# Sensitivity improvement using optimal design

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### Synopsis

The central aim of this paper is to show that, under broad but physically meaningful conditions, an optimally designed linear-regulator control system is also a system in which sensitivity to plant parameter variation is reduced by the application of feedback. The theory is applicable to multiple-input multiple-output systems. The result established thus links modern control-theory concepts to a concept of classical control theory.


## List of symbols

$x$ = state vector
$x_0$ = initial value (at time $t_0$) of $x$
$t_0, t_1$ = initial and final times
$u(t), \hat{u}(t)$ = control vector and modified control vector
$G(t), \hat{G}(t)$ = system input matrix and modified system input matrix
$K(t)$ = feedback law
$F(t)$ = system matrix
$\Phi(t, \tau)$ = transition matrix
$\delta(t-\tau)$ = unit impulse
$I$ = unit matrix
$W(t, \tau)$ = impulse response matrix
$T(t, \tau)$ = return difference matrix
$T^a(t, \tau)$ = the adjoint of $T = T'(\tau, t)$, where $'$ is matrix transposition
$V(x_0, t_0, t_1)$, $V^0(x_0, t_0, t_1)$, $V(x_0, t_0)$ = values of the performance index for various boundary conditions
$Q(t), R(t), S(t), H(t)$ = weighting matrixes appearing in the optimal-control problem
$||x(t)||^2_{Q(t)}$ = weighted norm of $x$, equal to $x'Qx$ where $Q$ is symmetric positive, definite
$||F||$ = norm of $F$, e.g. $\sum_{i,j} |f_{ij}|$
$P(t), \pi(t; t_1, 0), \bar{P}(t)$ = various solutions of a matrix Riccati equation
$D(t, \tau), E(t, \tau), U(t, \tau)$ = various 2-dimensional kernels

## 1 Introduction

Viewed in retrospect, one of the aims of classical control theory can be seen as an attempt to control, or reduce the effects of, undesirable variations in plant parameters by feedback (the sensitivity-reduction problem); one of the aims of modern control theory can be seen as an attempt to minimise, again by feedback, the 'cost' of controlling a system, where the cost is generally some function, chosen by the designer, of the system states and control inputs (the optimal-control problem).

The optimal-control problem itself, and its relation to the associated problems of sensitivity reduction and control in the presence of noise, have been discussed informatively by Kalman.[1–3] In particular, Reference 1 considers the relation between the optimal-control and the sensitivity-reduction problems for a linear constant single-input finite-dimensional dynamical system; under certain broad and physically meaningful conditions, it is shown that an optimal design corresponds to a design where sensitivity reduction is achieved.

Of some interest is the question whether there can be any such relation in the multiple-input case and in the time-varying case, especially since more elegant expressions of a sensitivity-reduction criterion are now becoming available.[4] In this paper we show that an optimally designed multiple-input time-varying system is, under broad conditions, a system where sensitivity reduction is achieved. The treatment naturally subsumes the time-invariant case.

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## 2 Notations and plant description

Throughout we shall assume that the plant under consideration is linear, finite-dimensional and continuous, being described by the equations

$$\frac{dx}{dt} = F(t)x + G(t)u \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where $x$ is an $n$ vector (the state), $u$ is a $p$ vector (the control), $F(t)$ is an $n \times n$ matrix continuous in $t$, and $G$ is an $n \times p$ matrix continuous in $t$. Unless otherwise specified, $u$ is assumed to be piecewise-continuous.

In a regulator system we feed back certain linear combinations of the states all assumed available at the output, to act as an input to drive the system to the (desired) zero state. In other words, we have

$$u = -K'(t)x \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

with the superscript prime denoting matrix transposition. The system represented by eqns. 1 and 2 is shown diagrammatically in Fig. 1. The impulse response from points 1 to 2 is {with $1(t)$ the unit step}

$$W(t, \tau) = K'(t)\Phi(t, \tau)G(\tau)1(t-\tau) \quad . \; . \; . \; . \quad (3)$$

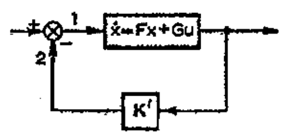


**Fig. 1**
*System with state-variable feedback*

where $\Phi(t, \tau)$ is the transition matrix associated with the homogeneous equation derived from eqn. 1:

$$\dot{x}(t) = F(t)x(t) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

It is possible to define a (time-domain) matrix return difference, such that, in the time-invariant case, the Laplace transform of this time-domain matrix is the usual return-difference matrix. The natural definition is (with $I$ the unit matrix and $\delta$ the unit impulse)

$$T(t, \tau) = I\delta(t-\tau) + W(t, \tau) \quad . \; . \; . \; . \; . \quad (5a)$$

$$= I\delta(t-\tau) + K'(t)\Phi(t, \tau)G(\tau)1(t-\tau) \quad (5b)$$

We note that, although $\{F, G, K\}$ is a particular realisation of $W(t, \tau)$, the return-difference matrix associated with this realisation, as determined from eqn. 5a, is co-ordinate-free by eqn. 5a; i.e. it is independent of the actual realisation used for $W(t, \tau)$. This would seem to indicate that it is not necessary to use a frequency-domain description for this co-ordinate-free quantity, in contrast to a suggestion made in Reference 1.

It is convenient in the sequel to use some operational notation. For a vector function $x$, assumed zero until some initial time $t_0$, we define a new function

$$y(t) = Tox = \int_{-\infty}^{+\infty} T(t, \tau)x(\tau)d\tau \quad . . . . . . . . . (6)$$

There is only a finite interval over which the integrand in eqn. 6 is nonzero, so that $y(t)$ is always well defined.

The adjoint operator to $T(t, \tau)$, written $T^a(t, \tau)$, is defined by

$$T^a(t, \tau) = T'(\tau, t) \quad . . . . . . . . . . (7)$$

and we can then define, at least formally,

$$T^a o T = \int_{-\infty}^{+\infty} T'(\lambda, t)T(\lambda, \tau)d\lambda \quad . . . . . . . (8)$$

The preceding definitions then imply (again formally)

$$z'o(Tox) = (T^a oz)'ox \quad . . . . . . . . . (9)$$

and $$x'oT^a oTox = (Tox)'o(Tox) \quad . . . . . . . (10)$$

Finally, we shall say that $T^a oT - I\delta(t - \tau)$ is nonnegative definite, written

$$T^a oT - I\delta(t - \tau) \geqslant 0 \quad . . . . . . . . (11a)$$

if, for all $x$ for which the left-hand side is well defined,

$$x'o\{T^a oT - I\delta(t - \tau)\}ox \geqslant 0 \quad . . . . . . (11b)$$

Note that, by eqn. 10, $T^a oT \geqslant 0$ for any $T$.

Reference 4 treats the sensitivity problem for time-varying systems, and shows that a necessary and sufficient condition to achieve improved sensitivity is for the return difference to satisfy eqn. 11a.

## 3 Optimal-control-regulator problem

One of the better known optimal-control problems which lead to a linear control law (and there are, indeed, very few of them) is the regulator problem involving a quadratic loss function. More specifically, the following function of the initial state $x_0$, initial time $t_0$ and final time $t_1$:

$$V(x_0, t_0, t_1) = \int_{t_0}^{t_1} \{||x(t)||^2_{Q(t)} + 2\langle x(t), S(t)u(t)\rangle + ||u(t)||^2_{R(t)}\}dt \quad . (12)$$

{where $||x(t)||^2_{Q(t)} = x'Qx$ and $\langle x, y\rangle = x'y$} is to be minimised by selecting $u(t)$ as some function $k\{x(t), t\}$ of the instantaneous state $x$ and the time $t$.

By a suitable change in the feedback law, it is possible to neglect the term involving $S(t)$; i.e. state and control are penalised, but only separately. For the problem to be a sensible one physically, both $Q$ and $R$ must be nonnegative definite, and, to avoid the possibility of the existence of a conjugate point, we take $R$ as positive definite. Writing $Q(t) = H'(t)H(t)$, we thus have

$$V(x_0, t_0, t_1) = \int_{t_0}^{t_1} \{||H(t)x(t)||^2_I + ||u(t)||^2_{R(t)}\}dt \quad . (13)$$

An extensive study of this problem may be found in Reference 2, where it is established that the minimum is

$$V^0(x_0, t_0, t_1) = ||x_0||^2_{\pi(t_0; t_1, 0)} \quad . . . . . . (14)$$

where $\pi(t; t_1, 0) = P(t)$ is the unique symmetric solution of the matrix Riccati differential equation

$$-\frac{dP}{dt} = F'(t)P + PF(t) - PG(t)R^{-1}(t)G'(t)P + H'(t)H(t) \quad . . . . (15)$$

satisfying the initial condition

$$\pi(t_1; t_1, 0) = P(t_1) = 0 \quad . . . . . . . . (16)$$

The control law yielding this minimum is

$$u(t) = -R^{-1}(t)G'(t)\pi(t; t_1, 0)x(t) \quad . . . . (17)$$

i.e. $$K'(t) = R^{-1}(t)G'(t)\pi(t; t_1, 0) \quad . . . . . . (18)$$

Of considerable engineering interest is the performance index obtained by letting the final time $t_1$ approach infinity, i.e.

$$V(x_0, t_0) = \lim_{t_1 \to \infty} \int_{t_0}^{t_1} \{||H(t)x(t)||^2_I + ||u(t)||^2_{R(t)}\}dt \quad . (19)$$

For the problem to have a well defined solution, we require $\bar{P}(t) = \lim_{t_1 \to \infty} \pi(t; t_1, 0)$ to be well defined, and this will be the case if:

(a) the plant is completely controllable

The earlier equations are still valid, with $\bar{P}(t)$ replacing $P(t)$.

Stability of the control law (i.e. stability of the closed loop system with the given feedback law) is not guaranteed by (a). A simple application of theorem (6–10) of Reference 2 shows that stability will be ensured if, for constants $\alpha_i > 0$,

(b) the plant $\{F, G, H\}$ is uniformly completely controllable
(c) the plant $\{F, G, H\}$ is uniformly completely observable
(d) $\alpha_1 I \leqslant H'H \leqslant \alpha_2 I$ (i.e. $H'H - \alpha_1 I$ is nonnegative definite etc.)
(e) $\alpha_3 I \leqslant R \leqslant \alpha_4 I$

While the physical motivation for uniformity in (b) and (c) is discussed in Reference 2, we comment that (b)–(e) ensure that, no matter how long we wait to control or observe, we cannot use an arbitrarily small control energy, or have to use an arbitrarily large energy, to change the state; nor, if the state is nonzero, does it show up in the output in an arbitrarily large or vanishingly small manner. It should be further noted that (b) to (e) do not merely ensure stability, but in fact uniform asymptotic stability, if we require additionally that (with $|| \ ||$ having now its usual meaning)

(f) $||F|| \leqslant \alpha_5$

(see Reference 3, theorem 4). In this case we also have

$$||\Phi(t, \tau)|| \leqslant \alpha_5 e^{-\alpha_6(t-\tau)}, \alpha_5, \alpha_6 > 0 \quad . . . (20)$$

Because of (e) we can simplify the problem further to the case $R(t) = I$ by replacing $u(t)$ with $\hat{u}(t)$, defined by

$$\hat{u}(t) = R^{1/2}(t)u(t) \quad . . . . . . . . . . (21)$$

and $G(t)$ with $\hat{G}(t)$, defined by

$$\hat{G}(t) = G(t)R^{-1/2}(t) \quad . . . . . . . . . (22)$$

Dropping the superscript ^, we then note that a minimisation of

$$V(x_0, t_0) = \lim_{t_1 \to \infty} \int_{t_0}^{t_1} \{||H(t)x(t)||^2_I + ||u(t)||^2_I\}dt \quad . (23)$$

is achieved with a feedback law

$$K'(t) = G'(t)\bar{P}(t) \quad . . . . . . . . . . (24)$$

with $\bar{P}(t)$ a solution of the equation

$$-\frac{dP}{dt} = F'(t)P + PF(t) - PG(t)G'(t)P + H'(t)H(t) \quad (25)$$

defined by the limiting process described earlier.

## 4 Analysis of sensitivity

In this Section our task is to compare the sensitivities of the open- and closed-loop systems with the feedback law predicted from the optimal theory. We assume the system satisfies (*b*)–(*f*) of Section 3, with $K'$ defining the feedback law.

The criterion for a sensitivity improvement when feedback is applied is

$$T^a o T - I\delta(t-\tau) \geqslant 0 \quad \ldots \quad (26)$$

where $T$ is the return difference (see Reference 4). We may calculate, using eqn. 5*b*,

$$T^a o T - I\delta(t-\tau) = D(t,\tau)1(t-\tau) + D'(\tau,t)1(\tau-t) \quad \ldots \quad (27)$$

where

$$D(t,\tau) = K'(t)\Phi(t,\tau)G(\tau) + G(t)\int_t^\infty \Phi'(\lambda,t)K(\lambda)K'(\lambda)\Phi(\lambda,\tau)\,d\lambda G(\tau) \quad . \quad (28)$$

Let us also define

$$U(t,\tau) = H(t)\Phi(t,\tau)G(\tau)1(t-\tau) \quad \ldots \quad (29)$$

Then $U^a o U = E(t,\tau)1(t-\tau) + E'(\tau,t)1(\tau-t) \quad . \quad (30)$

Assuming for the moment the existence of all integrals,

$$E(t,\tau) = G'(t)\int_t^\infty \Phi'(\lambda,t)H'(\lambda)H(\lambda)\Phi(\lambda,\tau)d\lambda G(\tau) \quad . \quad (31a)$$

$$= -G'(t)\int_t^\infty \{\Phi'(\lambda,t)\frac{d\bar{P}(\lambda)}{d\lambda}\Phi(\lambda,\tau) + \Phi'(\lambda,t)F'(\lambda)\bar{P}(\lambda)\Phi(\lambda,\tau) + \Phi'(\lambda,t)\bar{P}(\lambda)F(\lambda)\Phi(\lambda,\tau)\}d\lambda G(\tau)$$

$$+ G'(t)\int_t^\infty \Phi(\lambda,t)\bar{P}(\lambda)G(\lambda)G'(\lambda)\bar{P}(\lambda)\Phi(\lambda,\tau)d\lambda G(\tau)$$

using eqn. 15,

$$= -G'(t)\int_t^\infty \frac{d}{d\lambda}\{\Phi'(\lambda,t)\bar{P}(\lambda)\Phi(\lambda,\tau)\}d\lambda G(\tau) + G'(t)\int_t^\infty \Phi'(\lambda,t)\bar{P}(\lambda)G(\lambda)G'(\lambda)\bar{P}(\lambda)\Phi(\lambda,\tau)d\lambda G(\tau) \quad \ldots \quad (31b)$$

by using the elementary properties of the transition matrix. The upper limit of the first integral in eqn. 31*b* is shown below to be zero. Then from eqn. 24 and $\Phi(t,t) = I$, we have

$$E(t,\tau) = -G'(t)P(t)\Phi(t,\tau)G(\tau) + G'(t)\int_t^\infty \Phi'(\lambda,t)K(\lambda)K'(\lambda)\Phi(\lambda,\tau)d\lambda G(\tau)$$

$$= -K'(t)\Phi(t,\tau)G(\tau) + G'(t)\int_t^\infty \Phi'(\lambda,t)K(\lambda)K'(\lambda)\Phi(\lambda,\tau)d\lambda G(\tau)$$

$$= D(t,\tau) \quad \ldots \quad (32)$$

The existence of the integral in eqn. 31*a* follows from (*f*) in Section 3 and eqn. 20. The upper limit of the first integral in eqn. 31*b* will yield zero contribution if $\bar{P}(\infty)$ is finite, by eqn. 20. We note, however, that from eqn. 14

$$V^0(x_0, t) = \|x_0\|^2_{\bar{P}(t)}$$

while uniform complete controllability implies the existence of $\sigma$ independent of $t$, such that $x_0$ can be brought to zero by time $(t+\sigma)$, with a control $u$, bounded independently of $t$ (see Reference 2). From eqn. 19 and (*d*) of Section 3, it then follows that $\|\bar{P}(t)\|$ is bounded by a constant depending on $\sigma$, but independent of $t$. The existence of the second integral in eqn. 31*b* follows from the equality of eqn. 31*a* and the boundedness of the first integral in eqn. 31*b*. From eqns. 27, 30 and 32 we may then conclude

$$T^a o T - I\delta(t-\tau) = U^a o U \quad \ldots \quad (33)$$

The very nature of the right-hand-side member then requires eqn. 26:

$$T^a o T - I\delta(t-\tau) \geqslant 0$$

i.e. an improvement in sensitivity to parameter variation results from using the feedback law derived from the optimal design. Moreover, the 'more positive' the right-hand side of eqn. 33 the greater the improvement in sensitivity, as discussed in Reference 4.

## 5 Conclusion

In this paper we have shown that the use of modern control theory in system design can fulfil one of the design objectives of classical theory, namely the reduction of the sensitivity of a system to plant-parameter variation by use of feedback, and the extent of this improvement can be calculated.

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## 7 References

1 KALMAN, R. E.: 'When is a linear control system optimal?' *Trans. Amer. Soc. Mech. Engrs.*, March 1964, 86, Pt. D, p. 1
2 KALMAN, R. E.: 'Contributions to the theory of optimal control,' *Boln. Soc. Mat. Mex.*, 1960, p. 102
3 KALMAN, R. E., and BUCY, R. S.: 'New results in linear filtering and prediction theory,' *Trans. Amer. Soc. Mech. Engrs.*, March 1960, 82, Pt D, p. 95
4 ANDERSON, B. D. O., and NEWCOMB, R. W.: 'An approach to the time-varying sensitivity problem', Report SEL-66-06 (TR No. 6560-1), Stanford Electronics Laboratories, Stanford, Calif.