess parameters.

Before proceeding, it will be helpful to define a few terms which will be used later. The definitions given are based on those given in reference [10].

The *degree* of an n-tuple $(g_1, g_2, \ldots, g_n)^T$ of polynomials, is the greatest degree of its components g_j, $1 \leq j \leq n$.

If G is an $n \times k$ polynomial matrix with columns c_i, the ith *index* or column degree of G is defined as the degree of c_i, $1 \leq i \leq k$, and the *order* of G is defined as

$$\sum_{i=1}^{k} \text{degree } (c_i).$$

Note: (1) A matrix whose columns constitute a basis of a vector space will also be referred to as a basis of the vector space.

(2) All vector spaces referred to in this paper will be vector spaces of n-tuples over $F(s)$, the field of rational functions of s.

If V is a k-dimensional vector space of n-tuples over $F(s)$, then a *minimal basis* of V is an $n \times k$ polynomial matrix G such that G is a basis of V and G has least order among all polynomial bases of V.

The *invariant dynamical indices* (or just indices) of a vector space V of n-tuples over $F(s)$, are the indices of any minimal basis of V. The invariant dynamical indices will be denoted by $\nu(i)$, $i = 1, 2, \ldots, \dim V$.

If G is a $n \times k$ polynomial matrix with indices $\nu(i)$, $1 \leq i \leq k$, its *high order coefficient matrix*, denoted $[G]_h$, is the $n \times k$ real matrix whose ith *column consists of the coefficients of* $x^{\nu(i)}$ *in the* ith column of G.

Note: Since we will usually be referring to indices of basis matrices which are the same as the indices of the vector space, the alternative usages of the term index (and indices) will not create confusion. In the event of mention of the index of a matrix, which is not a minimal basis, being required, we will use the term *column degree*.

2. PARAMETRIC REPRESENTATION OF MINIMAL BASES AND MDP SOLUTIONS VIA THE ν-FORM MATRIX

In this section, we describe known background results.

Let all minimal polynomial bases of a k-dimensional rational vector space V be ordered according to increasing indices $\nu(1) \leq \nu(2) \leq \ldots \nu(k)$, and let

$$\nu(1) = \nu(2) = \ldots = \nu(r_1) < \nu(r_1+1) = \nu(r_1+2) = \ldots = \nu(r_1+r_2) < \nu(r_1+r_2+1) \ldots \leq \nu(k).$$

That is, the columns of the matrix of basis vectors are arranged in groups of r_i columns with the same index, the group of least index being first etc. For a particular vector space V the $\nu(i)$ are invariant[10] and hence so are the r_i. Let the number of groups of columns with equal indices be s, then $\nu(r_1+r_2+\ldots+r_s) = \nu(k)$. Define

$$\sigma_i = \sum_{j=1}^{i-1} r_j + 1, \quad i = 1, 2, \ldots, s. \tag{2.1}$$

So there are r_1 columns in the basis with index $\nu(\sigma_1) = \nu(1)$, r_2 columns with index $\nu(\sigma_2) = \nu(r_1 + 1), \ldots, r_i$ columns with index $\nu(\sigma_i)$.

A ν-form matrix corresponding to the indices of V is a $k \times k$ upper block triangular, unimodular matrix with the following structure.

$$N = \begin{bmatrix} D_1 & U_{12} & U_{13} & \ldots & U_{1s} \\ 0 & D_2 & U_{23} & \ldots & U_{2s} \\ 0 & 0 & D_3 & \ldots & U_{3s} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & D_s \end{bmatrix}. \tag{2.2}$$

The diagonal blocks D_i are real, non-singular $r_i \times r_i$ matrices. The entries above the diagonal blocks are polynomials with bounded degrees, those in the $r_i \times r_j$ block U_{ij} having degree less than or equal to $\nu(\sigma_j) - \nu(\sigma_i)$. The ν-form matrix has been termed a "properly indexed unimodular matrix" by Wolovich[11, p. 182] but we use the name ν-form for brevity. Note that a ν-form matrix may correspond to any ordered set of integers $\{v(i)\}$, but in this paper v-form matrices will always correspond to the indices of an appropriate vector space. We will now set out some of the properties of v-form matrices.

Lemma 1. The v-form matrices possess the following group structure: The product of two v-form matrices corresponding to the same set of indices $\{v(i)\}$ is a v-form matrix corresponding to $\{v(i)\}$, and any v-form matrix corresponding to a set of indices $\{v(i)\}$ has an inverse and this inverse is a v-form matrix corresponding to $\{v(i)\}$.

Proof. These results are established in reference [11, p. 185].

Lemma 2. If $\bar{G}$ is any degree ordered minimal basis of a k-dimensional vector space V of n-vectors over the field of rational functions of s, then $G = \bar{G}N$ is a degree ordered minimal basis of V for any v-form matrix N corresponding to the indices of V. Conversely, any minimal basis G can be expressed in the form $G = \overline{GN}$ for some v-form matrix $\bar{N}$.

Proof. Let $G = \bar{G}N$ where N is any v-form matrix. Now, since $\bar{G}$ is a polynomial basis of V and N is a unimodular polynomial matrix, then G is a polynomial basis of V (not necessarily minimal)[10, p. 491]. That G is degree ordered and minimal follows from the v-form structure, and the uniqueness of column degrees of a minimal basis; the calculations are straightforward.

To demonstrate the converse, use can be made of Forney's echelon form of a minimal basis[10, p. 499]. Denote the (unique) Forney echelon form minimal basis for V by G_E. For any minimal basis G of V there exists a unimodular matrix $\tilde{N}$ such that $G = G_E\tilde{N}$ and an examination of Forney's algorithm shows $\tilde{N}$ is v-form. Now G_E is unique[10, p. 500] so for a given basis $\bar{G}$ there is a v-form matrix N such that $\bar{G} = G_E\bar{N}$. Hence $G = \overline{GN}^{-1}\tilde{N}$ and $N = \bar{N}^{-1}\tilde{N}$ is v-form matrix by Lemma 1.

Lemma 2 tells us that for a vector space V, all minimal bases can be expressed as $\bar{G}N$ where $\bar{G}$ is any given minimal basis and N is a v-form matrix of parameters. A $\bar{G}$ can be computed as outlined in reference [10] (in refs. [3, 7] mention is made of a computer program for performing this computation). So the collection of minimal bases can thus be expressed using a finite number of parameters.

Forney[10] has shown how to proceed from a minimal basis of a particular vector space to a solution of the MDP. Having seen above how to obtain all minimal bases of a vector space we can obtain all solutions of the MDP. The relevant theorem[3, 7, 10] for constructing MDP solutions is as follows.

Lemma 3. Let

$$K(s) = \left[\frac{K_m(s)}{K_q(s)}\right] \quad (2.3)$$

be a column degree ordered $(m+q)\times(m+q-p)$ matrix† whose columns are a minimal basis of the vector space kernel $[T_1(s)\vdots -T_2(s)]$. $K_m(s)$ is the first m rows and $K_q(s)$ the last q rows of $K(s)$. $T_1(s)$ and $T_2(s)$ are as described in equation (1.1). Write the high order coefficient matrix of $K(s)$ as

$$[K(s)]_h = \left[\frac{K_{m\gamma}}{K_{q\gamma}}\right]. \quad (2.4)$$

The MDP has a solution if and only if

$$\operatorname{rank}[K_{q\gamma}] = q. \quad (2.5)$$

If condition (2.5) holds, consider the first q columns of $K(s)$ for which the corresponding columns of $K_{q\gamma}$ are linearly independent. Denote these columns by

$$S(s) = \left[\frac{Q(s)}{P(s)}\right]. \quad (2.6)$$

Then the minimal dynamic order (or minimal McMillan degree) of a $T(s)$ which solves equation (1.1) is the sum of the column degrees of $S(s)$. A proper minimal order solution of (1.1) is

$$T(s) = Q(s)P^{-1}(s) \quad (2.7)$$

and the column degrees of $S(s)$ are the controllability indices ($v^*(i)$, $i=1,2,\ldots,q$) of $T(s)$[9, p. 512], that is the controllability indices in any state-space minimal realization of $T(s)$.

All solutions $Q(s)P^{-1}(s)$ of the MDP are associated with a matrix $[Q^T(s)\vdots P^T(s)]^T$ whose columns are members of the vector space kernel $[T_1(s)\vdots -T_2(s)]$; the columns are linearly independent because $P(s)$ is non-singular and the columns are a subset of a minimal basis by uniqueness of controllability indices[10]. Thus, given all minimal bases, all solutions of the MDP can be found, and given any minimal basis $K(s)$ all minimal bases $K(s)N(s)$ are known where $N(s)$ is a v-form parameter matrix.

This means that given any minimal basis of kernel $[T_1\vdots -T_2]$, all solutions of the MDP can be expressed in terms of a finite number of parameters. In the next section, we shall seek a more efficient parameterisation.

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†The dimensions follow from those of $T_1(s)$ and $T_2(s)$, as well as the fact that rank $T_1 = p$, see the introduction.

3. PSEUDO ν-FORM MATRICES AND THE REDUCTION IN NUMBERS OF PARAMETERS

While it is easy to select q columns from the known minimal basis $K(s)$ satisfying the conditions of Lemma 3, it is not immediately obvious which q columns should be chosen from the parametric matrix $K(s)N(s)$.

However, instead of multiplying $K(s)$ by $N(s)$ and then selecting q columns, we may select q columns of $N(s)$ and multiply $K(s)$ by these only. As it turns out, the selection is straightforward, while also the number of parameters needing consideration is reduced. So instead of using a ν-form parameter matrix we may use an appropriate $(m+q-p)\times q$ 'pseudo' ν-form matrix. A pseudo ν-form matrix $\bar{N}_A(s)$ corresponding to *two* ordered sets of integers $\{\nu(i)\}$ and $\{\nu^*(i)\}$, is a matrix with the following structure

$$\bar{N}_A = \begin{bmatrix} \bar{D}_1 & \bar{N}_{12} & \dots & \bar{N}_{1s} \\ 0 & \bar{D}_2 & \dots & \bar{N}_{2s} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \bar{D}_s \end{bmatrix} \tag{3.1}$$

where $\{\nu(i)\}$ is usually the set of indices of the relevant vector space. The $\bar{D}_i$ are real full rank $r_i \times t_i$ matrices with r_i defined as before and t_i similarly defined for the set $\{\nu^*(j)\}$. The $\bar{N}_{ij}$ are $r_i \times t_j$ polynomial matrices whose entries have bounded degrees as follows. For the ordered set of $q=\sum_{i=1}^{s} t_i$ integers, denoted $\{\nu^*(j)\}$, the degree of the (i,j) entry of $\bar{N}_A$ is less than or equal to $\nu^*(j)-\nu(i)$ when $\nu^*(j)-\nu(i)$ is non-negative, otherwise the corresponding entry of $\bar{N}_A$ is identically zero.

Note that for purposes of solving the MDP, the set $\nu^*(j)$ must always be a subset of the indices $\nu(i)$ of V and moreover, they must be the controllability indices of $T(s)$ referred to in Section 2. It is not hard to check that such an abbreviated ν-form matrix, when multiplied on the left by the minimal basis matrix $K(s)$, will produce a matrix with q columns satisfying the condition of Lemma 3. Thus, if the $\nu^*(j)$ are the controllability indices of $T(s)$, $K(s)\bar{N}_A(s)$ will always provide a minimal order solution of (1.1). We remark that we shall later consider solutions of higher order than the minimum using pseudo ν-form matrices, at which time the $\nu^*(j)$ need no longer be the controllability indices of $T(s)$, but we shall always have $\nu^*(i) \geqq \nu(i)$.

The reduction in the number of columns in the parameter matrix represents a reduction in the number of parameters needed to find all solutions of the MDP. However, even more reductions in required parameter numbers are possible. This will be demonstrated using the following lemma.

Lemma 4. Let V be a k-dimensional vector space with $(m+q-p)$ associated indices $\nu(i)$, arranged with $\nu(i)\leqq \nu(i+1)$. Let $\{\nu^*(j), j=1,2,\dots,q\}$ be a subset of $\{\nu(i)\}$ with $\nu^*(j) \leqq \nu^*(j+1)$ and let $\bar{V}$ be any vector space with $\{\nu^*(j)\}$ as indices. If $\bar{N}_A(s)$ is a pseudo ν-form matrix corresponding to $\{\nu(i)\}$ and $\{\nu^*(j)\}$, if L is a ν-form matrix corresponding to $\{\nu(i)\}$ and M a ν-form matrix corresponding to $\{\nu^*(j)\}$, then $N_A = L\bar{N}_A M$ is a pseudo ν-form matrix corresponding to $\{\nu(i)\}$ and $\{\nu^*(j)\}$.

Proof. The proof is straightforward and will be omitted.

The same solution (2.7) to the MDP is obtained if the q columns of $S(s)$ in equation (2.6) are replaced by $S(s)M(s)$ where $M(s)$ is an arbitrary $q\times q$ ν-form matrix of the vector space $\bar{V}$, the space spanned by the columns of $S(s)$:

$$S(s)M(s) = \begin{bmatrix} Q(s) \\ \hline P(s) \end{bmatrix} M(s) \tag{3.2}$$

$$= K(s)\bar{N}_A(s)M(s) \tag{3.3}$$

where $K(s)$ is the known minimal basis of kernel $[T_1(s) \vdots -T_2(s)]$ and $\bar{N}_A(s)$ is an appropriate *pseudo* ν-form parameter matrix. Hence all solutions (2.7) could be obtained using $N_A(s) = \bar{N}_A(s)M(s)$ instead of $\bar{N}_A(s)$. We may then choose $M(s)$ to reduce the number of parameters required to represent all solutions of the MDP.

Let $\bar{N}_A(s)$ be as described in (3.1) and choose $M(s)=\hat{M}^{-1}(s)$ where $\hat{M}(s)$ is a $q\times q$ ν-form matrix of $\bar{V}$, determined as follows. For the first t_1 rows of $\hat{M}(s)$ select, from the first r_1 rows of $\bar{N}_A(s)$, the last t_1 rows for which the corresponding rows of $\bar{D}_1$ are linearly independent. For the next t_2 rows of $\hat{M}(s)$ select, from the next r_2 rows of $\bar{N}_A(s)$, the last t_2 rows for which the corresponding rows of $\bar{D}_2$ are linearly independent. Continue in this way until q rows have been selected for $\hat{M}(s)$. Then $\hat{M}(s)$ is a $q\times q$ ν-form matrix corresponding to the indices of $\bar{V}$ with $t_i \times t_i$ diagonal blocks. Then by Lemma 1, the inverse of $\hat{M}(s)$ is a permissible choice of $M(s)$. So

$$N_A(s) = \bar{N}_A(s)M(s) = \bar{N}_A(s)\hat{M}^{-1}(s)$$

$$= \begin{bmatrix} W_1 & V_{12} & \dots & V_{1s} \\ 0 & W_2 & \dots & V_{2s} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & W_s \end{bmatrix} \tag{3.4}$$

with W_i a real $r_i \times t_i$ block (of some free parameters) in column reduced echelon form, a standard canonical form for real matrices[12]. The $r_i \times t_j$ blocks V_{ij} have zero rows corresponding to the rows in W_i which have the 1's characteristic of the echelon form. The remaining entries of V_{ij} are polynomials satisfying the usual bounds on degrees and whose coefficients are free parameters. Clearly $N_A(s)$ has fewer parameters than $\bar{N}_A(s)$; the reduction is achieved by eliminating overlap of solutions (that is the same solution (2.7) occurring when different sets of numbers are substituted for the parameters in $\bar{N}_A(s)$).

When we consider the construction of the original pseudo ν-form matrix $\bar{N}_A(s)$ it is clear that every entry in and above the diagonal blocks $\bar{D}_i$ is non-zero, i.e. not identically zero. Consequently the very last t_i rows of all the above sets of r_i rows of $\bar{N}_A$ will, generically, satisfy the condition of linear independence of the corresponding rows of $\bar{D}_i$. *So W_i and V_{ij} will generically have the form*†

$$W_i=\left[\begin{array}{c} Y_i \\ \hline I_{t_i} \end{array}\right]; \quad V_{ij}=\left[\begin{array}{c} Z_{ij} \\ \hline 0_{ij} \end{array}\right] \tag{3.5}$$

where I_{ti} is the $t_i\times t_i$ unit matrix, 0_{ij} is the $t_i\times t_j$ zero matrix, Y_i is a $(r_i-t_i)\times t_i$ real matrix, all of whose entries are free parameters, and Z_{ij} is a $(r-t_i)\times t_j$ polynomial matrix whose entries have degree $\nu^*(\sigma_j)-\nu(\sigma_i)$ and whose real coefficients are all free parameters.

However, one can readily conceive of cases where, upon assigning numerical values to the free parameters in $\bar{N}_A$, the very last t_i rows of $\bar{D}_i$ will not be linearly independent and (3.5) will not be the correct echelon form. Nevertheless if we allow the free parameters to be real or tend to infinity, in a sense made clear below in an example, then the pathological cases are covered by the structure of (3.5).

For example, if $r_1=3$ and $t_1=2$, a particular $N_A(s)$ could have any of the following three echelon forms with r_{ij} being real and P_{ij} real polynomials.

$$\left[\begin{array}{cc|ccc} r_{11} & r_{12} & p_{13} & p_{14} & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \hline \multicolumn{2}{c|}{0} & \ddots & & \end{array}\right] \tag{3.6a}$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & r_{22} & p_{23} & p_{24} & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \hline \multicolumn{2}{c|}{0} & \ddots & & \end{array}\right] \tag{3.6b}$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & p_{33} & p_{34} & \cdots \\ \hline \multicolumn{2}{c|}{0} & \ddots & & \end{array}\right] \tag{3.6c}$$

Further column manipulation of (3.6a) by means of a slightly altered ν-form $M(s)$ yields the following two equivalent matrices

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{r_{11}} & -\frac{r_{12}}{r_{11}} & -\frac{p_{13}}{r_{11}} & -\frac{p_{14}}{r_{11}} & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \hline \multicolumn{2}{c|}{0} & \ddots & & \end{array}\right] \tag{3.7a}$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{r_{12}} & -\frac{r_{11}}{r_{12}} & -\frac{p_{13}}{r_{12}} & -\frac{p_{14}}{r_{12}} & \cdots \\ \hline \multicolumn{2}{c|}{0} & \ddots & & \end{array}\right]. \tag{3.7b}$$

If r_{11}, r_{12} and certain coefficients of the polynomials p_{1j} approach infinity at related rates, then (3.7a) approaches the same form as (3.6b) with

$$r_{22}=-\frac{r_{12}}{r_{11}}, \quad p_{23}=-\frac{p_{13}}{r_{11}}, \quad \text{etc.}$$

Likewise if in (3.7b), r_{12} and certain coefficients of p_{1j} approach infinity then (3.7b) can tend to the same form as (3.6c) with

$$p_{33}=-\frac{p_{13}}{r_{12}}, \quad \text{etc.}$$

So by allowing parameters to be real and to tend to infinity the structure (3.6a) covers all cases.

Let us summarize:

MDP Parametric Construction Procedure. The parametrically expressed set of all solutions to the MDP (1.1) is constructed as follows. First compute a minimal basis of kernel $[T_1(s)\vdots -T_2(s)]$. Then multiply this minimal basis by an appropriate pseudo ν-form matrix $N_A(s)$ as described above, see (3.4) and (3.5). Partition the resulting matrix, $K(s)N_A(s)$, as $[Q(s)^T \vdots P(s)^T]^T$ and obtain the solutions $T(s)=Q(s)P^{-1}(s)$. All solutions of

†A reviewer has pointed out that a similar effect may be achieved by arranging to have the basis itself in a similar form, obtainable from the ideas of [10, 13].

the MDP are obtained, provided parameters are allowed to tend to infinity in an appropriate manner.

The question arises as to whether these solutions will be proper. The answer is that they will be at least generically proper. There is a choice for $N_A(s)$ which simply picks out the first q columns of $K(s)$ for which the corresponding columns of $K_{q\gamma}$ are linearly independent. Because one $N_A(s)$ has this property, it is not hard to show that almost all have it. On the other hand, if there are choices of q columns of $K(s)$ with the correct controllability indices for which the corresponding matrix $K_{q\gamma}$ does not have linearly independent columns, as well there may be in a specific problem, this means that there exist specific choices of $N_A(s)$ which would result in an improper $T(s)$.

Finally in this section, we comment on the question of stability. The condition for stability is that the determinant of $P(s)$ be Hurwitz, i.e. have all its roots in the open left half plane). This defines a set of polynomial inequalities in the parameters, the solution of which defines that set of real numbers which can be inserted into the parameters of $T_i(s)$ to give the set of stable solutions of the MDP. Solution of a set of polynomial inequalities has been dealt with in [14] and [15]. Of course, the evaluation of the determinant of a matrix with literal entries, while in principle possible, may not be an easy task. A discussion of the problem can be found in [16].

As discussed in [3], it may be that all solutions of the MDP have certain readily computable poles in common. Of course, if any are unstable, the stable MDP problem has no solutions. The computational simplifications which this fixed pole concept makes possible are discussed in Section 7; they involve further theory.

4. EXAMPLE OF THE STABLE SOLUTION OF THE MDP

To illustrate the above discussion, let us examine the example presented in references [1, 2].

Consider the MDP with

$$T_1 = \frac{1}{s^2+3s+2}\begin{bmatrix} s+1 & s+3 & s^2+3s \\ s+2 & s^2+2s & 0 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.1}$$

A minimal basis for kernel $[T_1(s) \vdots T_2(s)]$ is

$$K(s) = \begin{bmatrix} s+2 & -1 & 1 \\ s+2 & 2s+1 & 1 \\ s+2 & -1 & s+2 \\ s+4 & 1 & s+2 \\ s+2 & 2s-1 & 1 \end{bmatrix}. \tag{4.2}$$

The indices are $\nu(1) = \nu(2) = \nu(3) = 1$. From Lemma 3 it can readily be seen that solution to the MDP exists and that a set of solution columns is the first two columns (any two columns could have been chosen). So the corresponding pseudo ν-form parameter matrix is

$$\bar{N}_A = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

where n_{ij} are real indeterminates. The echelon form matrix derived from this structure is

$$N_A = \begin{bmatrix} a_{11} & a_{12} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The subsets of the minimal basis which lead to solution of the MDP (4.1) are

$$K(s)N_A = \begin{bmatrix} a_{11}(s+2)-1 & a_{12}(s+2)+1 \\ a_{11}(s+2)+(2s+1) & a_{12}(s+2)+1 \\ a_{11}(s+2)-1 & a_{12}(s+2)+(s+2) \\ a_{11}(s+4)+1 & a_{12}(s+4)+(s+2) \\ a_{11}(s+2)+(2s-1) & a_{12}(s+2)+1 \end{bmatrix}$$

So all solutions of MDP (4.1) are given by substituting real numbers for a_{11}, a_{12} and a_{22} in the following

$$T(s) = \begin{bmatrix} a_{11}(s+2)-1 & a_{12}(s+2)+1 \\ a_{11}(s+2)+(2s+1) & a_{12}(s+2)+1 \\ a_{11}(s+2)-1 & a_{12}(s+2)+(s+2) \end{bmatrix} \begin{bmatrix} a_{11}(s+4)+1 & a_{12}(s+4)+(s+2) \\ a_{11}(s+2)+(2s-1) & a_{12}(s+2)+1 \end{bmatrix}^{-1} \tag{4.3}$$

$$= \frac{1}{-(a_{11}+2a_{12}+2)s^2-3(a_{11}+2a_{12}+1)s+3(2a_{12}+1)}$$

$$\times \left[\begin{array}{l|l} -2a_{12}s^2-2(2a_{12}+1)s & -a_{11}s^2+(-3a_{11}+2a_{12}+1)s+3(2a_{12}+1) \\ \hline 2a_{12}s+2(2a_{12}+1) & -(a_{11}+2a_{12}+2)s^2-(3a_{11}+8a_{12}+5)s-(2a_{12}+1) \\ \hline (-a_{11}-2a_{12}-2)s^2+(-3a_{11}-4a_{12}-3)s & 2(a_{11}+a_{12}+1)s+2(2a_{11}+3a_{12}+2) \\ \quad +(-2a_{11}+1) & \end{array}\right] \tag{4.4}$$

For $T(s)$ to be stable it is necessary that

$$\begin{vmatrix} a_{11}(s+4)+1 & a_{12}(s+4)+(s+2) \\ a_{11}(s+2)+(2s-1) & a_{12}(s+2)+1 \end{vmatrix}$$
$$= -(a_{11}+2a_{12}+2)s^2 -3(a_{11}+2a_{12}+1)s+3(2a_{12}+1) \quad (4.5)$$

be Hurwitz. That is

$$\left.\begin{array}{l} (a_{11}+2a_{12}+1)>0 \text{ and } 2a_{12}+1<0 \\ \text{or } a_{11}+2a_{12}+2<0 \text{ and } 2a_{12}+1>0 \end{array}\right\} \quad (4.6)$$

The inequalities (4.6) are satisfied for parameter values a_{11}, a_{12} satisfying

$$\text{or} \quad \left.\begin{array}{l} a_{12}>-\tfrac{1}{2} \\ a_{11}<-(2a_{12}+2) \end{array}\right\} \quad (4.7)$$

So a stable solution to the MDP (4.1) is obtained by selecting any two real numbers a_{11}, a_{12} satisfying (4.7) and substituting these values into (4.3).

Remarks. (1) The solutions (4.4) are proper for all parameter values except for those satisfying the polynomial equation $a_{11}+2a_{12}+2=0$. Hence the $T(s)$ is only generically proper.

(2) The set of solutions defined by (4.4) above is identical with the set of solutions of this example given in [1,2] by Wang and Davison and the parametric representations are equivalent. Of course, easy hand solutions were possible here; but note that Collins, author of [16], has developed a great deal of sophisticated software relevant for tackling decision–algebra types of problems.

5. NONMINIMAL SOLUTIONS TO THE MODEL MATCHING PROBLEM AND CONSTRUCTION OF A LEAST ORDER STABLE SOLUTION

If there are no stable solutions to the MDP, it may well be of interest to examine nonminimal solutions to obtain a stable solution. (Recall that the existence of a stable solution of some suitably high order can always be checked moderately easily, as noted in Section 1.) Accordingly, we devote most of this section to describing a parametrisation of nonminimal solutions to the model matching problem. Incorporation of the stability constraint, at least in formal terms, is then easily achieved.

We shall first see how, with an excess number of parameters, a solution to the model-matching problem with prescribed column degrees can be achieved.

Lemma 5. With notation as in earlier sections, let $v^*(1) \leqq v^*(2) \ldots \leqq v^*(q)$ be a set of column degrees with $v(i) \leqq v^*(i)$. Let $\bar{N}(s)$ be a $(m+q-p)\times q$ polynomial matrix with $\partial[\bar{n}_{ij}(s)] = v^*(j)-v(i)$. (If $v^*(j)-v(i)<0$, we take $\bar{n}_{ij}(s)=0$.) Then, generically the column degrees of $K(s)\bar{N}(s)$ are $v^*(1),\ldots,v^*(q)$. Conversely, suppose $\bar{N}(s)$ is a polynomial matrix such that $K(s)\bar{N}(s)$ has column degrees $v^*(1),\ldots,v^*(q)$. Then $\partial[\bar{n}_{ij}(s)] \leqq v^*(j)-v(i)$ with, for each j, equality for at least one i.

Note: we have in mind a partitioning of $K(s)\bar{N}(s)$ as $[Q^{\mathrm{T}}(s) \vdots P^{\mathrm{T}}(s)]^{\mathrm{T}}$; then $T(s)=Q(s)P^{-1}(s)$ is a solution to the model matching problem, assuming $P(s)$ is nonsingular. It may or may not be proper; if it is proper, then its controllability indices are the $v^*(i)$.

Proof. For the first part, we have

$$t_{ij}(s) = \sum k_{il}(s)\bar{n}_{lj}(s)$$

and

$$\partial[t_{ij}(s)] \leqq \max_l \, [v(l)+(v^*(j) -v(l))|v^*(j)-v(l) \geqq 0] = v^*(j).$$

Equality holds generically, since failure of the equality depends on cancellations which do not occur generically. For the second part, let $t_j(s)$ be the jth column of $K(s)\bar{N}(s)$, so that $t_j(s) = K(s)\bar{n}_j(s)$. By the predictable degree property[10], applicable because $K(s)$ is column proper,

$$v^*(j) = \partial[t_j(s)] = \max_i \, [v_i+\partial(\bar{n}_{ij})].$$

So

$$\partial[\bar{n}_{ij}(s)] \leqq v^*(j)-v(i)$$

with equality for at least one i.

The following corollary is a consequence of Lemma 5 and the note following the lemma.

Corollary. All proper solutions $T(s)$ of the model matching problem with controllability indices $v^*(1),\ldots,v^*(q)$ are obtainable as

$$T(s) = Q(s)P^{-1}(s)$$

where

$$[Q^{\mathrm{T}}(s) \vdots P^{\mathrm{T}}(s)]^{\mathrm{T}} = K(s)\bar{N}(s)$$

and

$$\partial[\bar{n}_{ij}(s)] \leqq v_j^*-v_i$$

with, for each j, equality for at least one i.

We now see how we may, in principle, proceed to form all $T(s)$ with certain controllability indices. But we need to discuss the elimination of surplus parameters from $\bar{N}(s)$, and the question of whether or not $T(s)$ is proper. To eliminate surplus parameters, we essentially generalize the procedure given in studying minimal degree solutions.

Let

$$\nu(1)=\nu(2)=\ldots=\nu(\sigma_1),$$
$$\nu(\sigma_1+1)=\ldots=\nu(\sigma_2)$$

and so on. Define $\nu^*(\rho_1), \nu^*(\rho_2), \ldots$ similarly. Then the degree structure of $\bar{N}(s)$ can be viewed as

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \ldots & \delta_{1\alpha} \\ \delta_{21} & \delta_{22} & \ldots & \delta_{2\alpha} \\ \vdots & \vdots & & \vdots \\ \delta_{\beta 1} & \delta_{\beta 2} & \ldots & \delta_{\beta\alpha} \end{bmatrix}$$

where δ_{ij} denotes a block with $\sigma_{i+1}-\sigma_i$ rows, $\rho_{j+1}-\rho_j$ columns; and with degree $\delta_{ij}=\nu^*(\rho_j)-\nu(\sigma_i)$. If the degree is negative, the corresponding block of $\bar{N}(s)$ is actually zero. Notice that $\delta_{ij}<\delta_{i,j+1}$ and $\delta_{ij}>\delta_{i+1,j}$; moving right causes a strict increase and moving down a strict decrease in degree.

Denote by a ϕ a block which formally has negative degree. Then the degree structure of $\bar{N}(s)$ will be of one of the following two forms:

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \ldots & \delta_{1\bar{\alpha}} & \ldots & \delta_{1\alpha} \\ \vdots & \vdots & & & & \\ \hdashline \delta_{\beta_1 1} & \delta_{\beta_1 2} & \ldots & & & \\ \hdashline \phi & \vdots & & & & \\ \hdashline \vdots & \delta_{\beta_2 2} & \ldots & & & \\ \hdashline \vdots & \phi & & & & \\ \vdots & \vdots & & & & \\ \hdashline \phi & \phi & & \delta_{\beta\bar{\alpha}} & \ldots & \delta_{\beta\alpha} \end{bmatrix}$$

or

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{1\alpha} \\ \vdots & \vdots & \\ \hdashline \delta_{\beta_1 1} & \delta_{\beta_2 2} & \\ \hdashline \phi & \vdots & \\ \hdashline & \delta_{\beta_2 2} & \\ \hdashline & \phi & \\ \hdashline & \vdots & \delta_{\beta\alpha} \\ \hdashline & & \phi \end{bmatrix}$$

In obtaining a canonical $N(s)$, the only block rows of $\bar{N}(s)$ in which simplifications will be made are those corresponding to the block rows shown boxed in the degree matrix. Beneath the first nonzero block in such block rows of $\bar{N}(s)$ there stands a zero block. Simplifications, i.e. reductions in the number of parameters, will be made through right multiplication by a ν-form matrix which preserves the column degree structure of $K(s)\bar{N}(s)$. Thus the (i,j) block of the ν-form matrix will have degree $\nu^*(\rho_j)-\nu^*(\rho_i)$.

Let us focus now on a submatrix of $\bar{N}(s)$ with one of the two following degree structures

$$\begin{bmatrix} \delta_{\beta_1 1} & \delta_{\beta_1 2} & \ldots & & & \delta_{\beta_1\alpha} \\ \phi & \delta_{\beta_2 2} & \ldots & & & \\ \vdots & \vdots & & & & \\ \phi & \phi & \ldots & \delta_{\beta\bar{\alpha}} & \ldots & \delta_{\beta\alpha} \end{bmatrix}$$

or

$$\begin{bmatrix} \delta_{\beta_1 1} & \delta_{\beta_1 2} & \ldots & \delta_{\beta_1\alpha} \\ \phi & \delta_{\beta_2 2} & \ldots & \delta_{\beta_2\alpha} \\ \vdots & \vdots & & \vdots \\ \phi & \phi & & \delta_{\beta_{\bar{\alpha}}\alpha} \\ \vdots & \vdots & & \vdots \\ \phi & \phi & \ldots & \phi \end{bmatrix}.$$

We shall describe the operations for only the first structure. The variations for the second are trivial. The reader should check that the operations are equivalent to postmultiplication by a ν-form matrix, which has no effect on the degree structure nor the number of free parameters in the remaining block rows of $\bar{N}(s)$.

1. Change the coefficient matrix of powers of $s^{\nu^*(\rho_{\bar{\alpha}})-\nu(\sigma_\beta)}$ in the block of $\bar{N}(s)$ corresponding to the $\delta_{\beta\bar{\alpha}}$ block of the degree matrix to

$$\begin{bmatrix} X \\ I \end{bmatrix} \quad \text{or} \quad [I \vdots 0]$$

as appropriate. This is generically possible. Here X denotes a matrix with free parameters.

2. Subtract multiples of columns in the $\bar{\alpha}$th block column from all later columns to reduce the degree of all terms in the same rows as occupied by the identity matrix in step 1 to less than $\nu^*(\rho_{\bar{\alpha}})-\nu(\sigma_\beta)$.

3. Pass to the second last block row of $N(s)$ and repeat the procedure of introducing an identity matrix as a coefficient of highest degree terms in the first nonzero block and obtaining terms of lower degree in appropriate rows of blocks to the right of this nonzero block.

4. Then pass to the third last block row, etc.

In this way, $\bar{N}(s)$ is replaced by $N(s)$, a matrix with fewer parameters. No solutions of the model matching problem are thrown away in the process.

We remark that as with the minimal design problem, one can obtain certain solutions by letting parameters tend to infinity in a certain way, and by this means avoid the apparent gap in step 1, in which a certain adjustment is only generically possible.

Proper solutions $T(s)$ will, for a given set of $\nu^*(j)$, be either obtained generically, or not at all. It is easy in a given instance to establish whether proper solutions will be obtained.

It is clear that, as with the parametrised minimal degree solutions, we are in a position to write down polynomial inequalities involving the free parameters in $\bar{N}(s)$ which, if satisfied by particular values of those parameters, constitute stability conditions for $T(s)$.

Solutions will have order $\sum_{j=1}^{q} \nu^*(j)$ and so by considering all sets $\{\nu^*(j)\}$ with a given sum we can construct all solutions of a given order. Hence, if there is no stable minimal order solution we may systematically test for stable solutions of progressively higher order until the least order stable solutions are found.

In this searching process, certain combinations of $\nu^*(j)$ can be omitted. As illustrated by an example in the next section, it is sometimes possible to choose the $\nu^*(j)$ so that $\bar{N}(s)$ has the form

$$\bar{N}(s) = \begin{bmatrix} \tilde{N}(s) \\ 0 \end{bmatrix}$$

where $\tilde{N}(s)$ is a square matrix. [This will happen precisely when $\nu^*(q) < \nu(q+1)$ as may be checked.] Then if

$$K(s) = \begin{bmatrix} \tilde{Q}(s) & X \\ \tilde{P}(s) & X \end{bmatrix}$$

(where X denotes entries whose particular form is not important), and if

$$K(s)\bar{N}(s) = [Q^{\mathrm{T}}(s) \vdots P^{\mathrm{T}}(s)]^{\mathrm{T}},$$

it follows that

$$\begin{aligned} T(s) &= Q(s)P^{-1}(s) \\ &= (\tilde{Q}(s)\tilde{N}(s)\,(\tilde{P}(s)\tilde{N}(s))^{-1} = \tilde{Q}(s)\tilde{P}^{-1}(s). \end{aligned}$$

Thus no new solutions $T(s)$ to the model matching problem are really obtained; rather, only new matrix fraction decompositions of known solutions are obtained.

6. EXAMPLE OF LEAST ORDER STABLE SOLUTION OF THE EXACT MODEL MATCHING PROBLEM

To illustrate the procedure, described in the previous section, for finding least order stable solutions, consider the example in [3].

$$T_1(s) = \begin{bmatrix} \dfrac{s}{s^2+3s+2} & 0 & \dfrac{s^2+2s+2}{s^2+3s+2} \\ \dfrac{2s+1}{s+2} & \dfrac{s-1}{s+2} & 0 \end{bmatrix};$$

$$T_2(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (7.1)$$

A minimal basis of kernel $[T_1(s) \vdots -T_2(s)]$ is

$$K(s) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & s+2 & 0 \\ 1 & 0 & s^2+3s+2 \\ 1 & 0 & s^2+2s+2 \\ 1 & s-1 & 0 \end{bmatrix}.$$

The indices are $\nu(1) = 0$, $\nu(2) = 1$, $\nu(3) = 2$. It is easy to establish that the minimal order is 1. Initially we wish to find all solutions of the model matching problem, whose controllability indices total 1 ($q = 2$ so there are two controllability indices). The only set of two integers totalling 1 is $\{0, 1\}$ so $\nu^*(1) = 0$, $\nu^*(2) = 1$.

The pseudo ν-form matrix corresponding to $\{\nu(i)\}$ and $\{\nu^*(j)\}$ is

$$\bar{N}_1(s) = \begin{bmatrix} a_{11} & a_{12}s + b_{12} \\ 0 & a_{22} \\ 0 & 0 \end{bmatrix}$$

where a_{ij}, b_{12} are real parameters. This reduces to the parameter matrix

$$N_1(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has no parameters so there is only 1 solution of order 1. Right multiplying the minimal basis matrix

$$K(s)N_1(s) = \begin{bmatrix} 1 & 0 \\ -1 & s+2 \\ 1 & 0 \\ 1 & 0 \\ 1 & s-1 \end{bmatrix}.$$

The solution is stable if

$$\begin{vmatrix} 1 & 0 \\ 1 & s-1 \end{vmatrix} = s-1$$

is Hurwitz. Clearly it is not, so there is no stable minimal order solution of (1.1). The next step is to consider solutions of order 2. Sets of two indices whose sum is 2 are $\{0,2\}$ and $\{1,1\}$. The corresponding pseudo ν-form matrices are

$$\bar{N}_{21}(s) = \begin{bmatrix} a_{11} & a_{12}s^2+b_{12}s+c_{12} \\ 0 & a_{22}s+b_{22} \\ 0 & a_{32} \end{bmatrix}$$

and

$$\bar{N}_{22}(s) = \begin{bmatrix} a_{11}s+b_{11} & a_{12}s+b_{12} \\ a_{21} & a_{22} \\ 0 & 0 \end{bmatrix}$$

where the a_{ij}, b_{ij} are real parameters. As noted at the end of the previous section, no new solution of the model matching problem will follow from $\bar{N}_{22}(s)$. We therefore concentrate on $N_{21}(s)$, which can be reduced to

$$N_{21}(s) = \begin{bmatrix} 1 & 0 \\ 0 & a_{22}s+b_{22} \\ 0 & 1 \end{bmatrix}$$

Right multiplying the minimal basis matrix $K(s)$ by $N_{21}(s)$ yields

$$K(s)N_{21}(s) = \begin{bmatrix} 1 & 0 \\ -1 & (a_{22}s+b_{22})(s+2) \\ 1 & s^2+3s+2 \\ 1 & s^2+2s+2 \\ 1 & (a_{22}s+b_{22})(s-1) \end{bmatrix}.$$

The solution generated by the above matrix is stable if

$$D_1 = \begin{vmatrix} 1 & s^2+2s+2 \\ 1 & (a_{22}s+b_{22})(s-1) \end{vmatrix}$$

is Hurwitz. It can easily be shown that D_1 is Hurwitz if the parameters satisfy the following conditions

$$\begin{aligned} b_{22} &> -2 \\ b_{22}-2 < a_{22} &< 1 \end{aligned} \tag{6.2}$$

which are clearly easily satisfied. The set of all solutions of order 2 is given by

$$T(s) = \frac{1}{(a_{22}-1)s^2-(a_{22}-b_{22}+2)s-(b_{22}+2)} \times \begin{bmatrix} a_{22}s^2+(b_{22}-a_{22})s-b_{22} & -(s^2+2s+2) \\ -2a_{22}s^2+(2b_{22}-a_{22})s+b_{22} & (a_{22}+1)s^2+(2a_{22}+b_{22}+2)s+2(b_{22}+1) \\ (a_{22}-1)s^2+(b_{22}-a_{22}-3)s-(b_{22}+2) & s \end{bmatrix} \tag{6.3}$$

and these are stable if the parameters satisfy conditions (6.2).

So (6.2) and (6.3) represent all least order stable solutions of the exact model matching problem (1.1) where $T_1(s)$ and $T_2(s)$ are given by (6.1).

7. AN ALTERNATIVE PROBLEM

For the problem (1.1) Wolovich[3] has introduced the notions of 'common poles' of $T_1(s)$ and $T_2(s)$ as well as 'common zeros' of $T_1(s)$ and $T_2(s)$. If $T_1(s)$ and $T_2(s)$ are scalar, and have a common zero or common pole, then the same multiplicative factor can be eliminated from the numerator or denominator of $T_1(s)$ and $T_2(s)$ without affecting $T(s)$. Wolovich's idea is to generalize this concept to the matrix case. Thus he has shown how to reduce the problem (1.1) to an equivalent problem with no common poles or zeros, that is

$$M_m(s)T(s) = M_q(s) \tag{7.1}$$

where, furthermore, $M_m(s)$ and $M_q(s)$ are polynomial matrices with the same dimensions as $T_1(s)$ and $T_2(s)$ respectively. Thus $M_m(s)$ and $M_q(s)$ have no poles at all and no common zeros.

Wolovich[3] has also introduced the notion of the "fixed poles" of $T(s)$. If $T_1(s)$ and $T_2(s)$ are scalar, it is clear that any pole of $T_2(s)$ which is not a pole of $T_1(s)$ must be a pole of $T(s)$; furthermore, any zero of $T_1(s)$ which is not a zero of $T_2(s)$ must be a pole of $T(s)$. This idea is extended in [3] to the matrix case. The fixed poles of $T(s)$ are poles which occur in all solutions of (1.1) [or (7.1)] and are the multivariable zeros of $T_1(s)$ not common to $T_2(s)$ and the poles of $T_2(s)$ not common to $T_1(s)$. Thus the fixed poles of (1.1), which are also the fixed poles of (7.1), are simply the zeros of $M_m(s)$. The existence or otherwise of unstable fixed poles of $T(s)$ determines whether or not a stable solution can be found for the exact model matching problem.

Given that these poles are present in all $T(s)$ solving (7.1), it is likely that computational saving might be effected if they were isolated from the 'variable' part of $T(s)$, so that this variable part alone was then the subject of the procedures of previous sections. Let us see how such poles can be isolated.

Let $G_L(s)$ be a greatest left divisor of (or greatest common left divisor of the columns of) $M_m(s)$. That is

$$M_m(s) = G_L(s)\bar{M}_m(s) \tag{7.2}$$

where $G_L(s)$ is a right multiple of every left divisor of $M_m(s)$ and $\bar{M}_m(s)$ is a polynomial matrix with no nontrivial left divisor and whose dimensions are the same as $M_m(s)$. The zeros of $|G_L(s)|$ are the fixed poles of $T(s)$[3].

The purpose of this section is to define a problem similar to (7.1) in which the 'new' $T(s)$ has no fixed poles but in which the remaining poles and the zeros are unchanged. Besides offering computational advantages, this would be useful because we know from a theorem in [3] for a problem with no fixed poles we can arbitrarily position the poles of the solution, provided we go to high enough order (c.f. the stable observer problem where we know that for an observable system we can always devise an observer with arbitrary poles provided we allow high enough order). Define

$$\tilde{T}(s) = |G_L(s)|T(s). \tag{7.3}$$

Substituting (8.2) and (8.3) into (8.1) and rearranging leads to

$$\bar{M}_m(s)\tilde{T}(s) = G_L^+(s)M_q(s) \tag{7.4}$$

where $G_L^+(s)$ is the adjoint matrix of $G_L(s)$. Since the matrices $\bar{M}_m(s)$ and $G_L^+(s)M_q(s)$ are polynomial matrices they have no poles and hence no common poles. Furthermore, since $\bar{M}_m(s)$ has no left divisor $\bar{M}_m(s)$ and $G_L^+(s)M_q(s)$ have no common zeros and $\tilde{T}(s)$ has no fixed poles. Nevertheless the zeros and non-fixed poles of $T(s)$ and $\tilde{T}(s)$ are identical since these matrices are related by a scalar quantity which 'cancels' with the fixed poles of $T(s)$. So the equation (7.4) represents a problem with no fixed poles from which solutions to (7.1) can readily be obtained.

Unfortunately, as is evident from (7.3), there are some proper solutions of the problem (7.1) which would be derived from improper solutions of (7.4). In other words, if (7.4) is solved as an MDP and the solutions are substituted into (7.3), some proper solutions of (7.1) will be missed out. This implies that the methods previously outlined in this paper must be varied to allow improper solutions in which the degree of the numerator exceeds that of the denominator by an amount equal to the number of fixed poles (i.e. the degree of $|G_L(s)|$). The relevant variation is that, when columns satisfying condition (2.5) are being selected instead of considering the high order coefficient matrix of $K(s)$ we must consider the high order coefficient matrix of

$$\begin{bmatrix} I_m & 0 \\ 0 & s^f I_q \end{bmatrix} K(s) = \begin{bmatrix} K_m(s) \\ s^f K_q(s) \end{bmatrix}$$

where f is the number of fixed poles.

As well as the ability to arbitrarily position poles (and hence to stabilise) there is another advantage associated with this alternate problem, viz., the entries in the denominator of the solution will have lower degrees than for the original problem. This is an advantage because in the procedure of Section 5 we always checked the stability of a solution by evaluating the determinant of the denominator to see if it is Hurwitz. The determinant of the denominator of the alternative problem will then have lower degree, and consequently the Hurwitz test will be simpler.

8. CONCLUSION

In this paper we have shown how to find all solutions of the MDP in parametric form and then how to select stable solutions from among these. If there is no stable solution of the MDP we have shown how to find all solutions, of a given order, of the exact model matching problem. We have further indicated how, by considering orders which are progressively larger than the minimum, the least order stable solutions can be determined. Finally an alternative problem to the standard MDP or exact model matching problem is presented. The various demonstrations of course would require the development of substantial software for their execution.

To approach here to finding a least order stable solution can be contrasted with that suggested in [9]: parameterisation first of all stable solutions, followed by the imposition of sets of polynomial equalities, with the property that there exists a real solution to a given set if and only if there exists a stable solution of a certain degree. Put another way, in this paper we first parameterise all solutions of the exact model matching problem of certain dimensional complexity, and then search for a stable solution; in [9], we parameterise all stable solutions, and then search for a solution of a certain dimensional complexity.

It is conceivable that state variable descriptions and methods could still be used. A substantial extension of [8] would be necessary, and it is our conviction that decision algebra would at some stage be required, just as in the schemes of this paper and [9], for the stable least order problem.

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